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# A dispersive treatment of $K_{\ell 4}$ decays

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**Abstract.**  $K_{\ell 4}$  decays have several features of interest: they allow an accurate measurement of  $\pi\pi$ -scattering lengths; the decay is the best source for the determination of some low-energy constants of chiral perturbation theory ( $\chi$ PT); one form factor of the decay is connected to the chiral anomaly.

We present the results of our dispersive analysis of  $K_{\ell 4}$  decays, which provides a resummation of  $\pi\pi$ - and  $K\pi$ -rescattering effects. The free parameters of the dispersion relation are fitted to the data of the high-statistics experiments E865 and NA48/2. By matching to  $\chi$ PT at NLO and NNLO, we determine the low-energy constants  $L_1^r$ ,  $L_2^r$  and  $L_3^r$ . In contrast to a pure chiral treatment, the dispersion relation describes the observed curvature of one of the  $K_{\ell 4}$  form factors, which we understand as an effect of rescattering beyond NNLO.

## 1. Motivation

$K_{\ell 4}$ , the semileptonic decay of a kaon into two pions and a lepton-neutrino pair, plays a crucial role in the context of low-energy hadron physics, because it provides almost unique information about some of the  $SU(3)$   $\mathcal{O}(p^4)$  low-energy constants (LECs) of chiral perturbation theory, the low-energy effective theory of QCD [1, 2, 3]. Although the  $K_{\ell 4}$  decay offers similar information as  $K\pi$  scattering, it happens at lower energies, where the chiral expansion is more reliable. Besides, as the hadronic final state contains two pions,  $K_{\ell 4}$  is also one of the best sources of information on  $\pi\pi$  interaction [4, 5, 6].

On the experimental side, we are confronted with impressive precision from the high-statistics measurements of the E865 experiment at BNL [7, 8] and the NA48/2 experiment at CERN [6, 9]. The statistical errors of the  $S$ -wave of one form factor reach in both experiments the sub-percent level. Matching this precision requires a theoretical treatment beyond one-loop order in the chiral



expansion. A first treatment beyond one loop, based on dispersion relations, was already done twenty years ago [10]. The full two-loop calculation became available in 2000 [11]. However, as we will show below, even at two loops  $\chi$ PT is not able to predict the curvature of one of the form factors.

Here, we present the results of a new dispersive treatment of  $K_{\ell 4}$  decays [12, 13]. We do not solve an exact dispersion relation for this process, but an approximate form, which follows if the contribution of  $D$ - and higher waves to the discontinuities are neglected.<sup>1</sup> This approximation is violated only at  $\mathcal{O}(p^8)$  in the chiral counting. The effects due to  $\pi\pi$  and  $K\pi$  rescattering in  $S$ - and  $P$ -wave are resummed to all orders. We expect this to capture the most important contributions beyond  $\mathcal{O}(p^6)$ . Indeed it turns out that the dispersive description is able to reproduce the curvature of the form factor.

Our final analysis of  $K_{\ell 4}$  decays represents an extension and a major improvement of our previous dispersive framework [14, 15, 16]. Instead of a single linear combination of form factors, now we describe the two form factors  $F$  and  $G$  simultaneously, including more experimental data in the fits. The new framework is valid also for non-vanishing invariant energies of the lepton pair. We apply corrections for isospin-breaking effects in the fitted data that have not been taken into account in the experimental analyses [17]. Besides a matching to one-loop  $\chi$ PT, we also study the matching at two-loop level.

## 2. Dispersion relation for $K_{\ell 4}$

In this section, we provide only a very short summary of our dispersive treatment of the  $K_{\ell 4}$  form factors. For more details, we kindly refer the reader to [12, 13].

### 2.1. Matrix element and form factors

We consider the charged decay mode

$$K^+(p) \rightarrow \pi^+(p_1)\pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu), \quad (1)$$

where  $\ell \in \{e, \mu\}$  is either an electron or a muon. So far, experimental data is only available on the electron mode.

After integrating out the  $W$  boson, we end up with a Fermi type current-current interaction and the matrix element splits up into a leptonic times a hadronic part. The leptonic matrix element can be treated in a standard way. The hadronic matrix element exhibits the usual  $V - A$  structure of weak interaction. Its Lorentz structure allows us to write the two contributions as

$$\langle \pi^+(p_1)\pi^-(p_2) | V_\mu(0) | K^+(p) \rangle = -\frac{H}{M_K^3} \epsilon_{\mu\nu\rho\sigma} L^\nu P^\rho Q^\sigma, \quad (2)$$

$$\langle \pi^+(p_1)\pi^-(p_2) | A_\mu(0) | K^+(p) \rangle = -i\frac{1}{M_K} (P_\mu F + Q_\mu G + L_\mu R), \quad (3)$$

where  $P = p_1 + p_2$ ,  $Q = p_1 - p_2$ ,  $L = p - p_1 - p_2$ . The form factors  $F$ ,  $G$ ,  $H$  and  $R$  are dimensionless scalar functions of the usual Mandelstam variables  $s$ ,  $t$  and  $u$ . In experiments on the electron mode  $K_{e4}$ ,  $R$  is not accessible.  $H$  gets a first contribution only at  $\mathcal{O}(p^4)$  due to the chiral anomaly. Here we focus on the form factors  $F$  and  $G$ .

### 2.2. Reconstruction theorem and integral equations

Let us for the moment regard the di-lepton invariant squared energy  $s_\ell = L^2$  as a fixed parameter. Based on fixed- $s/t/u$  dispersion relations, one can derive a decomposition of the form factors

<sup>1</sup> This means that  $D$ - and higher partial waves have no right-hand unitarity cut. However, they are still non-zero and have left-hand cuts due to unitarity in the crossed channels.

into functions of only one Mandelstam variable, known as ‘reconstruction theorem’ [18, 19]. In [20] the reconstruction theorem is used for a similar dispersive description of the  $K_{\ell 4}$  form factors in order to study isospin-breaking effects in the phases at two loops. The derivation neglects the imaginary parts of  $D$ - and higher partial waves, an  $\mathcal{O}(p^8)$  effect:

$$\begin{aligned} F(s, t, u) &= M_0(s) + \frac{u-t}{M_K^2} M_1(s) + (\text{terms involving functions of } t \text{ or } u) + \mathcal{O}(p^8), \\ G(s, t, u) &= \tilde{M}_1(s) + (\text{terms involving functions of } t \text{ or } u) + \mathcal{O}(p^8), \end{aligned} \quad (4)$$

where the functions of one variable  $M_0, \dots$  are defined to contain only the right-hand cut of the partial waves of the form factors  $F$  and  $G$  in the three channels. E.g. the function  $M_0$  contains the right-hand cut of the  $s$ -channel  $S$ -wave  $f_0$  of the form factor  $F$ :

$$M_0(s) = P(s) + \frac{s^2}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im} f_0(s)}{(s' - s - i\epsilon)s'^2}, \quad (5)$$

where  $P(s)$  is a subtraction polynomial. Eight more functions  $M_1, \dots$  take care of the right-hand cuts of  $S$ - and  $P$ -waves in all channels, such that all the discontinuities are divided up into functions of a single variable. They satisfy inhomogeneous Omnès equations with the solution

$$M_0(s) = \Omega_0^0(s) \left\{ \tilde{P}(s) + \frac{s^3}{\pi} \int_{4M_\pi^2}^{\Lambda^2} ds' \frac{\hat{M}_0(s') \sin \delta_0^0(s')}{|\Omega_0^0(s')|(s' - s - i\epsilon)s'^3} \right\}, \quad (6)$$

where  $\tilde{P}(s)$  is a new subtraction polynomial and the Omnès function is given by

$$\Omega_0^0(s) = \exp \left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_0^0(s')}{(s' - s - i\epsilon)s'} \right\}. \quad (7)$$

In total, 9 subtraction constants appear. We need the following elastic  $\pi\pi$  and  $K\pi$  phase shifts as input, which we assume to reach a multiple of  $\pi$  at the cut-off  $\Lambda^2$ :

- $\delta_0^0, \delta_1^1$ : elastic  $\pi\pi$ -scattering phase shifts [21, 22],
- $\delta_0^{1/2}, \delta_1^{1/2}, \delta_0^{3/2}, \delta_1^{3/2}$ : elastic  $K\pi$ -scattering phase shifts [23, 24].

The inhomogeneities in the Omnès problem are given by the differences of the functions  $M_0, \dots$  and the corresponding partial wave, e.g.  $\hat{M}_0(s) = f_0(s) - M_0(s)$ . These ‘hat functions’ contain the left-hand cut of the partial wave and we compute them by projecting out the partial wave of the decomposed form factor (4). The inhomogeneities  $\hat{M}_0, \dots$  are then given as angular averages of all the functions  $M_0, \dots$ . Hence, we face a set of coupled integral equations: the functions  $M_0, \dots$  are defined by dispersion integrals involving the inhomogeneities  $\hat{M}_0, \dots$ , which are again defined as angular integrals of the functions  $M_0, \dots$ . This system can be solved by iteration.

### 2.3. Numerical solution of the dispersion relation

We note that the integral equations are linear in the subtraction constants. Therefore, for each subtraction constant we construct a basis solution, which we obtain by solving numerically the integral equations in an iterative procedure. The final result is a linear combination of these basis solutions.

We determine the subtraction constants using three sources of information: first, we fit the experimental data on the form factors  $F$  and  $G$  from the high-statistics experiments NA48/2 [6, 9] and E865 [7, 8]. Secondly, we use as an additional constraint the well-known

soft-pion theorem [25, 26, 27], which establishes the following relations between  $F$ ,  $G$  and  $f_+$ , the  $K_{\ell 3}$  vector form factor:

$$\begin{aligned} F(M_\pi^2, M_K^2, M_\pi^2 + s_\ell) - G(M_\pi^2, M_K^2, M_\pi^2 + s_\ell) &= \mathcal{O}(M_\pi^2), \\ F(M_\pi^2, M_\pi^2 + s_\ell, M_K^2) + G(M_\pi^2, M_\pi^2 + s_\ell, M_K^2) &= \frac{\sqrt{2}M_K}{F_\pi} f_+(M_\pi^2 + s_\ell) + \mathcal{O}(M_\pi^2). \end{aligned} \quad (8)$$

Finally, we fix the subtraction constants that are not well determined by the data with chiral input.

### 3. Results

#### 3.1. Fits to data

We perform a fit of the dispersion relation to both, the E865 [7, 8] and NA48/2 data sets [6, 9], corrected for additional isospin-breaking effects that were not taken into account in the experimental analyses [17]. Recently, a two-dimensional data set on the  $S$ -wave of  $F$  has become available (addendum to [9]): in this set, not only a single bin but up to 10 bins are used in  $s_\ell$ -direction. If we allow for varying values of the di-lepton invariant squared energy  $s_\ell$ , the subtraction constants become functions of this parameter and the functions  $M_0, \dots$  depend on two variables, e.g.  $M_0(s, s_\ell)$ . We perform our fits in the two-dimensional  $(s, s_\ell)$ -plane using the full available data sets on the  $S$ - and  $P$ -waves of the form factors, given by

$$\begin{aligned} F_s(s, s_\ell) &= \left( M_0(s, s_\ell) + \hat{M}_0(s, s_\ell) \right) e^{-i\delta_0^0(s)}, \\ \tilde{F}_p(s, s_\ell) &= \left( M_1(s, s_\ell) + \hat{M}_1(s, s_\ell) \right) e^{-i\delta_1^1(s)}, \\ G_p(s, s_\ell) &= \left( \tilde{M}_1(s, s_\ell) + \hat{\tilde{M}}_1(s, s_\ell) \right) e^{-i\delta_1^1(s)}. \end{aligned} \quad (9)$$

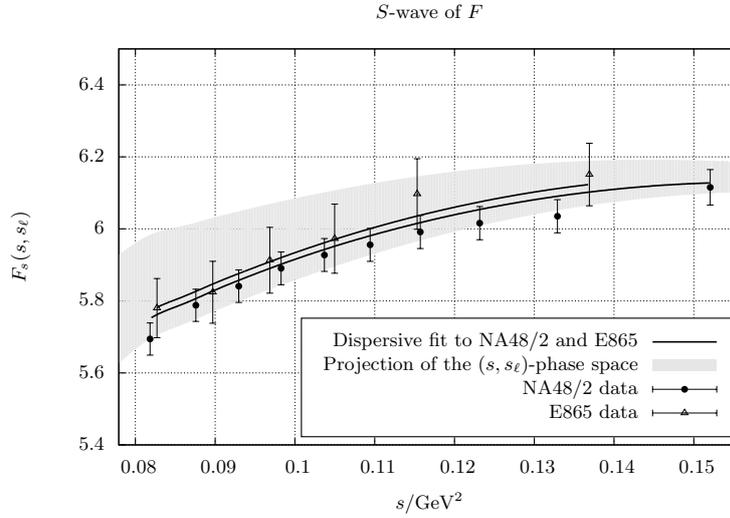
Figure 1 shows the fit results for the  $S$ -wave of the form factor  $F$ . The two-dimensional phase space is projected on the  $s$ -axis and only the data sets with a single bin in  $s_\ell$ -direction are plotted. The dispersive description reproduces beautifully the observed curvature of the form factor  $F_s$ . Note that  $\chi$ Pt alone is not able to describe this curvature, which can be understood as a higher-order effect of  $\pi\pi$  rescattering, fully taken into account in the dispersive Omnès representation of the form factor.

#### 3.2. Matching to $\chi$ Pt

We perform the matching to  $\chi$ Pt directly on the level of the subtraction constants. This means that we decompose the chiral expression at NLO or NNLO according to the reconstruction theorem and write the functions  $M_0, \dots$  in a chirally expanded Omnès form. This allows us to directly identify the subtraction polynomials  $\tilde{P}(s)$  in (6) with chiral expressions. Such a procedure separates the resummation of rescattering effects from the chiral matching. Note also that we subtract all the functions  $M_0, \dots$  at zero energy.

By matching the dispersion relation to  $\chi$ Pt, we are able to determine the LECs  $L_1^r, L_2^r$  and  $L_3^r$ . Using the information on the  $s_\ell$ -dependence of the form factors, also  $L_9^r$  can be extracted, though the present experimental data does not allow a precise determination.

If we perform the matching at two-loop level, many NNLO LECs  $C_i^r$  enter the matching equations. We compare different input values for the  $C_i^r$  [28, 29, 30] and assign a 50% uncertainty to the contribution of the  $C_i^r$  to the subtraction constants. This  $C_i^r$  contribution is then fitted as well, using constraints on the chiral convergence of the subtraction constants. We find that the  $C_i^r$  input values of the BE14 global fit [30] lead to the best chiral convergence and a good  $\chi^2$  of the whole fit.



**Figure 1.** Fit result for the  $S$ -wave of the form factor  $F$ . The dispersive description reproduces beautifully the curvature of the form factor. The  $(s, s_\ell)$ -phase space is projected on the  $s$ -axis, the plotted lines correspond to splines through the  $(s, s_\ell)$ -values of the data sets with a single bin in  $s_\ell$ -direction.

In table 1, we show the results of the matching at NLO and NNLO for the low-energy constants  $L_1^r$ ,  $L_2^r$  and  $L_3^r$ . For the NNLO matching, the BE14  $C_i^r$  input values were used. For comparison, we also quote the values of the BE14 global fit [30].

**Table 1.** Results for the LECs ( $\mu = 770$  MeV).

	$10^3 \cdot L_1^r$	$10^3 \cdot L_2^r$	$10^3 \cdot L_3^r$	$\chi^2/\text{dof}$
Dispersive treatment, NLO matching	0.51(6)	0.89(9)	-2.82(12)	141/116 = 1.2
Dispersive treatment, NNLO matching	0.69(18)	0.63(13)	-2.63(46)	122/122 = 1.0
BE14 global fit [30]	0.53(6)	0.81(4)	-3.07(20)	

At NLO, a large contribution to the uncertainties comes from the high-energy behaviour of the phase shifts, either from the  $\pi\pi$  phases in the case of  $L_1^r$  and  $L_2^r$  or the  $K\pi$  phases in the case of  $L_3^r$ . At NNLO, the largest uncertainty is due to the fitted contribution of the  $C_i^r$ .

#### 4. Conclusions

We have presented a dispersive representation of  $K_{\ell 4}$  decays that provides a model independent parametrisation valid up to and including  $\mathcal{O}(p^6)$ . The dispersion relation is based on unitarity, analyticity and crossing. It includes a full resummation of  $\pi\pi$ - and  $K\pi$ -rescattering effects. The dispersion relation is parametrised by subtraction constants that we determine by fitting experimental data and by using the soft-pion theorem as well as chiral input.

In contrast to a pure chiral description, the dispersion relation describes perfectly the experimentally observed curvature of the  $S$ -wave of the form factor  $F$ , which we interpret as a result of significant  $\pi\pi$ -rescattering effects. This is yet another case in which high-precision data clearly call for effects which go even beyond NNLO in  $\chi$ PT.

By using the matching equations to  $\chi$ PT we have extracted the values of the low-energy constants  $L_1^r$ ,  $L_2^r$  and  $L_3^r$ . The correction from NLO to NNLO, when matching the chiral and

dispersive representations and fitting the latter to the data are smaller than the corrections from NLO to NNLO observed in direct  $\chi$ PT fits. Constraints on the chiral convergence of the subtraction constants allow us to reduce the dependence on the input values for the  $C_i^r$ . Still, the poorly known values of the  $C_i^r$  are responsible for the larger uncertainties in the matching at NNLO.

The two-dimensional NA48/2 data set for the  $S$ -wave of  $F$ , which shows both the  $s$ - as well as the  $s_\ell$ -dependence, has allowed us to extract a value for  $L_9^r$  [13], which is roughly compatible with previous determinations. In accuracy, however, it cannot compete yet, as it reflects the low precision in the measurement of the  $s_\ell$ -dependence of  $F$ . A new high-statistics measurement of  $K_{e4}$  could improve on this and provide a new determination of  $L_9^r$ . Regarding prospects for forthcoming experiments, we refer to [31].

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