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High Energy Physics – Theory

Two-point correlator of twist-2 light-ray operators in N=4 SYM in BFKL approximation

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Abstract

We generalize local operators of the leading twist-2 of $\mathcal{N}=4$ SYM theory to the case of complex Lorentz spin j using principal series representation of $sl(2,R)$. We give the direct computation of correlation function of two such non-local operators in the BFKL regime when $j \to 1$. The correlator appears to have the expected conformal coordinate dependence governed by the anomalous dimension of twist-2 operator in NLO BFKL approximation predicted by Kotikov and Lipatov.

1. Introduction

The idea of analytical continuation in spin has a long history in theoretical physics going back to the early Regge's works on non relativistic quantum mechanics [1]. In the context of gauge theories these ideas turned into the what is now known as Regge theory. The BFKL approximation [2–4] was originally proposed for the study of the Regge (collinear) limit of hadron deep inelastic scattering amplitudes in QCD, when $g_{YM}^2 N \to 0$, the Mandelstam variable $s \to \infty$

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with $g^2_{YM} N \log \frac{s}{M^2}$ - fixed. Kotikov and Lipatov [5] applied a similar approximation to the study of anomalous dimensions of twist-2 operators tr $[F^{+\mu}_{\perp} g^{\perp}_{\mu\nu} D^{j-2} F^{\nu+}_{\perp} + ...]$ in the BFKL limit, when the Lorentz spin is analytically continued to $j-1=\omega \to 0$, the 't Hooft coupling $g^2 = \frac{g^2_{YM} N}{16\pi^2} \to 0$ and $\frac{g^2}{j-1} = const$. The anomalous dimension as a function of spin is understood as analytic continuation of the anomalous dimension of local operators. This immediately raises the question about the nature of such generalized object. What is the definition of operator with complex spin in terms of elementary fields? Is it possible to rederive the result of [5] directly calculating the correlation function between two such generalized operators? The goal of this paper is to construct an explicit form of twist-2 operators for arbitrary complex Lorentz spin and to perform the direct calculation of their two-point correlation function in the Leading Logarithmic Approximation (LLA) BFKL with Leading Order (LO) accuracy for the impact factors and NLO for the anomalous dimension.

We will achieve it defining certain light-ray operators with complex spin and conformal weight, which naturally generalize local twist-2 operators, namely, at integer values of the spin it turns into the integral of local operator along the light-ray. This allows us to resolve a subtle question about analytic continuation from local operators. The last has been extensively used in comparing of results [5] with integrability based calculations in $\mathcal{N}=4$ SYM [6–8]. To make such continuation, one usually uses the principle of maximal transcendentality within each order of the perturbation theory, however, this prescription was never been proven.[1]

At the same time, one can think about these light-ray operators as one-dimensional defects in superconformal $\mathcal{N}=4$ SYM and this paper as a first step in the direction of conformal bootstrap of nonlocal operators in BFKL limit. The Conformal Bootstrap Programme initiated in the classical works [10–14], received the substantial developing during the last years after the seminal paper [15]. Many impressive results was obtained from the bootstrap of the correlation functions of local operators (see [16] for review) and more recently CFT with boundaries and defects attracted a lot of attention particularly in $\mathcal{N}=4$ SYM [17–19] where one can also use Integrability. Nonlocal operators impose new symmetry restrictions on the CFT data and can let a further insight into the dynamics of conformal theories. Actually nonlocal light-ray operators already appear on the level of 4-point correlation function of local operators in Regge limit [20,21] efficiently resumming the intermediate local operators in OPE channel. Recently it was connected [21] with the Lorentzian inverse formula [22]. However, the explicit definition of nonlocal operators through the elementary fields is available in a very few cases and consideration mainly based on the symmetry analysis. At the same time, with our explicit definition of light-ray operators we can go beyond kinematics and get a deeper look into the dynamics of nonlocal operators in the gauge theories. Particularly we calculate the correlation function of three such light-ray operators in the partner papers [23,24].

Let us now describe the logic of our approach. We start with construction of a non-local light-ray twist-2 operator which transforms according to the principal series representation of $sl(2,R)$ with conformal spin $J = \frac{1}{2} + i\nu$, $\nu \in \mathbb{R}$ and even parity. This operator diagonalizes the renormalization group Hamiltonian. It is constructed from two local fields, with the coordinates x_{1-} and x_{2-} on the same light-ray, connected by the adjoint Wilson line factor. The operator is then integrated over the positions of both local operators along the light-ray.

[1] In the [9] a certain analyticity condition for the Baxter Q-function of $sl(2)$ Heisenberg spin chain were proposed reproducing the analytic continuations of harmonic sums w.r.t. the spin, at one and two loops.

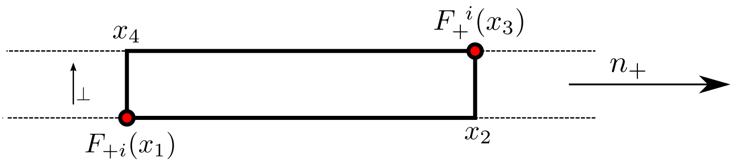

Fig. 1. The "frame" configuration for the regularized light ray operator: long sides are stretched along the light ray with the direction n_+, short sides oriented in an orthogonal direction.

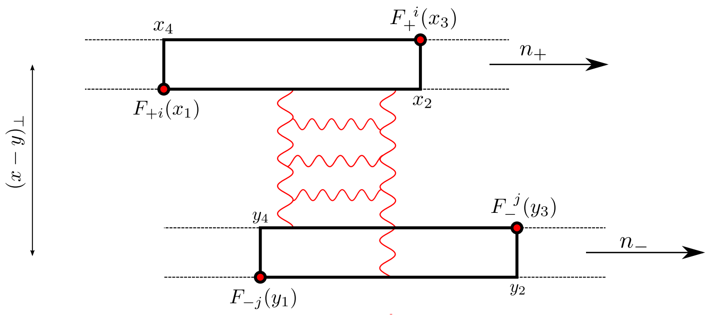

Fig. 2. Two Wilson frames, at a distance $|x-y|_\perp$ from each other, stretched in two different light-cone directions n_+ and n_- and a typical gluon exchange between them.

Constructed in this way, the light-ray operator is a singular, not well defined object in the BFKL regime. To avoid the singularity, we regularize it by placing the local operators on two different, but very close parallel light rays separated by a small distance $\delta x_\perp = |x_{1\perp} - x_{3\perp}|$ from each other, in a direction orthogonal to the light rays (see Fig. 1). We close it into a rectangular Wilson loop, with two fields inserted at the diagonally opposite corners of the loop, as depicted in Fig. 1. We will call this loop the Wilson frame.[2] Note that under a generic conformal transformation the frame will look almost the same: the distance between two light-lines will be slightly changed and the short lines connecting the ends of light ray intervals will be only slightly deformed. One can show that this deformation does not change the final results in our approximation: one can neglect the gluons emitted by infinitesimally short sides of the frame.

We will calculate in this paper the correlation function of two such objects separated by a certain distance in orthogonal space and stretched along two different light-like directions given by vectors n_+ and n_-, as shown in Fig. 2. We will use for that the OPE[3] decomposition over "color dipoles" in the limit when $(x_1 - x_3)^2 \to 0$, proposed by one of the authors [31] (see also the review [32]). The "color dipole" is a pair of parallel infinite light-like Wilson lines, with a cut-off σ on the momenta of gauge field in the light-cone direction. After such decomposition, symbolically depicted in Fig. 4, we calculate the correlator between two color dipoles. This calculation is done in two steps: first, for each dipole we compute the correlator for small values of the cutoff $\tilde{\sigma}$, such that $g^2 \log \frac{\tilde{\sigma}}{\sigma_0} \ll 1$, where $\sigma_0 < \tilde{\sigma} \ll \sigma$, when the lowest order of perturbation theory dominates in the LLA approximation, and then we evolve the result w.r.t. $\tilde{\sigma}$ to its final

[2] we hope the reader will avoid the confusion between this Wilson frame and the coordinate frame.

[3] For the recent development of these ideas see [25]. Another type of OPE for Wilson Loops with null edges was elaborated in [26–30].

value σ. It is important to stress that the evolution with respect to the scale $\sigma_\pm$ for each color dipole is governed by the BFKL equation [31]. The ratio of cut-offs $\frac{\sigma_+\sigma_-}{\sigma_{0+}\sigma_{0-}}$, due to the conformal invariance, appears to be related to certain anharmonic ratios defined by the shapes of our configuration of frames. The last step is the integration over the coordinates of Wilson frame along each of the light-rays. In what follows, we are going to precise each step of this calculation.

2. Generalization of twist-2 operators to the case of principal series $sl(2, R)$

The twist-2 supermultiplet of local operators was explicitly constructed in [33]. For example, one of the components at zero order in g_{YM} reads as follows (j is even):

$$\mathcal{S}^j_{\rm loc}(x) = 6\mathcal{O}^j_{gg}(x) + \frac{j-1}{2}\mathcal{O}^j_{qq}(x) + \frac{j(j-1)}{4}\mathcal{O}^j_{ss}(x), \tag{1}$$

where

$$\mathcal{O}^j_{gg}(x) = \text{tr}\, \mathcal{G}^{\frac{5}{2}}_{j-2,x_1,x_2} F_{+\perp}^{\ \ \mu}(x_1) g^{\perp}_{\mu\nu} F_{+\perp}^{\ \ \nu}(x_2)|_{x=x_1=x_2}, \tag{2}$$

$$\mathcal{O}^j_{qq}(x) = \text{tr}\, \mathcal{G}^{\frac{3}{2}}_{j-1,x_1,x_2} \bar\lambda_{\dot\alpha A}\sigma^{+\dot\alpha\beta}(x_1)\lambda^A_\beta(x_2)|_{x=x_1=x_2}, \tag{3}$$

$$\mathcal{O}^j_{ss}(x) = \text{tr}\, \mathcal{G}^{\frac{1}{2}}_{j,x_1,x_2} \bar\phi_{AB}(x_1)\phi^{AB}(x_2)|_{x=x_1=x_2}. \tag{4}$$

We introduced here the differential operator $\mathcal{G}^\alpha_{n,x_1,x_2} = i^n(\nabla_{x_2} + \nabla_{x_1})^n C^\alpha_n(\frac{\nabla_{x_2}-\nabla_{x_1}}{\nabla_{x_2}+\nabla_{x_1}})$, where $C^\alpha_n(x)$ is the Gegenbauer polynomial of order n with index α. ∇_x are covariant derivatives in the light-like direction n_+: $\nabla_x = n^\mu_+(\partial_\mu - ig_{YM}A_\mu) = \partial_+ - ig_{YM}A_+$. The fields entering the operators belong to the set $X = \{F_{+\perp}^{\ \ \mu}, \lambda^A_{+\alpha}, \bar\lambda^{\dot\alpha}_{+A}, \phi^{AB}\}$ which contains the field components with maximal spin (see appendix A).

Let us note that all components of twist-2 supermultiplet are constructed from so called primary conformal operators, in the sense that they realize the highest-weight representation of $sl(2, R)$. For example, in the case of $\mathcal{S}^j_{\rm loc}$ the operators $\mathcal{O}^j_{gg}$, $\mathcal{O}^j_{qq}$, $\mathcal{O}^j_{ss}$ are primaries, with conformal spin $J = j + 1$. Due to supersymmetry we should work with superconformal operators transforming under an irreducible representation of $sl(2|4)$. It leads to the superprimary operators which are a linear combination of conformal operators as in (1). It is important to stress that the coefficients in this combination do not depend on the Yang-Mills coupling constant g^2_{YM} and the renormalization takes place for each conformal operator separately.[4] These superconformal operators diagonalize the one-loop dilatation operator given by the Hamiltonian:

$$H = g^2[H_{12} + H_{21}], \tag{5}$$

$$H_{i,i+1}\phi(z_i, z_{i+1}) = 2\Big[\psi(J^{\mathfrak{G}}_{i,i+1}) - \psi(1)\Big], \tag{6}$$

where $J^{\mathfrak{G}}_{i,i+1}$ is defined through the Casimir operator $J^2_{i,i+1} = J^{\mathfrak{G}}_{i,i+1}(J^{\mathfrak{G}}_{i,i+1} - 1)$ of the full $\mathfrak{G} = PSU(2, 2|4)$ group [34].

We start with generalization of local conformal operators. Our logic will be close to the logic of [35]. Local conformal operators correspond to the discrete unitary irreps of $sl(2, R)$. Let us construct a nonlocal light-ray operator[5] which realizes the principal series irrep of $sl(2, R)$ with

[4] It is so, because the supercharges don't depend on g_{YM}.

[5] An interesting dual conformal symmetry on the light-cone was discovered in [36].

the conformal spin $J = \frac{1}{2} + i\nu,\ \nu \in \mathbb{R}$ and even parity. A general light-ray operator with local fields χ^s of the same conformal spin s looks as follows:

$$S^s_\phi(x_{1\perp}) = \int\limits_{-\infty}^{\infty} dx_{1-} \int\limits_{x_{1-}}^{\infty} dx_{2-} \phi(x_{1-}, x_{2-}) \chi^s(x_1)[x_1, x_2]_{Adj} \chi^s(x_2), \tag{7}$$

where

$$[x_1, x_2]_{Adj} = P\exp[ig_{YM} \int\limits_0^1 du (x_2 - x_1)_\mu A^\mu_{Adj}(x_1(1-u) + x_2 u)], \tag{8}$$

and the function $\phi(x_1, x_2)$ is an arbitrary function of two variables. We are looking for the operators S^s_ϕ which are the eigenfunctions of $sl(2, R)$ Casimir operator defined in the following way: Take the generators of $sl(2, R)$ satisfying standard relations:

$$\begin{aligned} &\left[J_3, J_\pm\right] = \pm J_\pm, \\ &\left[J_+, J_-\right] = -2J_3. \end{aligned} \tag{9}$$

and realize them on the fields with conformal spin s:

$$\begin{aligned} J_+ &= \frac{i}{\sqrt{2}} P_+ = \frac{1}{\sqrt{2}} \frac{d}{dx}, \\ J_- &= \frac{i}{\sqrt{2}} K_+ = \sqrt{2}(2sx + x^2 \frac{d}{dx}), \\ J_3 &= \frac{i}{2}(D + M_{-+}) = s + x\frac{d}{dx}. \end{aligned} \tag{10}$$

Here x is a coordinate along the light ray. The equation on the eigenvalues and eigenfunctions $\phi(x_{1-}, x_{2-})$ of the Casimir operator

$$\vec{J}^2 S^s_\phi = J(J-1) S^s_\phi = (j+1) j S^s_\phi \tag{11}$$

can thus be rewritten as a partial differential equation

$$\left[\beta^2 \left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2}\right) - 2s\beta\frac{\partial}{\partial\beta} + s(s+1)\right]\phi(\alpha, \beta) = J(J-1)\phi(\alpha, \beta), \tag{12}$$

where $\alpha = x_{1-} + x_{2-},\ \beta = x_{2-} - x_{1-}$. Separating the variables $\phi(\alpha, \beta) = f(\alpha) g(\beta)$ we get:

$$\begin{cases} \dfrac{\partial^2}{\partial\alpha^2} f(\alpha) = -k^2 f(\alpha), \\ \left(\beta^2 \dfrac{\partial}{\partial\beta^2} - 2s\beta\dfrac{\partial}{\partial\beta} + s(s+1) + k^2\beta^2\right) g(\beta) = J(J-1) g(\beta). \end{cases} \tag{13}$$

The general solution for the eigenfunctions reads as follows:

$$\begin{aligned} &\frac{\phi(x_{1-}, x_{2-})}{(x_{2-} - x_{1-})^{2s-\frac{3}{2}}} = \\ &= \int dk \left[\eta_1(k) e^{ik(x_{1-}+x_{2-})} (C_{11} \mathbf{J}_{-\frac{1}{2}+J}(k(x_{2-} - x_{1-})) + C_{12} \mathbf{J}_{\frac{1}{2}-J}(k(x_{2-} - x_{1-}))) + \right. \\ &\left. + \eta_2(k) e^{-ik(x_{1-}+x_{2-})} (C_{21} \mathbf{J}_{-\frac{1}{2}+J}(k(x_{1-} - x_{2-})) + C_{22} \mathbf{J}_{\frac{1}{2}-J}(k(x_{2-} - x_{1-}))\right], \end{aligned} \tag{14}$$

where $\mathbf{J}_\nu(x)$ - is a Bessel function and $\eta_1(k),\ \eta_2(k)$ are arbitrary functions of k. In addition, we should impose a set of constraints on the operator S^s_ϕ. First of all, it should be an entire function of J, to allow for an unambiguous analytic continuation of the light-ray operator in J, and the dimension of this operator should coincide with the standard local twist-2 operator for any integer J. Both of these conditions are satisfied if we choose a linear combination of Bessel functions as the Hankel function of the second order:

$$C_{i1}\mathbf{J}_{-\frac{1}{2}+J}(k(x_{2-}-x_{1-})) + C_{i2}\mathbf{J}_{\frac{1}{2}-J}(k(x_{2-}-x_{1-})) \to \mathbf{H}^2_{J-\frac{1}{2}}(k(x_{2-}-x_{1-})),\quad i\in\{1,2\} \tag{15}$$

In this way we obtain an operator which is an entire function of spin: It is well defined for $J=\frac{1}{2}+i\nu$ and thus it can be uniquely continued to the whole complex plane of J. It is natural to choose the so far arbitrary coefficient functions as $\eta_1(k)=\eta_2(k)=\frac{1}{2}\delta(k)(\frac{k}{2})^{J-\frac{1}{2}}$ which naturally sets to zero the center-of-mass momentum k and cancels the singularity at $k\to 0$.[6]

Now, using the asymptotics of Hankel function at $k\to 0$:

$$\mathbf{H}^2_{J-\frac{1}{2}}(k(x_{2-}-x_{1-})) \to -(\frac{k(x_{2-}-x_{1-})}{2})^{-J+\frac{1}{2}}\frac{\Gamma(J-\frac{1}{2})}{\pi}, \tag{16}$$

we get the following form of the light-ray operators (denoted by $\breve{S}$) for scalars, fermions and gluons:

$$\breve{S}^J_{sc}(x_{1\perp}) = \int_{-\infty}^{\infty} dx_{1-}\int_{x_{1-}}^{\infty} dx_{2-}(x_{2-}-x_{1-})^{-J}\ \mathrm{tr}\,\bar{\phi}_{AB}(x_1)[x_1,x_2]_{Adj}\phi^{AB}(x_2), \tag{17}$$

$$\breve{S}^J_{f}(x_{1\perp}) = \int_{-\infty}^{\infty} dx_{1-}\int_{x_{1-}}^{\infty} dx_{2-}(x_{2-}-x_{1-})^{-J+1}\ \mathrm{tr}\,\bar{\lambda}_{\dot{\alpha}A}(x_1)\sigma^{+\dot{\alpha}\beta}[x_1,x_2]_{Adj}\lambda^A_\beta(x_2), \tag{18}$$

$$\breve{S}^J_{gl}(x_{1\perp}) = \int_{-\infty}^{\infty} dx_{1-}\int_{x_{1-}}^{\infty} dx_{2-}(x_{2-}-x_{1-})^{-J+2}\ \mathrm{tr}\,F_{+\perp}^{\ \mu}(x_1)g^\perp_{\mu\nu}[x_1,x_2]_{Adj}F_{+\perp}^{\ \nu}(x_2). \tag{19}$$

Let us clarify the correspondence of nonlocal operators (17)-(19) to the local operators, using gluons as an example. We take an odd integer J in (19) and define the integral over x_2-x_1, with the prescription analogous to the eq.(3.19) of [35]. This gives, e.g. for the gluons:

$$\breve{S}^J_{gl}(x_{1\perp}) = \frac{2^{3-J}2\pi i}{\Gamma(J-2)}\int_{-\infty}^{\infty} dx_-\ \mathrm{tr}\left[(\overleftarrow{\nabla}-\overrightarrow{\nabla})^{J-3}F_+^{\ i}(x)F_{+i}(x)\right], \tag{20}$$

where $\overrightarrow{\nabla}$ and $\overleftarrow{\nabla}$ are covariant derivatives which act on the left and right scalars, correspondingly. On the other hand, the local gluon operator (for odd J) has the following form:

$$\mathcal{O}^j_{gl}(x) = \mathcal{O}^{J-1}_{gl}(x) = \mathrm{tr}\left[i^{J-3}(\overrightarrow{\nabla}+\overleftarrow{\nabla})^{J-3}C^{\frac{5}{2}}_{J-3}\left(\frac{\overrightarrow{\nabla}-\overleftarrow{\nabla}}{\overrightarrow{\nabla}+\overleftarrow{\nabla}}\right)F_+^{\ i}(x)F_{+i}(x)\right]. \tag{21}$$

[6] such choice of $\eta_1(k),\ \eta_2(k)$ also leads to the light-ray operator transforming as a primary w.r.t. SO(4,2), see the related discussion in [44].

Integrating it over the coordinate, we get:

$$\int_{-\infty}^{\infty} dx_- \, \mathcal{O}_{gl}^{J-1}(x) = i^{J-3} \frac{\Gamma(J-\frac{1}{2}) 2^{J-3}}{\Gamma(\frac{5}{2})\Gamma(J-2)} \int_{-\infty}^{\infty} dx_- \mathrm{tr}\left[(\overrightarrow{\nabla} - \overleftarrow{\nabla})^{J-3} F_+{}^i(x) F_{+i}(x) \right]. \quad (22)$$

All terms in $\mathcal{O}_{gl}^{J-1}(x)$ with nonzero power $\overrightarrow{\nabla} + \overleftarrow{\nabla}$ disappear because they are full derivatives. Now we want to construct a nonlocal superconformal operator. Nonlocal operator which corresponds to $\int dx\, \mathcal{S}_{loc}^{j}(x)$ is a sum $\mathcal{S}_{nloc}^{J} = c_{sc} S_{sc}^{J} + c_f S_f^J + c_{gl} S_{gl}^J$ with some so far unknown coefficients. They can be fixed by comparing the local and nonlocal operators in the case of integer J. For example for gluons, using (20), (22) we conclude that

$$c_{gl} = \frac{i^J 2^{2J-4} \Gamma(J-\frac{1}{2})}{\pi \Gamma(\frac{1}{2})}. \quad (23)$$

And similarly we get the other coefficients:

$$c_f = i \frac{J-2}{2} \frac{i^J 2^{2J-4} \Gamma(J-\frac{1}{2})}{\pi \Gamma(\frac{1}{2})}, \quad (24)$$

$$c_{sc} = -(J-2)(J-1) \frac{i^J 2^{2J-4} \Gamma(J-\frac{1}{2})}{\pi \Gamma(\frac{1}{2})}. \quad (25)$$

Finally, the superconformal operator with conformal spin J reads as follows:

$$\breve{\mathcal{S}}^J(x_{1\perp}) = \frac{i^J 2^{2J-4} \Gamma(J-\frac{1}{2})}{\pi \Gamma(\frac{1}{2})} \left(-(J-1)(J-2) \breve{S}_{sc}^J(x_{1\perp}) + i \frac{J-2}{2} \breve{S}_f^J(x_{1\perp}) + \breve{S}_{gl}^J(x_{1\perp}) \right). \quad (26)$$

Now let us omit the common factor and redefine the operator as follows:

$$\breve{\mathcal{S}}^J(x_{1\perp}) = -(J-1)(J-2) \breve{S}_{sc}^J(x_{1\perp}) + i \frac{J-2}{2} \breve{S}_f^J(x_{1\perp}) + \breve{S}_{gl}^J(x_{1\perp}). \quad (27)$$

In what follows, we will be interested in the BFKL limit when the Lorentz spin j goes to one: $j = J - 1 = 1 + \omega \to 1$. In this limit, the operator takes the following form:

$$\breve{\mathcal{S}}^{2+\omega}(x_{1\perp}) = -\omega \breve{S}_{sc}^{2+\omega}(x_{1\perp}) + i \frac{\omega}{2} \breve{S}_f^{2+\omega}(x_{1\perp}) + \breve{S}_{gl}^{2+\omega}(x_{1\perp}), \quad (28)$$

where we kept in the coefficients only the leading order in ω.

These light ray operators, regularized by counterterms in $\overline{MS}$ scheme, are well defined objects and one can calculate their anomalous dimensions order-by-order in perturbation theory. However, if one wants to calculate $j \to 1$ asymptotics of anomalous dimensions in all orders by BFKL approximation, the light-ray operators regularized with counterterms are not convenient since direct application of BFKL analysis gives divergent result. The way out is to introduce point-splitting regularization of UV-divergencies of light-ray operators. Namely we will introduce the following regularization mentioned in the introduction: we replace the light-ray operator (27) by a non-local rectangular Wilson loop with two opposite sides stretched along the light-cone direction and two other sides (whose length tends to zero) being orthogonal to the light-cone. The fields are placed in two opposite corners of the Wilson frame. The configurations of Wilson frame and the positions of operators resulting from these operations are shown in Fig. 1. For example, for the gluons we get:

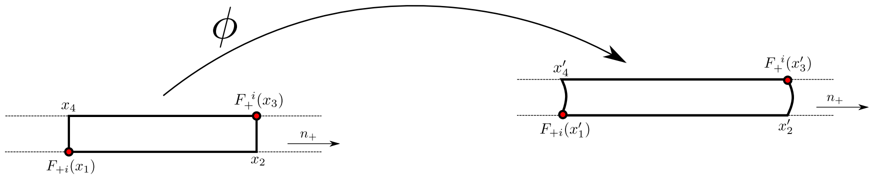

Fig. 3. A generic conformal transformation $\phi : x \to x'$ acting on infinitesimally narrow Wilson frame almost conserves its shape.

$$
\begin{aligned}
&\check{S}^J_{gl}(x_{1\perp}) = \lim_{(x_{1\perp}-x_{3\perp})^2\to 0} S^J_{gl}(x_{1\perp}, x_{3\perp}) = \\
&= \lim_{(x_{1\perp}-x_{3\perp})^2\to 0} \int_{-\infty}^{\infty} dx_{1-} \int_{x_{1-}}^{\infty} dx_{3-}(x_{3-} - x_{1-})^{-J+2} \operatorname{tr} F_+{}^i(x_1)[x_1, x_3]_\square F_{+i}(x_3), \qquad (29)\\
&x_1 = (x_{1-}, 0, x_{1\perp}), \quad x_3 = (x_{3-}, 0, x_{3\perp})
\end{aligned}
$$

and the full regularized operator reads as follows:

$$
\mathcal{S}^J(x_{1\perp}, x_{3\perp}) = -(J-1)(J-2)S^J_{sc}(x_{1\perp}, x_{3\perp}) + i\frac{J-2}{2}S^J_f(x_{1\perp}, x_{3\perp}) + S^J_{gl}(x_{1\perp}, x_{3\perp}), \qquad (30)
$$

where $[x_1, x_3]_\square$ - Wilson frame with the local fields placed at the corners x_1, x_3 on a diagonal of the frame and the short sides x_{23}, x_{41} directed into the orthogonal space. The operation "lim" is understood in the following sense: at first we should carry out all calculation with the fixed length of short sides $|x_{1\perp} - x_{3\perp}|^2 \neq 0$, and only after that we take the limit. In this sense we can treat our infinitesimally narrow Wilson frame as a conformal object. Namely if we carry out any conformal transformation this infinitesimally narrow Wilson frame almost conserves its shape (see Fig. 3).

3. OPE over color dipoles for nonlocal operators $\mathcal{S}^J_{nloc}$

Let us introduce two nonlocal super-primary operators defined above: the first one, $\mathcal{S}^{2+\omega_1}_+$, is placed along n_+ and the second, $\mathcal{S}^{2+\omega_2}_-$, along n_-. Our goal is to calculate their correlation function in the BFKL limit $\omega_1, \omega_2 \to 0$, $\frac{g^2}{\omega_1}$, $\frac{g^2}{\omega_2}$ $\to$ fixed. The main contribution in this case comes from large distances $L_+(L_-)$ along $n_+(n_-)$. The integral over $L_+(L_-)$ entering into the definition of the regularized light-ray operator leads to the Regge pole $\frac{1}{\omega_{1,2}}$. This pole is analogous to the large log $\frac{s}{M^2}$ in the Regge approximation for high energy scattering amplitudes.[7] Summing all contributions in $\frac{g^2}{\omega_{1,2}}$ in our setup corresponds to summing the powers $g^2 \log \frac{s}{M^2}$ in the LLA in high energy scattering - the key feature of the BFKL approximation.

In this case we can apply the OPE over color dipoles, which was elaborated for the scattering amplitudes in [31]. Let us remind the logic of this approach within the scattering theory and then relate it to our calculation.

[7] In high energy scattering M is a reference scale such that $m^2 \ll M^2 \ll s$, where m is a characteristic hadron mass.

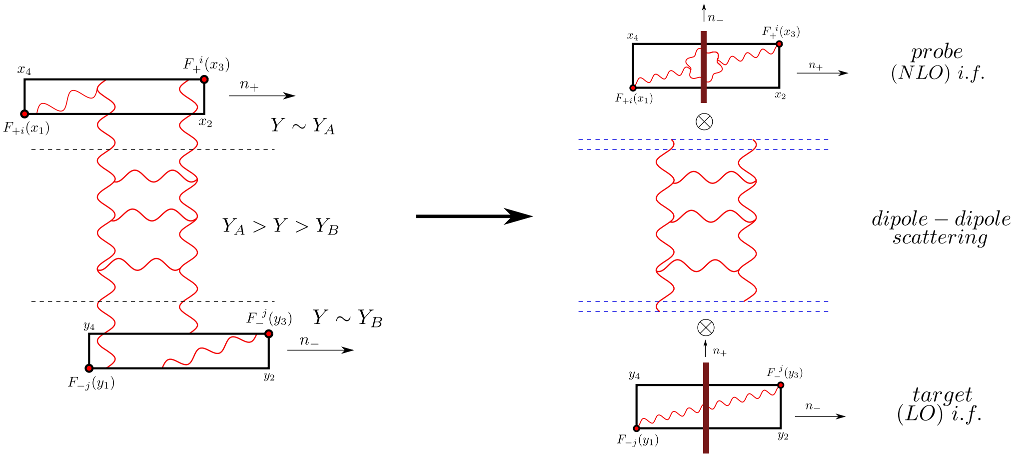

Fig. 4. Color dipole decomposition for the correlator of two frames. Due to the separation of scales w.r.t. the rapidity Y in BFKL approximation, the correlator factorizes into the "probe impact factors", the dipole-dipole interaction and the "target impact factor". The analogue of rapidity Y in the current paper is $\log \sigma$ – the logarithm of cutoff for the momenta of the gauge fields in the light cone direction. As an example, the probe is represented by an NLO graph where as the target – by an LO graph.

The high-energy behavior of the amplitudes can be studied in the framework of the rapidity evolution of Wilson-line operators forming color dipoles. The main idea is the factorization in rapidity: we separate a typical functional integral describing scattering of two particles into (i) the integral over the gluon (and gluino) fields with rapidity close to the rapidity of the "probe" Y_A, (ii) the integral over the gluons with rapidity close to the rapidity of the target Y_B, and (iii) the integral over the intermediate region of rapidities $Y_A > Y > Y_B$, see Fig. 4. The result of the first integration is a certain coefficient function (impact factor) times color dipole (ordered in the direction of the probe velocity) with rapidities up to Y_A. Similarly, the result of the second integration is again the impact factor times the color dipole ordered in the direction of target' velocity with rapidities greater than Y_B. The result of the last integration is the correlation function of two dipoles which can be calculated using the evolution equation for color dipoles, known in the leading and next-to-leading order [31].

To factorize in rapidity, it is convenient to use the background field formalism: we integrate over gluons with $Y > Y_A$ and leave the gluons with $Y < Y_A$ as a background field, to be integrated over later. Since the rapidities of background gluons are very different from the rapidities of gluons in our Feynman diagrams, the background field is seen by the probe in the form of a shock wave (pancake) due to the Lorentz contraction. To derive the expression of a quark or gluon propagator in this shock-wave background we represent the propagator as a path integral over various trajectories, each of them weighed with the gauge factor $\mathrm{Pexp}(ig \int dx_\mu A^\mu)$ ordered along the propagation path. Now, since the shock wave is very thin, quarks or gluons emitted by the probe do not have time to deviate in transverse direction so their trajectory inside the shock wave can be approximated by a segment of the straight line. Moreover, since there is no external field outside the shock wave, the integral over the segment of straight line can be formally extended to $\pm\infty$ limits yielding the Wilson-line gauge factor

$$U_{x_\perp}^{\sigma_+} = P\exp[ig_{YM}\int_{-\infty}^{\infty} dx_+ A_-^{\sigma_+}(x)], \tag{31}$$

where we have used the gauge field with a cutoff $\sigma = e^Y$ w.r.t. the longitudinal momenta k_+

$$A_\mu^{\sigma_+}(x) = \int d^4k\theta(\sigma_+ - |k_+|)e^{ikx}A_\mu(k). \tag{32}$$

Now let us adopt this scheme of calculation to our correlator. In our case, the Wilson frames play the role of the probe and the target, respectively. The gluons emitted and absorbed within each frame contribute to their "impact factors". The correlator factorizes into these two impact factors and the BFKL evolution of color dipoles appearing in the OPE of the frames. The BFKL evolution corresponds to the evolution of the cutoff from some minimal[8] σ_{0+} to the final value σ_+ for the frame oriented in the n_- direction. Similarly, for the second dipole, the evolution goes from σ_{0-} to σ_-. As we will see later, the ratio $\frac{\sigma_+\sigma_-}{\sigma_{0+}\sigma_{0-}}$ will be identified with a certain anharmonic ratio of characteristic sizes of the configuration. The rest of the calculation is very similar to the case of scattering in Regge kinematics and is based on computations of graphs in the pancake background.

The propagators of gluons, scalars and fermions get modified by the presence of this pancake background. Denoting the corresponding average as $\langle\dots\rangle$ we represent these propagators as follows[9]:

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle = \frac{1}{4\pi^3}\int d^2z_\perp U_z^{\sigma_+ab}[x_+g_{\mu\xi}^\perp - n_{-\mu}(x-z)_\xi^\perp][y_+\delta_\nu^{\perp\xi} - n_{-\nu}(y-z)_\perp^\xi]\cdot$$
$$\cdot\frac{x_+|y_+|}{[-2(x-y)_-x_+|y_+| + x_+(y-z)_\perp^2 + |y_+|(x-z)_\perp^2 + i\epsilon]}, \tag{33}$$

$$\langle\hat\Phi_I^a(x)\hat\Phi_J^b(y)\rangle = \frac{\delta_{IJ}}{4\pi^3}\int d^2z_\perp\cdot$$
$$\cdot\frac{x_+|y_+|U_z^{ab}}{\left[-2(x-y)_-x_+|y_+| + (x-z)_\perp^2|y_+| + (y-z)_\perp^2 x_+ + i\epsilon\right]^2}, \tag{34}$$

$$\langle\lambda_\alpha^{aI}(x)\,\bar\lambda_{\dot\alpha}^{bJ}(y)\rangle = \frac{i}{2\pi^3}\int d^2z_\perp U_z^{ab}\left[x_+\bar n_- + (\bar x - \bar z)\right]n_+\left[|y_+|\bar n_- - (\bar y - \bar z)\right]\cdot$$
$$\cdot\frac{x_+|y_+|}{[-2(x-y)_-x_+|y_+| + (x-z)_\perp^2|y_+| + (y-z)_\perp^2 x_+ + i\epsilon]^3}, \tag{35}$$

where $\bar n_{\alpha\dot\alpha} \equiv n_\mu\bar\sigma_{\alpha\dot\alpha}^\mu$ and $n^{\dot\alpha\alpha} \equiv n^\mu\sigma_\mu^{\dot\alpha\alpha}$, and $U_z^{ab} = 2\mathrm{tr}(t^aU_zt^bU_z^\dagger)$.

The effective propagator for any field χ has a form of decomposition over Wilson lines $\langle\chi^a(x)\chi^b(y)\rangle = \int d^2z_\perp U_{z_\perp}^{\sigma_+ab}f(z,x,y)$, where $f(z,x,y)$ is a function which depends only on the coordinates and doesn't carry the color indices. Then any conformal nonlocal operator S_{sc}^J, S_f^J, S_{gl}^J can be decomposed over the color dipoles:

[8] The initial point σ_0 is an analog of the low normalization point $\mu^2 \sim Q_0^2 \sim 1$ GeV for usual DGLAP evolution. It should be chosen in such way that $\sigma_0 \gg M$ but $g^2\ln\frac{\sigma_0}{M} \ll 1$ where M is of order of the mass of colliding particles (in our case of M is of order of inverse transverse separations of Wilson frames).

[9] pancake is placed along n_+ direction.

$$\mathrm{tr}(\chi(x)[x,y]_{\square}\chi(y)) \to \int d^2z_\perp f(z,x,y) U^{\sigma+ab}_{z\perp}\mathrm{tr}(t^a U^{\sigma+}_{x\perp} t^b U^{\sigma+}_{y\perp}) \xrightarrow[N\to\infty]{}$$
$$\int d^2z_\perp f(z,x,y)\frac{N^2}{2}(1-\frac{1}{N}\mathrm{tr}(1-U^{\sigma+}_{x\perp}U^{\sigma+\dagger}_{z\perp}) - \frac{1}{N}\mathrm{tr}(1-U^{\sigma+}_{z\perp}U^{\sigma+\dagger}_{y\perp}) + O(g^2)), \tag{36}$$

where we have used the following sequence of equalities:

$$\begin{aligned}
&U^{\sigma+ab}_{z}\mathrm{tr}(t^a U^{\sigma+}_{x} t^b U^{\sigma+}_{y}) = 2\mathrm{tr}(t^a U^{\sigma+}_{z} t^b U^{\sigma+\dagger}_{z})\,\mathrm{tr}(t^a U^{\sigma+}_{x} t^b U^{\sigma+}_{y})\\
&= \mathrm{tr}(t^a U^{\sigma+}_{x} U^{\sigma+\dagger}_{z} t^a U^{\sigma+}_{z} U^{\sigma+\dagger}_{y}) =\\
&= \frac{1}{2}\mathrm{tr}(U^{\sigma+}_{x}U^{\sigma+\dagger}_{z})\,\mathrm{tr}(U^{\sigma+}_{z}U^{\sigma+\dagger}_{y}) - \frac{1}{2N}\mathrm{tr}(U^{\sigma+}_{x}U^{\sigma+\dagger}_{y}) \xrightarrow[N\to\infty]{}\\
&\frac{N^2}{2}\left[1-\mathbf{U}^{\sigma+}(x,z)-\mathbf{U}^{\sigma+}(z,y)+\mathbf{U}^{\sigma+}(x,z)\mathbf{U}^{\sigma+}(z,y)\right]=\\
&= \frac{N^2}{2}\left[1-\mathbf{U}^{\sigma+}(x,z)-\mathbf{U}^{\sigma+}(z,y)\right]\left(1+O(g^2,\frac{1}{N^2})\right),
\end{aligned} \tag{37}$$

where we have introduced the color dipole operator in fundamental representation:

$$\mathbf{U}^{\sigma+}(x_{1\perp},z_\perp) = 1 - \frac{1}{N}\mathrm{tr}(U^{\sigma+}_{x_{1\perp}}U^{\sigma+\dagger}_{z\perp}) \tag{38}$$

with $U^{\sigma+}_{x\perp}$ defined in (31).

The first two lines of (37) hold at any N. In the last line we dropped the term non-linear in color dipoles. This is valid since in the BFKL approximation we take into account only linear evolution of Wilson-line operators corresponding to the processes containing two reggeized gluons in t-channel.[10]

Let us now proceed with calculation of the gluonic part. Calculation for scalars and fermions can be done in the same way and it turns out that the main contribution for the "impact factor" in LO comes just from gluons. Naively, it can be explained from the fact that in the limit $\omega = J-2\to 0$ the scalar and fermionic terms enter with subleading coefficients into the eq. (28). The explicit computation of correlators confirms it.

Using the propagator (33) and the decomposition (36) the OPE for the gluon operator stretched in n_+ direction reads as follows[11]

$$\begin{aligned}
&\mathrm{tr}\, F_{+i}(x_1)[x_1,x_3]F_+{}^i(x_3) \to\\
&\frac{N^2}{2\pi^3}\int d^2z\Bigg(\frac{2}{(x_{3-}(x_1-z)_\perp^2 - x_{1-}(x_3-z)_\perp^2)^2}\\
&+\frac{x_{1-}x_{3-}(9(x_1-z)_\perp^2(x_3-z)_\perp^2 + 6(x_1-z,x_3-z)^2)}{(x_{3-}(x_1-z)_\perp^2 - x_{1-}(x_3-z)_\perp^2)^4}\cdot\\
&(1-\mathbf{U}^{\sigma-}(x_{1\perp},z_\perp)-\mathbf{U}^{\sigma-}(z_\perp,x_{3\perp}))\Bigg),
\end{aligned} \tag{39}$$

[10] The nonlinear terms are relevant for the high-energy in dense QCD regime like pA scattering on LHC.

[11] The representation for the impact factor as an integral of 4-point correlator of the external currents and the gluonic current was constructed in [37,38].

where we use in $\mathbf{U}^{\sigma_-}$ the gauge field with a cut-off σ_- for the light-cone momenta. Now we can collect the full expression for the correlator of operators $S_{gl}^{2+\omega_1}$ and $S_{gl}^{2+\omega_2}$ stretched along n_+ and n_- directions using (29) and (39):

$$
\begin{aligned}
&\langle S_{gl}^{2+\omega_1}(x_{1\perp}, x_{3\perp}) S_{gl}^{2+\omega_2}(y_{1\perp}, y_{3\perp})\rangle = \\
&= (\frac{N^2}{2\pi^3})^2 \int_{-\infty}^{\infty} dx_{1-} \int_{x_{1-}}^{\infty} dx_{3-} (x_{3-} - x_{1-})^{-\omega_1} \int_{-\infty}^{\infty} dy_{1+} \int_{y_{1+}}^{\infty} dy_{3+} (y_{3+} - y_{1+})^{-\omega_2} \cdot \\
&\cdot \int d^2 z \left(\frac{2}{((x_{3-}(x_1 - z)_\perp)^2 - x_{1-}(x_3 - z)_\perp^2)^2} \right. \\
&+ \left. \frac{x_{1-}x_{3-}(9(x_1 - z)_\perp^2 (x_3 - z)_\perp^2 + 6(x_1 - z, x_3 - z)^2)}{(x_{3-}(x_1 - z)_\perp^2 - x_{1-}(x_3 - z)_\perp^2)^4} \right) \\
&\cdot \int d^2 w \left(\frac{2}{((y_{3+}(y_1 - w)_\perp)^2 - y_{1+}(y_3 - w)_\perp^2)^2} \right. \\
&+ \left. \frac{y_{1+}y_{3+}(9(y_1 - w)_\perp^2 (y_3 - w)_\perp^2 + 6(y_1 - w, y_3 - w)^2)}{(y_{3+}(y_1 - w)_\perp^2 - y_{1+}(y_3 - w)_\perp^2)^4} \right) \\
&(\langle \mathbf{U}^{\sigma_-}(x_{1\perp}, z_\perp) \mathbf{V}^{\sigma_+}(y_{1\perp}, w_\perp)\rangle + \langle \mathbf{U}^{\sigma_-}(x_{1\perp}, z_\perp) \mathbf{V}^{\sigma_+}(w_\perp, y_{3\perp})\rangle + \\
&\langle \mathbf{U}^{\sigma_-}(z_\perp, x_{3\perp}) \mathbf{V}^{\sigma_+}(y_{1\perp}, w_\perp)\rangle + \langle \mathbf{U}^{\sigma_-}(z_\perp, x_{3\perp}) \mathbf{V}^{\sigma_+}(w_\perp, y_{3\perp})\rangle),
\end{aligned} \tag{40}
$$

where $\mathbf{V}^{\sigma_+}(y_{1\perp}, w_\perp)$ is the operator similar to (38) but for the second frame operator stretched along n_-.

All terms in the last brackets are similar and give the same contribution. We proceed with the first one. As was demonstrated in [31], the problem of calculation of correlator for two dipoles splits into two parts: we compute the correlator for relatively small values of the cutoff $\tilde{\sigma}$ such that $g^2 \log \frac{\tilde{\sigma}}{\sigma_0} \ll 1$, where $\sigma_0 < \tilde{\sigma} \ll \sigma$ for each dipole, when the lowest order of perturbation theory dominates in the LLA BFKL approximation, and then we evolve the result w.r.t. $\tilde{\sigma}$ to its final value σ. Let us elaborate it in detail.

3.1. BFKL evolution

As was demonstrated in [39] evolution w.r.t. the cutoff can be written in the form of BFKL equation[12]:

$$
\sigma \frac{d}{d\sigma} \mathbf{U}^\sigma(z_1, z_2) = \mathcal{K}_{\mathrm{BFKL}} * \mathbf{U}^\sigma(z_1, z_2), \tag{41}
$$

where $\mathcal{K}_{\mathrm{BFKL}}$ is the integral operator having the following form in LO BFKL approximation:

$$
\mathcal{K}_{\mathrm{LO\,BFKL}} * \mathbf{U}(z_1, z_2) = \frac{2g^2}{\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\mathbf{U}(z_1, z_3) + \mathbf{U}(z_3, z_2) - \mathbf{U}(z_1, z_2)]. \tag{42}
$$

In principle, we will use in what follows the NLO generalization of this kernel, or rather of its eigenvalues, to fix the NLO scaling of the correlator. To fix the right NLO normalization of the

[12] we dropped index $\perp$ for sake of brevity. Also we will omit index $\perp$ in all formulas where it doesn't lead to confusion.

correlator, we should also correct the operators $\mathbf{U}$, $\mathbf{V}$, but we will restrict ourselves to the LO in the normalization.

The BFKL kernel has $G = SL(2, C)$ symmetry and its eigenfunctions and the spectrum should be classified w.r.t. the irreps of this group. The $SL(2, C)$ group has three sets of unitary irreps. The color dipole operator can be expanded w.r.t. only one of them, the principal series, characterized by conformal weights $h = \frac{1+n}{2} + i\nu$, $\bar{h} = \frac{1-n}{2} + i\nu$, where $\nu \in \mathbb{R}$, $n \in \mathbb{Z}$ and a two-dimensional coordinate z_0. Explicitly, the eigenfunction reads as follows[13]:

$$E_{h,\bar{h}}(z_{10}, z_{20}) = \left[\frac{z_{12}}{z_{10}z_{20}}\right]^h \left[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\right]^{\bar{h}}. \tag{43}$$

Let us introduce the projection of dipole on the E-function:

$$\mathcal{U}_{\nu,n}(z_0) = \frac{1}{\pi^2}\int \frac{d^2z_1 d^2z_2}{z_{12}^4} E^*_{\nu,n}(z_{10}, z_{20})\mathbf{U}(z_1, z_2) \tag{44}$$

and the inverse transformation is

$$\mathbf{U}(z_1, z_2) = \sum_{n=-\infty}^{\infty} \int d\nu \int d^2z_0 \frac{\nu^2 + n^2/4}{\pi^2} E_{\nu,n}(z_{10}, z_{20})\mathcal{U}_{\nu,n}(z_0). \tag{45}$$

The solutions of BFKL equation (41) in terms of this projection can be explicitly written in the following form:

$$\mathcal{U}^{\sigma}_{\nu,n}(z_0) = \left(\frac{\sigma}{\sigma_0}\right)^{\aleph(\nu,n)} \mathcal{U}^{\sigma_0}_{\nu,n}(z_0), \tag{46}$$

where $\aleph(n, \nu)$ are the eigenvalues of $\mathcal{K}_{BFKL}$. Let us give their expression already in the NLO approximation [5]

$$\begin{aligned}
&\aleph(\nu, n = 0) = 4g^2(\chi(\nu) + g^2\delta(\nu)),\\
&\chi(\nu) = 2\Psi(1) - \Psi(\frac{1}{2} + i\nu) - \Psi(\frac{1}{2} - i\nu),\\
&\delta(\nu) = \chi''(\nu) + 6\zeta(3) - 2\zeta(2)\chi(\nu) - 2\Phi(\frac{1}{2} + i\nu) - 2\Phi(\frac{1}{2} - i\nu),
\end{aligned} \tag{47}$$

where $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ and function $\Phi(x)$ has the following representation:

$$\Phi(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Psi'(\frac{k+2}{2}) - \Psi'(\frac{k+1}{2})}{k + x}. \tag{48}$$

The transformation (44) can be expressed graphically as in Fig. 5.

[13] where we pass to the complex coordinates for 2-dimensional space: $z = (x, y) \to x + iy$. Complex conjugation of z is denoted by $\bar{z}$.

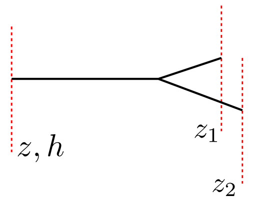

Fig. 5. Graphical representation of the projection (44) of color dipole on the E-eigenfunction. Dotted lines represent the Wilson lines and all the coordinates correspond to the transverse 2-dimensional space.

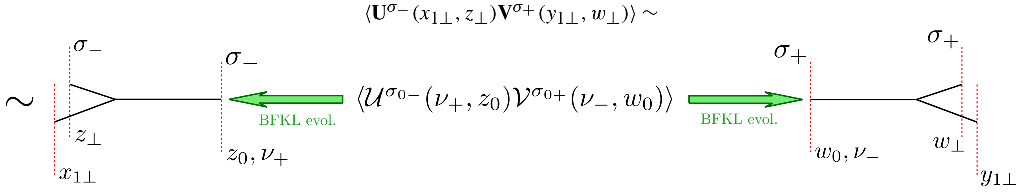

Fig. 6. The logic of our calculation of the dipole-dipole correlation function: the projection of the color dipoles onto the E-functions at each end-point, the BFKL evolution from relatively small cutoffs (green arrows) and, finally, the calculation of the dipole-dipole correlation function at small cutoffs, in the middle.

3.2. Correlator of dipoles with a small cutoff

Now let us introduce projections of $\mathbf{U}^{\sigma_-}(x_{1\perp}, z_\perp)$ and $\mathbf{V}^{\sigma_+}(y_{1\perp}, w_\perp)$ to the eigenfunctions:

$$\mathcal{U}^{\sigma_-}_{\nu_+,n_+}(z_0) = \frac{1}{\pi^2}\int \frac{d^2x_{1\perp}d^2z_\perp}{(|x_{1\perp}-z_\perp|^2)^2} E^*_{\nu_+,n_+}(x_{1\perp}-z_0, z_\perp - z_0)\mathbf{U}^{\sigma_-}(x_{1\perp}, z_\perp), \tag{49}$$

$$\mathcal{V}^{\sigma_+}_{\nu_-,n_-}(w_0) = \frac{1}{\pi^2}\int \frac{d^2y_{1\perp}d^2w_\perp}{(|y_{1\perp}-w_\perp|^2)^2} E^*_{\nu_-,n_-}(y_{1\perp}-w_0, w_\perp - w_0)\mathbf{V}^{\sigma_+}(y_{1\perp}, w_\perp). \tag{50}$$

It is important to stress that the contribution corresponding to the lowest twist-2 comes from the projections with $n = 0$. Using BFKL evolution (46) we can reduce the correlator with arbitrary cut-off to the case of small[14] cutoffs $\sigma_{\pm 0}$:

$$\langle \mathcal{U}^{\sigma_-}_{\nu_+}(z_0), \mathcal{V}^{\sigma_+}_{\nu_-}(w_0)\rangle = \left(\frac{\sigma_-}{\sigma_{-0}}\right)^{\aleph(\nu_+)} \left(\frac{\sigma_+}{\sigma_{+0}}\right)^{\aleph(\nu_-)} \langle \mathcal{U}^{\sigma_{0-}}_{\nu_+}(z_0), \mathcal{V}^{\sigma_{0+}}_{\nu_-}(w_0)\rangle. \tag{51}$$

Graphically the logic of our calculation can be represented as in Fig. 6.

The correlation function between two dipoles with relatively small cutoffs $\sigma_\pm \gtrsim \sigma_{\pm 0}$ can be calculated perturbatively. In one loop it reads as follows [39]

$$\langle \mathcal{U}^{\sigma_-}_{\nu_+}(z_0), \mathcal{V}^{\sigma_+}_{\nu_-}(w_0)\rangle = \frac{-4\pi^4 g^4}{N^2\nu_-^2(\nu_-^2+\frac{1}{4})^2} \times$$
$$\left(\delta(z_0-w_0)\delta(\nu_+ + \nu_-) + \frac{2^{1-4i\nu_-}\delta(\nu_+-\nu_-)}{\pi|z_0-w_0|^{2-4i\nu_-}} \frac{\Gamma(\frac{1}{2}+i\nu_-)\Gamma(1-i\nu_-)}{\Gamma(i\nu_-)\Gamma(\frac{1}{2}-i\nu_-)}\right) \times$$

[14] But large enough to use LLA and only two-region contribution to $\aleph(n, \nu)$.

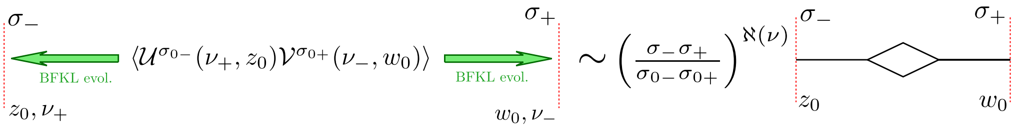

Fig. 7. The scheme of calculation of the dipole-dipole correlator for small cutoffs and the BFKL evolution (shown by green arrows). In the r.h.s. we use the orthogonality condition for the E-functions.

$$(1 + O\left(g^2 \log\left(\frac{\sigma_-\sigma_+}{\sigma_{0-}\sigma_{0+}}\right)\right). \tag{52}$$

Graphically, the last calculation, together with the BFKL evolution, looks as depicted in the Fig. 7.

4. Calculation of correlation function

Now using the inversion formula (45) and the eq. (51) we obtain the correlator of two color dipoles with the original finite cut-offs $\sigma_\pm$:

$$\begin{aligned}
&\langle \mathbf{U}^{\sigma_-}(x_{1\perp}, z_\perp)\mathbf{V}^{\sigma_+}(y_{1\perp}, w_\perp)\rangle = \\
&= -\frac{4\pi^2 g^4}{N^2}\int d\nu_+ \int d^2 z_0 \frac{\nu_+^2}{\pi^2}\left(\frac{(x_1-z)_\perp^2}{(x_1-z_0)_\perp^2(z-z_0)_\perp^2}\right)^{\frac{1}{2}+i\nu_+}\cdot \\
&\cdot \int d\nu_- \int d^2 w_0 \frac{\nu_-^2}{\pi^2}\left(\frac{(y_1-w)_\perp^2}{(y_1-w_0)_\perp^2(w-w_0)_\perp^2}\right)^{\frac{1}{2}+i\nu_-}\left(\frac{\sigma_-}{\sigma_{0-}}\right)^{\aleph(\nu_+)}\left(\frac{\sigma_+}{\sigma_{0+}}\right)^{\aleph(\nu_-)}\cdot \\
&\cdot \frac{\pi^2}{\nu_-^2(\nu_-^2+\frac{1}{4})^2}\left(\delta(z_0-w_0)\delta(\nu_++\nu_-) + \frac{2^{1-4i\nu_-}\delta(\nu_+-\nu_-)}{\pi|z_0-w_0|^{2-4i\nu_-}}\frac{\Gamma(\frac{1}{2}+i\nu_-)\Gamma(1-i\nu_-)}{\Gamma(i\nu_-)\Gamma(\frac{1}{2}-i\nu_-)}\right).
\end{aligned} \tag{53}$$

Integrating over ν_- and[15] over w_0 we get:

$$\begin{aligned}
&\langle \mathbf{U}^{\sigma_-}(x_{1\perp}, z_\perp)\mathbf{V}^{\sigma_+}(y_{1\perp}, w_\perp)\rangle = \\
&= -\frac{8g^4}{N^2}\iint \frac{d\nu\,\nu^2\,d^2 z_0}{(\frac{1}{4}+\nu^2)^2}\left(\frac{(x_1-z)_\perp^2}{(x_1-z_0)_\perp^2(z-z_0)_\perp^2}\right)^{\frac{1}{2}+i\nu}\left(\frac{(y_1-w)_\perp^2}{(y_1-z_0)_\perp^2(w-z_0)_\perp^2}\right)^{\frac{1}{2}-i\nu}\cdot \\
&\cdot \left(\frac{\sigma_+\sigma_-}{\sigma_{+0}\sigma_{-0}}\right)^{\aleph(\nu)}.
\end{aligned} \tag{54}$$

Now we come to the subtlest point in our calculations: we should fix the physical value of the ratio of our cutoffs $\frac{\sigma_+\sigma_-}{\sigma_{-0}\sigma_{+0}}$. In general, they are some functions of our configuration of the frames and should depend on conformal ratios of 8 points characterizing the shapes and positions of two frames. But we expect that in the limit of very narrow frames which we will need, the cutoffs depend only on the distances between the positions of local fields x_1, x_3, y_1, y_3. Indeed, if we make a conformal transformation with generic parameters, which are not related to the shape of

[15] using the "star-triangle" relation, see e.g. the Appendix A of [40].

the frames, the frames remain rectangular (up to an insignificant, in our limit, deformation of their short sides, as shown in Fig. 2) and are still characterized by 4 points x_1', x_3', y_1', y_3' - new position of the local fields inserted into the frames. That means that the cutoffs can depend only on two conformally invariant ratios: $r_1 = \frac{(x_1-y_3)^2(x_3-y_1)^2}{x_{13}^2 y_{13}^2}$ and $r_2 = \frac{(x_1-y_1)^2(x_3-y_3)^2}{x_{13}^2 y_{13}^2}$ and we have to calculate this dependence explicitly. A natural assumption would be then to put $\frac{\sigma_+\sigma_-}{\sigma_{-0}\sigma_{+0}} \simeq r_2$ or $\frac{\sigma_+\sigma_-}{\sigma_{-0}\sigma_{+0}} \simeq r_1$. Both choices give the same result in the limit when the distance between the frames is much less than their lengths. It is demonstrated in the appendix B that precisely this cut-off occurs in the NLO graphs in this limit. But before this limit the result appears to be a bit subtler. Instead of doing explicit calculations, we will appeal to a similar calculation done in [41] for the 4-point correlator of local scalar fields $\langle \mathrm{tr}\, Z^2(x_1)\bar{Z}^2(x_3)Z^2(y_1)\bar{Z}^2(y_3)\rangle$. Its form is fixed by conformal symmetry so that it depends on the same conformal ratios as our configuration. Comparing the result of BFKL computation of this correlator, which also uses the OPE for the regularized color dipoles, with the Regge limit of the same quantity found in the papers [42,20], the following prescription was found for the cutoff dependence on x_1, x_3, y_1, y_3[16]:

$$\left(\frac{\sigma_+\sigma_-}{\sigma_{+0}\sigma_{-0}}\right)^{\aleph(\nu)} \to$$
$$\to \frac{i}{\sin\pi\aleph(\nu)}\left(\frac{((x_1-y_3)^2)^{\frac{\aleph(\nu)}{2}}((x_3-y_1)^2)^{\frac{\aleph(\nu)}{2}}}{(x_{13}^2)^{\frac{\aleph(\nu)}{2}}(y_{13}^2)^{\frac{\aleph(\nu)}{2}}} - \frac{((x_1-y_1)^2)^{\frac{\aleph(\nu)}{2}}((x_3-y_3)^2)^{\frac{\aleph(\nu)}{2}}}{(x_{13}^2)^{\frac{\aleph(\nu)}{2}}(y_{13}^2)^{\frac{\aleph(\nu)}{2}}}\right). \tag{55}$$

Since we are in a very similar kinematic situation we will assume here that our cutoff dependence on the coordinates of the frames is also governed by (55).

A motivation stemming from the Feynman graphs of NLO impact factor is given in Appendix B.

We denote the coordinates of the vertices of a frame as follows:

$$\begin{aligned} x_1 &= (-uL_-, 0, x_{1\perp}), \\ x_3 &= (\bar{u}L_-, 0, x_{3\perp}), \\ y_1 &= (0, -vL_+, y_{1\perp}), \\ y_3 &= (0, \bar{v}L_+, y_{3\perp}), \end{aligned} \tag{56}$$

where $\bar{u} = 1-u$, $\bar{v} = 1-v$. These parameters can be restricted to $u, v \in (0,1)$ since each frame should intersect the shock wave in order to give a non-zero contribution. If there is no intersection, we can make a scale transformation which sends the frame to an infinite distance from the shock wave, thus suppressing their interaction.

Using (55) we can rewrite the correlator of two dipoles (54) in the explicit way:

$$\langle \mathbf{U}^{\sigma_-}(x_{1\perp}, z_\perp)\mathbf{V}^{\sigma_+}(y_{1\perp}, w_\perp)\rangle =$$
$$= -\frac{8g^4}{N^2}\int d\nu \int d^2z_0 \frac{\nu^2}{(\frac{1}{4}+\nu^2)^2}\left(\frac{(x_1-z)_\perp^2}{(x_1-z_0)_\perp^2(z-z_0)_\perp^2}\right)^{\frac{1}{2}+i\nu}.$$

[16] By keeping each factor in the anharmonic ratio in a separate power $\frac{\aleph(\nu)}{2}$ we choose the right analytic branch giving the right signature factor. It could be explicited by an $i0$ prescription.

$$
\cdot\left(\frac{(y_1-w)_\perp^2}{(y_1-z_0)_\perp^2(w-z_0)_\perp^2}\right)^{\frac{1}{2}-i\nu}\frac{i}{\sin\pi\aleph(\nu)}\cdot
$$
$$
\cdot\left(\left(\frac{(2uL_-\bar{v}L_++\Delta_\perp^2)(2\bar{u}L_-vL_++\Delta_\perp^2)}{x_{13\perp}^2y_{13\perp}^2}\right)^{\frac{\aleph(\nu)}{2}}\right.
$$
$$
\left.-\left(\frac{(-2uL_-vL_++\Delta_\perp^2)(-2\bar{u}L_-\bar{v}L_++\Delta_\perp^2)}{x_{13\perp}^2y_{13\perp}^2}\right)^{\frac{\aleph(\nu)}{2}}\right), \tag{57}
$$

where $\Delta_\perp=(x-y)_\perp$ is the distance between the frames in the orthogonal direction. Then we plug it into (40) and thus obtain a closed expression for the correlator. Now we should carry out the remaining integrations. Let us start with the integrations over L_- and L_+. We can factor out the L-dependence in each of the two terms in (55) leading to the following two terms in (40):

$$
\int_0^\infty dL_-L_-^{-2-\omega_1}\int_{-L_-}^0 dx_{1+}\int_0^\infty dL_+L_+^{-2-\omega_2}\int_{-L_+}^0 dy_{1-}\times
$$
$$
\times\left((2uL_-\bar{v}L_++\Delta_\perp^2)(2\bar{u}L_-vL_++\Delta_\perp^2)\right)^{\frac{\aleph(\nu)}{2}}=
$$
$$
=2\pi\delta(\omega_1-\omega_2)\int_0^1\int_0^1 dudv(4u\bar{u}v\bar{v})^{\frac{\aleph(\nu)}{2}}(\frac{\Delta^2}{2u\bar{v}})^{\frac{\aleph(\nu)}{2}}(\frac{\Delta^2}{2\bar{u}v})^{-\omega+\frac{\aleph(\nu)}{2}}\times
$$
$$
\times B(-\omega,\omega-\aleph(\nu))_2F_1(-\frac{\aleph(\nu)}{2},-\omega;-\aleph(\nu);1-\frac{u\bar{v}}{\bar{u}v}) \tag{58}
$$

and

$$
\int_0^\infty dL_-L_-^{-2-\omega_1}\int_{-L_-}^0 dx^+\int_0^\infty dL_+L_+^{-2-\omega_2}\int_{-L_+}^0 dy^-\times
$$
$$
\times\left((-2uL_-vL_++\Delta_\perp^2)\right)^{\frac{\aleph(\nu)}{2}}\left(-2\bar{u}L_-\bar{v}L_++\Delta_\perp^2)\right)^{\frac{\aleph(\nu)}{2}}=
$$
$$
=2\pi e^{i\pi\aleph(\nu)}(-1)^{\aleph(\nu)-\omega}\delta(\omega_1-\omega_2)\int_0^1\int_0^1 dudv(4u\bar{u}v\bar{v})^{\frac{\aleph(\nu)}{2}}(\frac{\Delta^2}{2uv})^{\frac{\aleph(\nu)}{2}}(\frac{\Delta^2}{2\bar{u}\bar{v}})^{-\omega+\frac{\aleph(\nu)}{2}}\times
$$
$$
\times B(-\omega,\omega-\aleph(\nu))_2F_1(-\frac{\aleph(\nu)}{2},-\omega;-\aleph(\nu);1-\frac{uv}{\bar{u}\bar{v}}). \tag{59}
$$

The best way to do these integrals is to change the integration variables to L_+L_- and $\frac{L_+}{L_-}$. The integral over $\frac{L_+}{L_-}$ renders $\delta(\omega_1-\omega_2)$. Here $B(-\omega,\omega-\aleph(\nu))=\frac{\Gamma(-\omega)\Gamma(\omega-\aleph(\nu))}{\Gamma(-\aleph(\nu))}$, and thus it has a pole in ν, namely $\frac{1}{\omega-\aleph(\nu)}$. We postpone the ν-integration because we close the contour integration (in the upper or lower half-plane) depending on whether the modulo of the ratio

$$
\left|\left(\frac{(x_1-z)^2}{(x_1-z_0)^2(z-z_0)^2}\right)\left(\frac{(y_1-w)^2}{(y_1-z_0)^2(w-z_0)^2}\right)^{-1}\right| \tag{60}
$$

is greater or smaller than one. Hence we first carry out the coordinate integrations. First let us perform the u- and v-integrations. We can factor out all functions depending on u and do the u-integration:

$$\int_0^1 du(u(1-u))^{\frac{\aleph(\nu)}{2}} \frac{1}{u^{\frac{\aleph(\nu)}{2}}} \frac{1}{(1-u)^{\frac{\aleph(\nu)}{2}-\omega}} {}_2F_1(-\frac{\aleph(\nu)}{2}, -\omega; -\aleph(\nu); 1-\frac{u\bar{v}}{\bar{u}v})\cdot$$
$$\cdot\left(\frac{2}{(((1-u)(x_1-z)_\perp)^2+u(x_3-z)_\perp^2)^2}\right.$$
$$\left.-\frac{u(1-u)(9(x_1-z)_\perp^2(x_3-z)_\perp^2+6(x_1-z,x_3-z)_\perp^2)}{((1-u)(x_1-z)_\perp^2+u(x_2-z)_\perp^2)^4}\right)=$$
$$=\frac{1}{2}\left(\frac{1}{|x_3-z|_\perp^2|x_1-z|_\perp^2}-\frac{2[(x_3-z)_\perp\cdot(x_1-z)_\perp]^2}{(|x_3-z|_\perp^2|x_1-z|_\perp^2)^2}\right)(1+O(g^2,\omega)), \tag{61}$$

where by $\cdot$ we denote the scalar product of 2-dimensional vectors. The first line of the integrand is equal to $1+O(g^2,\omega)$ and the terms of the order $O(g^2,\omega)$ contribute only to the NLO impact factor. We will drop such terms since we limit ourselves to the LO to the impact factor only. Finally we obtain our correlator of the regularized light ray operators in the form:

$$\langle S_{gl+}^{2+\omega_1}(x_{1\perp},x_{3\perp})S_{gl-}^{2+\omega_2}(y_{1\perp},y_{3\perp})\rangle=$$
$$=-i\frac{4N^2g^4}{\pi^5}\delta(\omega_1-\omega_2)\cdot$$
$$\cdot\int d\nu\frac{(\Delta_\perp^2)^{\aleph(\nu)-\omega}}{(x_{13\perp}^2y_{13\perp}^2)^{\frac{\aleph(\nu)}{2}}}B(-\omega,\omega-\aleph(\nu))\frac{1-e^{i\pi(2\aleph(\nu)-\omega)}}{\sin\pi\aleph(\nu)}\frac{\nu^2}{(\frac{1}{4}+\nu^2)^2}\cdot$$
$$\cdot\int d^2z\left(\frac{1}{2|x_1-z|_\perp^2|x_3-z|_\perp^2}-\frac{(x_1-z,x_3-z)_\perp^2}{(|x_1-z|_\perp^2|x_3-z|_\perp^2)^2}\right)\cdot$$
$$\cdot\int d^2w\left(\frac{1}{2|y_1-w|_\perp^2|y_3-w|_\perp^2}-\frac{(y_1-w,y_3-w)_\perp^2}{(|y_1-w|_\perp^2|y_3-w|_\perp^2)^2}\right)\cdot$$
$$\cdot\int d^2z_0\left(\frac{|x_1-z|_\perp^2}{|x_1-z_0|_\perp^2|z-z_0|_\perp^2}\right)^{\frac{1}{2}+i\nu}\left(\frac{|y_1-w|_\perp^2}{|y_1-z_0|_\perp^2|w-z_0|_\perp^2}\right)^{\frac{1}{2}-i\nu}. \tag{62}$$

To be able to calculate these integrals over z and w we derived, using the dimensional regularization and Feynman parameterization, the following formula:

$$2\int\frac{d^2z}{\pi}\left(\frac{1}{(x-z)^2(y-z)^2}-\frac{2\langle x-z,y-z\rangle^2}{((x-z)^2(y-z)^2)^2}\right)\frac{(x-z)^{2\beta}}{z^{2\beta}}=$$
$$=-\frac{\beta}{1-\beta}\frac{2\langle x,y\rangle^2-x^2y^2}{x^2(y^2)^{1+\beta}((x-y)^2)^{1-\beta}}. \tag{63}$$

It leads to the following expression for our correlator:

$$\langle S_{gl+}^{2+\omega_1}(x_{1\perp},x_{3\perp})S_{gl-}^{2+\omega_2}(y_{1\perp},y_{3\perp})\rangle=$$

$$= -i\frac{N^2 g^4}{4\pi^3}\delta(\omega_1-\omega_2)\int d\nu(\Delta_\perp^2)^{\aleph(\nu)-\omega}B(-\omega,\omega-\aleph(\nu))\frac{1-e^{i\pi(2\aleph(\nu)-\omega)}}{\sin\pi\aleph(\nu)}\frac{\nu^2}{(\frac{1}{4}+\nu^2)^2}\cdot$$
$$\frac{1}{(|x_{13}|_\perp^2|y_{13}|_\perp^2)^{\frac{1}{2}+\frac{\aleph(\nu)}{2}}}\int d^2z_0\frac{(|x_{13}|_\perp^2)^{\frac{1}{2}+i\nu}(2\cos^2(\phi_x)-1)}{(|x_1-z_0|_\perp^2)^{\frac{1}{2}+i\nu}|x_3-z_0|_\perp^2)^{\frac{1}{2}+i\nu}}$$
$$\frac{(|y_{13}|_\perp^2)^{\frac{1}{2}-i\nu}(2\cos^2(\phi_y)-1)}{(|y_1-z_0|_\perp^2)^{\frac{1}{2}-i\nu}|y_3-z_0|_\perp^2)^{\frac{1}{2}-i\nu}}, \tag{64}$$

where ϕ_x is the angle between the vectors $z_0-x_{1\perp}$ and $z_0-x_{3\perp}$, ϕ_y is the angle between the vectors $z_0-y_{1\perp}$ and $z_0-y_{3\perp}$.

The last integration can be done directly in the limit $x_{13},\ y_{13}\to 0$. The calculations are given in Appendix C. Finally we get:

$$\langle S_{gl+}^{2+\omega_1}(x_{1\perp},x_{3\perp})S_{gl-}^{2+\omega_2}(y_{1\perp},y_{3\perp})\rangle \xrightarrow[x_{13\perp},\ y_{13\perp}\to 0]{}$$
$$\to -i\frac{N^2 g^4}{4\pi^3}\delta(\omega_1-\omega_2)\int d\nu(\Delta_\perp^2)^{\aleph(\nu)-\omega}B(-\omega,\omega-\aleph(\nu))\frac{1-e^{i\pi(2\aleph(\nu)-\omega)}}{\sin\pi\aleph(\nu)}\frac{\nu^2}{(\frac{1}{4}+\nu^2)^2}\cdot$$
$$\frac{1}{(|x_{13}|_\perp^2|y_{13}|_\perp^2)^{1+\frac{\aleph(\nu)}{2}}}\left(\frac{(|x_{13}|_\perp^2)^{\frac{1}{2}+i\nu}(|y_{13}|_\perp^2)^{\frac{1}{2}+i\nu}}{(|x-y|_\perp^2)^{1+2i\nu}}G(\nu)+(\nu\to-\nu)\right), \tag{65}$$

where

$$G(\nu) = -i\frac{4^{-1-2i\nu}\pi^3(i-2\nu)^2}{\Gamma^2(\frac{3}{2}-i\nu)\Gamma^2(1+i\nu)\sinh(2\pi\nu)}.$$

Now we can carry out the last integration over ν as the pole contribution at $\omega=\aleph(\nu)$. We pick here the first pole Ψ-functions in (47) which corresponds to the operator with the lowest possible twist = 2. Note that we omitted from our contour of integration the singularity at $\nu=-\frac{i}{2}$.[17] Finally, we arrive at the final result of our paper - the correlator of two light ray operators representing the analytic continuation of twist two operators to the Lorentz spin $j=1+\omega$, in the BFKL limit $\omega\to 0$ and $\frac{g^2}{\omega}$ - fixed:

$$\langle S_{+}^{2+\omega_1}(x_{1\perp},x_{3\perp})S_{-}^{2+\omega_2}(y_{1\perp},y_{3\perp})\rangle \xrightarrow[x_{13\perp},\ y_{13\perp}\to 0]{} \delta(\omega_1-\omega_2)\Upsilon(\tilde\gamma)\frac{(x_{13\perp}^2)^{\frac{\tilde\gamma}{2}-\frac{\omega}{2}}(y_{13\perp}^2)^{\frac{\tilde\gamma}{2}-\frac{\omega}{2}}}{((x-y)_\perp^2)^{2+\tilde\gamma}}, \tag{66}$$

where Υ is given by

$$\Upsilon(\tilde\gamma) = -N^2 g^4\frac{2^{-1-2\tilde\gamma}\pi}{\tilde\gamma^2\Gamma^2(1-\frac{\tilde\gamma}{2})\Gamma^2(\frac{1}{2}+\frac{\tilde\gamma}{2})\sin(\pi\tilde\gamma)\hat\aleph'(\tilde\gamma)} \tag{67}$$

and $\tilde\gamma=-1+2i\nu$ is the solution of $\omega=\hat\aleph(\tilde\gamma)$, where $\hat\aleph(\tilde\gamma)=\aleph(-i\frac{\tilde\gamma+1}{2})$ and $\aleph(\nu)$ is given by (47). Finally let us introduce the new quantity $\gamma=\tilde\gamma-\omega$ which is the anomalous dimension in NLO BFKL. It satisfies the following equation:

[17] As demonstrated in [43], the contribution of this pole cancels with the contribution of two lowest-order diagrams which are absent in Fig. 4.

$$\omega = \hat{\aleph}(\gamma+\omega) = \hat{\aleph}(\gamma) + \hat{\aleph}'(\gamma)\hat{\aleph}(\gamma) + o(g^4)\,. \tag{68}$$

This anomalous dimension γ is in the full correspondence with [5]. The correlator in terms of γ reads as follows:

$$\langle S_+^{2+\omega_1}(x_{1\perp},x_{3\perp})S_-^{2+\omega_2}(y_{1\perp},y_{3\perp})\rangle \xrightarrow[x_{13\perp},\,y_{13\perp}\to 0]{} \delta(\omega_1-\omega_2)\Upsilon(\gamma+\omega)\frac{(x_{13\perp}^2)^{\frac{\gamma}{2}}(y_{13\perp}^2)^{\frac{\gamma}{2}}}{((x-y)_\perp^2)^{2+\gamma+\omega}}\,. \tag{69}$$

Note that this formula correctly reproduces the tensor structure of the correlator corresponding to local twist-2 operators (1) integrated along light-rays and analytically continued to $j\to 1+\omega$. Indeed, the regularized operators enter with a multiplier Λ^γ where Λ is a scheme-dependent cutoff. We use the point-splitting regularization in the orthogonal direction for our light-ray operators and hence the cutoffs are defined as $\Lambda_x = \frac{1}{|x_{13\perp}|}$ and $\Lambda_y = \frac{1}{|y_{13\perp}|}$. Now if we redefine light ray operators as $\Lambda_x^\gamma \breve{S}_+^{2+\omega_1}(x_{1\perp}) \to \breve{S}_+^{2+\omega_1}(x_{1\perp})$, $\Lambda_y^\gamma \breve{S}_+^{2+\omega_1}(y_{1\perp}) \to \breve{S}_+^{2+\omega_2}(y_{1\perp})$ the correlation function acquires a standard form:

$$\langle \breve{S}_+^{2+\omega_1}(x_\perp)\breve{S}_-^{2+\omega_2}(y_\perp)\rangle = \delta(\omega_1-\omega_2)\frac{\Upsilon(\gamma+\omega)}{((x-y)_\perp^2)^{2+\gamma+\omega}}\,, \tag{70}$$

in agreement with analytical continuation of the integrated two-point correlator of local operators. Indeed, as was mentioned in section 2, at even values of the spin j the light-ray operator $\breve{\mathcal{S}}^{j+1}(x_\perp)$ can be written as a local operator $\mathcal{S}^j_{\rm loc}(x)$ with dimension $\Delta(j)$ integrated along the light ray direction n_+ (see (20)):

$$\breve{\mathcal{S}}^{j+1}(x_\perp) \sim \int_{-\infty}^{\infty} dx_- \mathcal{S}^j_{\rm loc}(x) \tag{71}$$

and the correlator of two light-ray operators stretched along n_+ and n_- vectors is just the double integral of two-point correlator of local operators w.r.t. light-ray directions $n_\pm$[18]:

$$\langle \breve{\mathcal{S}}^{j_1+1}(x_\perp)\breve{\mathcal{S}}^{j_2+1}(y_\perp)\rangle = \frac{\delta(j_1-j_2)b_{j_1}}{\left(|x-y|_\perp^2\right)^{\Delta(j_1)-1}} \tag{72}$$

with the same coordinate dependence as (70).

In the leading order of perturbation theory, when $\frac{g^2}{\omega}\to 0$, the coefficient $\Upsilon(\gamma+\omega)$ reads as follows:

$$\Upsilon(-8g^2/\omega) = \frac{\omega N^2}{\pi 2^7} \tag{73}$$

and our BFKL result (69) reduces to

$$\langle \breve{S}_+^{2+\omega_1}(x_\perp)\breve{S}_-^{2+\omega_2}(y_\perp)\rangle = \delta(\omega_1-\omega_2)\frac{\omega N^2}{\pi 2^7}\frac{1}{((x-y)_\perp^2)^{2+\omega}}\,. \tag{74}$$

[18] This formula also can be seen as a two point correlation function of two primaries with dimension $\Delta - 1$ wrt to the conformal group $SO(3,1)$ of the 2d orthogonal plane, see the related discussion in [44].

5. Conclusion

In this paper we have generalized the twist-2 operator for the case of principal series representation in terms of a nonlocal light ray operator. Then we have calculated the correlation function between two such operators in the BFKL limit. The correlator takes the form expected from conformal invariance, with the same anomalous dimension as predicted in [5].

As was already mentioned in the introduction, the method of [5] is rather indirect and is based on the comparison with the Bjorken scaling for the scattering amplitudes. It was suggested there that an analytic continuation of anomalous dimensions of local twist-2 operators gives the anomalous dimension of some non-local gluon operator $F_{-i}\nabla^{\omega-1}F_{-}^{\ i}$. This method, however, does not tell us the explicit form of this operator and in this paper we demonstrated that $F_{-i}\nabla^{\omega-1}F_{-}^{\ i}$ is actually a light-ray operator ($j\equiv\omega+1$):

$$\mathcal{F}_j(x_\perp)=\int_0^\infty dL_+\,L_+^{1-j}$$
$$\cdot\int dx_+\ \mathrm{tr}\ F_{-}^{\ i}(x_+n_-+x_\perp)[x,x+L_+n_-]F_{-i}\left((L_++x_+)n_-+x_\perp\right) \tag{75}$$

with the anomalous dimension of this operator $\gamma(j;g^2)$ being an analytic continuation of the anomalous dimension (68) of local twist two operators. The correlator we calculate is a physical quantity well adopted to the study of CFT. Recently a nonperturbative light-ray operators in general CFT were introduced in [21], however their definition is quite implicit, involves the smearing of two another local operators and assumes that the pole structure comes from the light-ray kinematics. Particularly it is not clear why so defined light-ray operator will not depend on the choice of smearing local operators. In contrast, our definition is very explicit and defines the light-ray operator in terms of elementary fields. It would be interesting to understand better the relation of our light-rays to the construction of [21].

Secondly, as we already discussed in the introduction, our work can be considered as a first step in the realization of conformal bootstrap program for nonlocal operators in gauge theories in BFKL limit. Particularly in companion papers [23,24] we calculate the correlation functions of three light-ray operators in BFKL limit. It can be treated as OPE coefficient in the OPE expansion of two light-ray operators[19] in BFKL limit. As an auxiliary step, we used the OPE over dipoles with cutoff which is not a conformal object, so would be very tempting to understand better the relation between OPE over dipoles and conformal light-ray operators, ideally finding an conformal invariant analog for dipoles.

Another promising direction would be to embed light-ray operators further into the integrable structure of $\mathcal{N}=4$ SYM what can lead to the further cross-fertilization of Integrability and Conformal Bootstrap. The recently introduced approach to (super)conformal blocks based on the relation to the integrable Calogero-Sutherland models [45–48] can be easily adapted to light-ray operators and potentially play an important role in such development.

And finally, let us stress again that our generalization of twist-2 operators based on principal series representation with continuous spin j allows us to circumvent a subtle question of analytic continuation in j. The well-known principle of maximal transcendentality, which often serves as a mnemonic prescription for such analytic continuation, notably in the perturbative expansion

[19] for OPE expansion of two ANEC operators see [44].

based on integrability [6,7], might naturally emerge in the framework of the extension of N=4 SYM physical space to the principal series of $PSU(2,2|4)$ or its subgroups. It is tempting to suggest that the principal series representation, in terms of nonlocal objects generalizing local operators, might fix at once the analytic continuation for all such observables.

CRediT authorship contribution statement

Ian Balitsky: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Vladimir Kazakov:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Evgeny Sobko:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Notations

In this section we set our notations. The lagrangian of N=4 SYM with the $SU(N_c)$ gauge group has the following form:

$$\mathfrak{L} = \mathrm{Tr}\left\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi^{AB})(D^\mu\bar{\phi}_{AB}) + \frac{1}{8}g^2[\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}] + \right.$$
$$\left. +2i\bar{\lambda}_{\dot{\alpha}A}\sigma_\mu^{\dot{\alpha}\beta}D^\mu\lambda_\beta^A - \sqrt{2}g\lambda^{\alpha A}[\bar{\phi}_{AB},\lambda_\alpha^B] + \sqrt{2}g\bar{\lambda}_{\dot{\alpha}A}[\phi^{AB},\bar{\lambda}_B^{\dot{\alpha}}]\right\}, \tag{A.1}$$

where field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ and covariant derivative $D_\mu = \partial_\mu - ig[A_\mu, ...]$. Notice that we work with Minkowski signature $(+,-,-,-)$ and all fields are taken in the adjoint representation of $SU(N_c)$. $SO(6)$-multiplet with scalars $\phi^a, a \in \{1 \div 6\}$ can be grouped into the antisymmetric tensor ϕ^{AB}, $A, B \in \{1 \div 4\}$:

$$\phi^{AB} = \frac{1}{\sqrt{2}}\Sigma^{aAB}\phi^a, \quad \bar{\phi}_{AB} = \frac{1}{\sqrt{2}}\bar{\Sigma}^a_{AB}\phi^a = (\phi^{AB})^*, \tag{A.2}$$

using Dirac matrices in 6-d Euclidian space:

$$\Sigma^{aAB} = (\eta_{1AB}, \eta_{2AB}, \eta_{3AB}, i\bar{\eta}_{1AB}, i\bar{\eta}_{2AB}, i\bar{\eta}_{3AB}),$$
$$\bar{\Sigma}^a_{AB} = (\eta_{1AB}, \eta_{2AB}, \eta_{3AB}, -i\bar{\eta}_{1AB}, -i\bar{\eta}_{2AB}, -i\bar{\eta}_{3AB}),$$

and 't Hooft symbols:

$$\eta_{iAB} = \epsilon_{iAB} + \delta_{iA}\delta_{4B} - \delta_{iB}\delta_{4A},$$
$$\bar{\eta}_{iAB} = \epsilon_{iAB} - \delta_{iA}\delta_{4B} + \delta_{iB}\delta_{4A},$$
$$\eta_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \text{(A.3)}$$

$$i\bar{\eta}_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, i\bar{\eta}_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 1 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, i\bar{\eta}_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \quad \text{(A.4)}$$

Explicit formula for scalars reads as follows

$$[\phi^{AB}] = \frac{1}{\sqrt{2}}(\phi^1\eta_{1AB} + \phi^2\eta_{2AB} + \phi^3\eta_{3AB} + \phi^4 i\bar{\eta}_{1AB} + \phi^5 i\bar{\eta}_{2AB} + \phi^6 i\bar{\eta}_{3AB}) =$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \phi^3 + i\phi^6 & -\phi^2 - i\phi^5 & \phi^1 - i\phi^4 \\ -\phi^3 - i\phi^6 & 0 & \phi^1 + i\phi^4 & \phi^2 - i\phi^5 \\ \phi^2 + i\phi^5 & -\phi^1 - i\phi^4 & 0 & \phi^3 - i\phi^6 \\ -\phi^1 + i\phi^4 & -\phi^2 + i\phi^5 & -\phi^3 + i\phi^6 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Z & -Y & \bar{X} \\ -Z & 0 & X & \bar{Y} \\ Y & -X & 0 & \bar{Z} \\ -\bar{X} & -\bar{Y} & -\bar{Z} & 0 \end{pmatrix}.$$

Fermions are realized as a two-component Weyl spinors λ^A_α with conjugated $\bar{\lambda}_{\dot{\alpha}A}$. Spinor index $\alpha \in \{1,2\}$ and $A \in \{1 \div 4\}$ is a $SU(4)$ index. Due to supersymmetry one can fix just the propagator of scalars and get the normalization for fermions and gauge fields acting by supercharges. In this article we set the normalization for free propagators as follows:

$$\langle Z(x)^a_b \bar{Z}(y)^c_d\rangle_0 = \mathcal{N}(\delta^a_d\delta^c_b - \frac{1}{N_c}\delta^a_b\delta^c_d)\frac{1}{(x-y)^2}, \quad \text{and the same for } X \text{and } Y, \quad \text{(A.5)}$$

$$\langle \lambda^A_\alpha(x)^a_b \bar{\lambda}_{\dot{\beta}B}(y)^c_d\rangle_0 = i\mathcal{N}\delta^A_B(\delta^a_d\delta^c_b - \frac{1}{N_c}\delta^a_b\delta^c_d)\bar{\sigma}^\mu_{\alpha\dot{\beta}}\frac{\partial}{\partial x^\mu}\frac{1}{(x-y)^2}, \quad \text{(A.6)}$$

$$\langle A_\mu(x)^a_b A_\nu(y)^c_d\rangle_0 = -\mathcal{N}(\delta^a_d\delta^c_b - \frac{1}{N_c}\delta^a_b\delta^c_d)\frac{g_{\mu\nu}}{(x-y)^2}, \quad \text{(A.7)}$$

where $\mathcal{N} = -\frac{1}{8\pi^2}$, $\{\sigma^\mu\} = \{1, \sigma\}$ and $\{\bar{\sigma}^\mu\} = \{1, -\sigma\}$ with ordinary Pauli matrices σ. Throughout the text we use the basis $\{n_+, n_-, e_{1\perp}, e_{2\perp}\}$ with two light-like vectors $n^\mu_+ = \{\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\}$, $n^\mu_- = \{\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\}$ normalized as $(n_-n_+) = 1$ and two orthogonal vectors $e_{1\perp}, e_{2\perp}$, which span 2-d plane $\{\perp\}$ orthogonal to $\{n_+, n_-\}$. The vector x reads in this basis as $x = x_-n_+ + x_+n_- + x_\perp$, with its square equal to $x^2 = 2x_+x_- - x^2_\perp$.

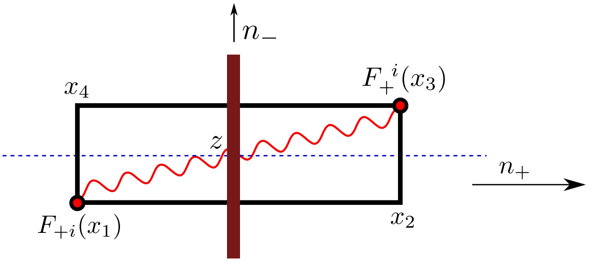

Fig. B.8. ImpactFactorLO.

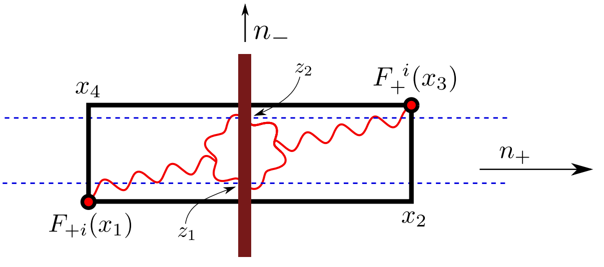

Fig. B.9. ImpactFactorNLO.

Field content of twist-2 operators All twist-2 operators, which were discussed in this paper, are constructed from the set of elementary fields $X = \{F_{+\perp}{}^{\mu}, \lambda^A_{+\alpha}, \bar{\lambda}^{\dot{\alpha}}_{+A}, \phi^{AB}\}$. Twist 2 is the minimal possible twist (defined as bare dimension minus spin). Gluon field $F_{+\perp}{}^{\mu}$ is obtained by projection of one of the indices of the field strength tensor $F^{\mu\nu}$ on n^+ direction where as the second index is automatically restricted to the transverse plane with the metric $g^{\perp}_{\mu\nu} = g_{\mu\nu} - n_{+\mu}n_{-\nu} - n_{+\nu}n_{-\mu}$. Weyl spinors $\lambda_{+\alpha}$ and $\bar{\lambda}^{\dot{\alpha}}_{+}$ correspond to the states with definite helicity $1, -1$, respectively and they are parameterized as $\lambda_{+\alpha} = \frac{1}{2}\bar{\sigma}^{-}_{\alpha\dot{\beta}}\sigma^{+\dot{\beta}\gamma}\lambda_{\gamma}$ and $\bar{\lambda}^{\dot{\alpha}}_{+} = \frac{1}{2}\sigma^{-\dot{\alpha}\beta}\bar{\sigma}^{+}_{\beta\dot{\gamma}}\bar{\lambda}^{\dot{\gamma}}$.

Appendix B. Explanation of the coordinate dependence of the cut-off ratio (55) using NLO impact factor

In principle, in the context of high energy scattering, the cutoffs σ in (55) should be obtained from the NLO impact factor for Wilson frame. In accordance with general logic of high-energy OPE we factorize any correlation function into a product of the “probe” impact factor, the “target” impact factor, and the amplitude of scattering of two (conformal) dipoles. The “rapidity divide” between the impact factor and the dipole-dipole scattering is determined from two conditions: (i) the properly defined impact factor should not scale with the energy, so that all the energy dependence is contained in the dipole-dipole scattering, and (ii) the impact factor should be Möbius invariant. The calculation of the NLO impact factor for frames is beyond the scope of present paper where we limit ourselves only to the LO impact factor, with a typical Feynman graph given in Fig. B.8 (but take into account the NLO dimension!); however, it is instructive to consider a typical Feynman graph in NLO to read off the cutoff dependence on the shape of the configuration of frames. A typical Feynman diagram for the NLO impact factor is shown in Fig. B.9 and the result is proportional to [49]

$$g^2 \int d^2 z_1 d^2 z_2 \int_0^\infty dp_{1-} e^{i\frac{p_{1-}}{2}\mathcal{Z}_1} \int_0^\infty \frac{dp_{2-}}{p_{2-}} e^{i\frac{p_{2-}}{2}\mathcal{Z}_2} + (z_1 \leftrightarrow z_2), \tag{B.1}$$

where $\mathcal{Z}_i \equiv \frac{(x_1 - z_i)_\perp^2}{x_{1-}} - \frac{(x_3 - z_i)_\perp^2}{x_{2-}}$. The integral over α_2 in the Eq. (B.1) diverges. This divergence reflects the fact that the eq. (B.1) is not exactly the NLO impact factor since we must subtract from it the matrix element of the leading-order contribution, the graph in Fig. B.8, which is proportional to

$$g^2 \int d^2 z_1 \int_0^\infty dp_{1-}\, e^{i\frac{p_{1-}}{2}\mathcal{Z}_1} \int_0^{\sigma_-} \frac{dp_{2-}}{p_{2-}}, \tag{B.2}$$

where the integral over p_{2-} is restricted by the "rigid cutoff" (55). The difference of these two expressions gives the typical logarithmic term in the NLO impact factor in the form

$$\begin{aligned} & g^2 \int d^2 z_1 d^2 z_2 \int_0^\infty dp_{1-}\, e^{i\frac{p_{1-}}{2}\mathcal{Z}_1} \Big(\int_0^\infty \frac{dp_{2-}}{p_{2-}}\, e^{i\frac{p_{1-}}{2}\mathcal{Z}_2} - \int_0^{\sigma_-} \frac{dp_{2-}}{p_{2-}} \Big) + (z_1 \leftrightarrow z_2) = \\ & = \frac{1}{\mathcal{Z}_1^2} \ln \sigma \mathcal{Z}_2 \ + \ (z_1 \leftrightarrow z_2). \end{aligned} \tag{B.3}$$

The logarithmic contribution is obviously not conformally invariant. As explained in [49] the reason is that while formally light-like Wilson lines are Möbius invariant, the rigid cutoff (55) violates the invariance. Since the conformally invariant cutoff for rapidity divergence of Wilson lines is not known (it may even not exist) we proceed with the rigid cutoff (55) but pay the price of correcting the "rigid-cutoff" dipoles by counterterms restoring the conformal invariance order-by-order in perturbation theory. In the NLO approximation such "composite conformal dipole" has the form

$$\begin{aligned} & \mathbf{U}(z_1, z_2)^{\text{conf}} = \mathbf{U}(z_1, z_2) + \\ & + \frac{g^2}{\pi} \int d^2 z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\mathbf{U}(z_1, z_3) + \mathbf{U}(z_3, z_2) - \mathbf{U}(z_1, z_2)] \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} \end{aligned} \tag{B.4}$$

is the "composite dipole" with the conformal longitudinal cutoff in the next-to-leading order and a is an arbitrary dimensional constant. The arbitrary dimensional constant a should be chosen in such a way that the impact factor (B.1) does not change with length of the frame. It is convenient to choose the rapidity-dependent constant $a \to a e^{-2\eta}$ so that the $[\mathrm{Tr}\{\hat{U}^\sigma_{z_1} \hat{U}^{\dagger\sigma}_{z_2}\}]^{\text{conf}}_a$ does not depend on $\eta = \ln \sigma_-$ and all the rapidity dependence is encoded into a-dependence:

$$\begin{aligned} & \mathbf{U}(z_1, z_2)^{\text{conf}} = \mathbf{U}(z_1, z_2) + \\ & + \frac{g^2}{\pi} \int d^2 z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\mathbf{U}(z_1, z_3) + \mathbf{U}(z_3, z_2) - \mathbf{U}(z_1, z_2)] \ln \frac{2 a z_{12}^2}{\sigma_+^2 z_{13}^2 z_{23}^2} \ + \ O(\alpha_s^2). \end{aligned} \tag{B.5}$$

We need to choose the new "rapidity cutoff" a in such a way that all the energy dependence is included into the matrix element(s) of Wilson-line operators so that the impact factor does not depend on energy (i.e. it should not scale with the length of frame.

Also, the NLO impact factor should be Möbius invariant. These two requirements fix the cutoff in the form $a_0 = \frac{2x_{1-}x_{3-}}{(x-y)^2}$ and we obtain that the typical logarithmic term in the NLO impact factor is proportional to

$$\frac{1}{\mathcal{Z}_1^2}\Big[\ln\frac{-x_{1-}x_{3-}z_{12}^2}{x_{13_\perp}^2(x_{1_\perp}-z_2)^2 z_{12}^2}\mathcal{Z}_2^2 + 2C\Big] + (x_1 \leftrightarrow x_3) + (z_1 \leftrightarrow z_2). \tag{B.6}$$

Thus, the "new rapidity cutoff" for the upper Wilson frame is $\sigma_- = \frac{2x_{1-}x_{3-}}{x_{13_\perp}^2}$ (for simplicity, we use the same notation σ since the meaning of a_0 is essentially the rapidity cutoff for the conformal dipole (B.4)). Similarly, for the lower Wilson frame the cutoff is $\sigma_- = \frac{2y_{1+}y_{3+}}{y_{13_\perp}^2}$ so we get $\sigma_+\sigma_- = r_1 = r_2$ at large longitudinal x, y.

Appendix C. Calculation of the integral in (64)

To carry out the integration over z_0 in the integral

$$A_{\mathbb{R}^2} = \int\limits_{\mathbb{R}^2} d^2 z_0 \frac{(|x_{13}|^2)^{\frac{1}{2}+i\nu}(2\cos^2(\phi_x)-1)}{(|x_1-z_0|^2)^{\frac{1}{2}+i\nu}|x_3-z_0|^2)^{\frac{1}{2}+i\nu}} \frac{(|y_{13}|^2)^{\frac{1}{2}-i\nu}(2\cos^2(\phi_y)-1)}{(|y_1-z_0|^2)^{\frac{1}{2}-i\nu}|y_3-z_0|^2)^{\frac{1}{2}-i\nu}} \tag{C.1}$$

let us define two functions

$$A_{\Omega} = \int\limits_{\Omega} d^2 z_0 \frac{(|x_{13}|^2)^{\frac{1}{2}+i\nu}(2\cos^2(\phi_x)-1)}{(|x_1-z_0|^2)^{\frac{1}{2}+i\nu}|x_3-z_0|^2)^{\frac{1}{2}+i\nu}} \frac{(|y_{13}|^2)^{\frac{1}{2}-i\nu}(2\cos^2(\phi_y)-1)}{(|y_1-z_0|^2)^{\frac{1}{2}-i\nu}|y_3-z_0|^2)^{\frac{1}{2}-i\nu}}, \tag{C.2}$$

$$B_{\Omega} = \int\limits_{\Omega} d^2 z_0 \frac{(|x_{13}|^2)^{\frac{1}{2}+i\nu}}{(|x_1-z_0|^2)^{\frac{1}{2}+i\nu}|x_3-z_0|^2)^{\frac{1}{2}+i\nu}} \frac{(|y_{13}|^2)^{\frac{1}{2}-i\nu}}{(|y_1-z_0|^2)^{\frac{1}{2}-i\nu}|y_3-z_0|^2)^{\frac{1}{2}-i\nu}} \tag{C.3}$$

and divide the full $\mathbb{R}^2$ space into three domains

(1) $\Omega_0 = |x_1 - z_0|, |x_3 - z_0| > q_x \wedge |y_1 - z_0|, |y_3 - z_0| > q_y$

(2) $\Omega_x = |x_1 - z_0|, |x_3 - z_0| < q_x$

(3) $\Omega_y = |y_1 - z_0|, |y_3 - z_0| < q_y$

where

$$q_x = \sqrt{|x_{13}||x-y|},\ q_y = \sqrt{|y_{13}||x-y|}$$

and calculate the difference $A_\Omega - B_\Omega$ for each of them.

In the case (1) we can expand $\cos^2 \approx 1 + o(|x_{13}|, |y_{13}|)$, then $2cos^2 - 1 \to 1$. The difference $A_{\Omega_0} - B_{\Omega_0}$ disappears in this domain. Now let us elaborate the case (2) (the case (3) is absolutely similar). In this case we integrate over z inside the circle centered at $x_1 \sim x_3$, with the radius q_x:

$$A_{\Omega_x} - B_{\Omega_x} = \frac{(|x_{13}|^2)^{\frac{1}{2}+i\nu}(|y_{13}|^2)^{\frac{1}{2}-i\nu}}{(|x-y|^2)^{1-2i\nu}} \int\limits_{|z-x|<q_x} d^2 z_0 \frac{2\cos^2(\phi_x)-1-1}{(|x_1-z_0|^2)^{\frac{1}{2}+i\nu}(|x_3-z_0|^2)^{\frac{1}{2}+i\nu}} \cdot$$

$$\cdot(1 + o(\frac{q_x}{|x-y|})) =$$

$$= -2\frac{(|x_{13}|^2)^{\frac{1}{2}+i\nu}(|y_{13}|^2)^{\frac{1}{2}-i\nu}}{(|x-y|^2)^{1-2i\nu}}\int_{\mathbb{R}^2} d^2z_0\frac{\sin^2(\phi_x)}{(|x_1-z_0|^2)^{\frac{1}{2}+i\nu}(|x_3-z_0|^2)^{\frac{1}{2}+i\nu}}(1+o(\frac{x_{13}}{q_x})). \tag{C.4}$$

The last integral can be calculated in elliptic coordinates

$$\begin{aligned}|x_1-z_0| &= \frac{|x_{13}|}{2}(\sigma+\tau),\\ |x_3-z_0| &= \frac{|x_{13}|}{2}(\sigma-\tau),\end{aligned} \tag{C.5}$$

which gives:

$$\begin{aligned}&\int_{\mathbb{R}^2} d^2z\frac{\sin^2(\phi_x)}{(|x_1-z_0|^2)^{\frac{1}{2}+i\nu}(|x_3-z_0|^2)^{\frac{1}{2}+i\nu}} =\\ &= 2^{3+4i\nu}(|x_{13}|^2)^{-2i\nu}\int_1^\infty d\sigma\int_{-1}^1 d\tau\frac{\sqrt{(\sigma^2-1)(1-\tau^2)}}{(\sigma^2-\tau^2)^{2+2i\nu}} =\\ &= -\pi 2^{-1+4i\nu}(|x_{13}|^2)^{-2i\nu}\frac{\Gamma(-\frac{1}{2}-i\nu)\Gamma(1+i\nu)}{\Gamma(1-i\nu)\Gamma(\frac{3}{2}+i\nu)},\end{aligned} \tag{C.6}$$

where we have used the formula:

$$\begin{aligned}&\int_1^\infty d\sigma\int_{-1}^1 d\tau\frac{\sqrt{(\sigma^2-1)(1-\tau^2)}}{(\sigma^2-\tau^2)^{2+2i\nu}} =\\ &= \int_1^\infty d\sigma\sqrt{-1+\sigma^2}\frac{1}{2}\pi\left(\sigma^2\right)^{-2-2i\nu}{}_2F_1\left(\frac{1}{2},2+2i\nu,2,\frac{1}{\sigma^2}\right) =\\ &= -\frac{\pi\Gamma\left(-\frac{1}{2}-i\nu\right)\Gamma(1+i\nu)}{16\Gamma(1-i\nu)\Gamma\left(\frac{3}{2}+i\nu\right)}.\end{aligned} \tag{C.7}$$

Finally we get:

$$\delta_x = A_{\Omega_x} - B_{\Omega_x} = \frac{(|x_{13}|^2)^{\frac{1}{2}-i\nu}(|y_{12}|^2)^{\frac{1}{2}-i\nu}}{(|x-y|^2)^{1-2i\nu}}\pi 2^{4i\nu}\frac{\Gamma(-\frac{1}{2}-i\nu)\Gamma(1+i\nu)}{\Gamma(1-i\nu)\Gamma(\frac{3}{2}+i\nu)}. \tag{C.8}$$

And similar expression for $A_{\Omega_y} - B_{\Omega_y}$:

$$\delta_y = A_{\Omega_y} - B_{\Omega_y} = \frac{(x_{13}|^2)^{\frac{1}{2}+i\nu}(|y_{13}|^2)^{\frac{1}{2}+i\nu}}{(|x-y|^2)^{1+2i\nu}}\pi 2^{-4i\nu}\frac{\Gamma(-\frac{1}{2}+i\nu)\Gamma(1-i\nu)}{\Gamma(1+i\nu)\Gamma(\frac{3}{2}-i\nu)}. \tag{C.9}$$

Expression for $B_{\mathbb{R}^2}$ (when $\Omega = \mathbb{R}^2$) in the limit $x_{13},\ y_{13} \to 0$ reads as follows:

$$B_{\mathbb{R}^2} = \int_{\mathbb{R}^2} d^2z_0\frac{(|x_{13}|^2)^{\frac{1}{2}+i\nu}}{(|x_1-z_0|^2)^{\frac{1}{2}+i\nu}(|x_3-z_0|^2)^{\frac{1}{2}+i\nu}}\frac{(|y_{13}|^2)^{\frac{1}{2}-i\nu}}{(|y_1-z_0|^2)^{\frac{1}{2}-i\nu}(|y_3-z_0|^2)^{\frac{1}{2}-i\nu}} =$$

$$= \frac{(|x_{13}|^2)^{\frac{1}{2}+i\nu}(|y_{13}|^2)^{\frac{1}{2}+i\nu}}{(|x-y|^2)^{1+2i\nu}} F(\nu) + (\nu \to -\nu), \tag{C.10}$$

where $F(\nu) = \frac{\pi 2^{-4i\nu}}{2i\nu} \frac{\Gamma(\frac{1}{2}+i\nu)\Gamma(-i\nu)}{\Gamma(\frac{1}{2}-i\nu)\Gamma(i\nu)}$. Finally, collecting the individual terms we obtain

$$A_{\mathbb{R}^2} = \frac{(|x_{13}|^2)^{\frac{1}{2}+i\nu}(|y_{13}|^2)^{\frac{1}{2}+i\nu}}{(|x-y|^2)^{1+2i\nu}} G(\nu) + (\nu \to -\nu),$$

where

$$G(\nu) = -i \frac{4^{-1-2i\nu}\pi^3(i-2\nu)^2}{\Gamma^2(\frac{3}{2}-i\nu)\Gamma^2(1+i\nu)\sinh(2\pi\nu)}.$$

Appendix D. Two-point correlator of Wilson frames

As was noticed before, the method of OPE over color dipoles is quite general and can be applied to many different operators. In this section we give the expression for the case of pure Wilson frames (with no field insertion). Namely, such an operator for a frame stretched along n_+ reads as follows:

$$S^{\omega}_{W.F.+}(x_{1\perp}, x_{3\perp}) = \int_{-\infty}^{\infty} dx_{1-} \int_{x_{1-}}^{\infty} dx_{3-} (x_{3-} - x_{1-})^{-2-\omega} \,\mathrm{tr}\, [x_1, x_2]_{\Box}. \tag{D.1}$$

The operator constructed from a pure Wilson rectangle collapses to one when it is reduced to light-ray, but it has a nontrivial correlation function when its transverse size is slightly different from zero. The OPE expansion of frames over color dipoles consists of simply replacement of a finite frame by an infinite dipole with a certain cutoff σ_+:

$$\mathrm{tr}\, [x_1, x_3]_{\Box} \to N(1 - \mathbf{U}^{\sigma_+}(x_{1\perp}, x_{3\perp})). \tag{D.2}$$

This formula is an analogue of (39). The rest of calculation almost directly repeats the calculations for the regularized light ray operators of the main text and the result reads as follows:

$$\langle S^{\omega_1}_{W.F.+}(x_{1\perp}, x_{3\perp}) S^{\omega_2}_{W.F.-}(y_{1\perp}, y_{3\perp})\rangle \sim \frac{g^4}{\omega} \frac{(x^2_{13\perp})^{2+\frac{\gamma}{2}}(y^2_{13\perp})^{2+\frac{\gamma}{2}}}{((x-y)^2_\perp)^{2+\gamma+\omega}}, \tag{D.3}$$

where γ is the anomalous dimension in the NLO BFKL given by the solution of (68). Let us stress that this result is in correspondence with (69). Namely let us check the weak coupling regime $\frac{g^2}{\omega} \to 0$. In this case we have:

$$\partial_{x_{1\perp}}\partial_{x_{3\perp}} \iint (x_{3-} - x_{1-})^{-2-\omega}[x_1, x_3]_{\Box} \simeq \frac{g^2_{YM}}{\omega} \iint (x_{3-} - x_{1-})^{-\omega} F(x_1)[x_1, x_3]_{\Box} F(x_3). \tag{D.4}$$

So it leads to the following correlator of two frames

$$\langle S^{2+\omega_1}_{+} S^{2+\omega_2}_{-}\rangle \sim (\frac{\omega}{g^2_{YM}})^2 \partial_{x_{1\perp}}\partial_{x_{3\perp}}\partial_{y_{1\perp}}\partial_{y_{3\perp}} \langle S^{\omega_1}_{W.F.+}(x_{1\perp}, x_{3\perp}) S^{\omega_2}_{W.F.-}(y_{1\perp}, y_{3\perp})\rangle, \tag{D.5}$$

which coincides with (74) and (D.3).

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