Small- x evolution in the next-to-leading order

Ian Balitsky

Phys. Dept., Old Dominion Univ.

and

Theory Group, JLab

12000 Jefferson Ave, Newport News, VA 23606, USA

The high-energy behavior of amplitudes in gauge theories can be reformulated in terms of the evolution of Wilson-line operators. In the leading order this evolution is governed by the non-linear BK equation. The NLO corrections define the scale of the running-coupling constant in the BK equation and in QCD, its kernel has both conformal and non-conformal parts. To separate the conformally invariant effects from the running-coupling effects, we calculate the NLO evolution of the color dipoles in the conformal $\mathcal{N} = 4$ SYM theory, then we define the “composite dipole operator” with the rapidity cutoff preserving conformal invariance, and the resulting Möbius invariant kernel for this operator agrees with the forward NLO BFKL calculation.

1 Small- x evolution of color dipoles

A general feature of high-energy scattering is that a fast particle moves along its straight-line classical trajectory and the only quantum effect is the eikonal phase factor acquired along this propagation path. In QCD, for fast quark or gluon scattering off some target, this eikonal phase factor is a Wilson line - an infinite gauge link ordered along the straight line collinear to particle's velocity n^μ (see e.g. the review [1])

$$U^\eta(x_\perp) = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} du \, n_\mu \, A^\mu(un + x_\perp) \right\} \quad (1)$$

Here A_μ is the gluon field of the target, $x_\perp$ is the transverse position of the particle which remains unchanged throughout the collision, and the index η labels the rapidity of the particle.

Let us consider the small- x behavior of structure functions of deep inelastic scattering (DIS) at small $x_B = Q^2/(2p \cdot q)$. The virtual photon decomposes into quark and antiquark which propagate along the straight lines separated by transverse distance. The propagation of this $q\bar{q}$ pair reduces to the “propagator of the color dipole” - two Wilson lines ordered along the direction collinear to quarks' velocity. The structure function of a hadron is proportional then to the matrix element of this color dipole operator

$$\hat{\mathcal{U}}^\eta(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{tr} \{ \hat{U}^\eta(x_\perp) \hat{U}^{\dagger\eta}(y_\perp) \} \quad (2)$$

switched between the target's states ($N_c = 3$ for QCD)

The energy dependence of the structure function is then translated into the dependence of the color dipole on the rapidity η . There are two ways to restrict the rapidity of Wilson lines: one can consider Wilson lines with the support line collinear to the velocity of the fast-moving particle or one can take the light-like Wilson line and cut the rapidity integrals “by hand”. While the former method appears to be more natural, it is technically simpler

to get the conformal results with the latter method of “rigid cutoff” in the longitudinal direction.

The small- x behavior of the structure functions is governed by the rapidity evolution of color dipoles [2, 3]. At relatively high energies and for sufficiently small dipoles we can use the leading logarithmic approximation (LLA) where $\alpha_s \ll 1$, $\alpha_s \ln x_B \sim 1$ and get the non-linear BK evolution equation for the color dipoles [4, 5]:

$$\begin{aligned} & \frac{d}{d\eta} \hat{\mathcal{U}}^\eta(z_1, z_2) \\ &= \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\hat{\mathcal{U}}^\eta(z_1, z_3) + \hat{\mathcal{U}}^\eta(z_3, z_2)) - \hat{\mathcal{U}}^\eta(z_1, z_3) - \hat{\mathcal{U}}^\eta(z_1, z_3) \hat{\mathcal{U}}^\eta(z_3, z_2)] \quad (3) \end{aligned}$$

where $\eta = \ln \frac{1}{x}$ and $z_{12} \equiv z_1 - z_2$ etc. (As usual, we denote operators by “hat”). The first three terms correspond to the linear BFKL evolution [6] and describe the parton emission while the last term is responsible for the parton annihilation. For sufficiently low x_B the parton emission balances the parton annihilation so the partons reach the state of saturation [7] with the characteristic transverse momentum Q_s growing with energy $1/x_B$ (for a review, see [8])

As usual, to get the region of application of the leading-order evolution equation one needs to find the next-to-leading order (NLO) corrections. In the case of the small- x evolution equation (3) there is another reason why NLO corrections are important. Unlike the DGLAP evolution, the argument of the coupling constant in Eq. (3) is left undetermined in the LLA, and usually it is set by hand to be Q_s . Careful analysis of this argument is very important from both theoretical and experimental points of view. From the theoretical viewpoint, we need to know whether the coupling constant is determined by the size of the original dipole $|x - y|$ or of the size of the produced dipoles $|x - z|$ and/or $|z - y|$ since we may get a very different behavior of the solutions of the equation (3). On the experimental side, the cross section is proportional to some power of the coupling constant so the argument determines how big (or how small) is the cross section. The typical argument of α_s is the characteristic transverse momenta of the process. For high enough energies, they are of order of the saturation scale Q_s which is $\sim 2 \div 3$ GeV for the LHC collider, so even the difference between $\alpha(Q_s)$ and $\alpha(2Q_s)$ can make a substantial impact on the cross section. The precise form of the argument of α_s should come from the solution of the BK equation with the running coupling constant, and the starting point of the analysis of the argument of α_s in Eq. (3) is the calculation of the NLO evolution.

2 NLO evolution of color dipoles in $\mathcal{N} = 4$ SYM

It is easy to see that the BK equation (3) is conformally invariant in the two-dimensional space. This follows from the conformal invariance of the light-like Wilson lines. Indeed, the Wilson line

$$U(x_\perp) = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\} \quad (4)$$

is invariant under the inversion $x^\mu \rightarrow x^\mu/x^2$ (with respect to the point with zero (-) component). It is easy to check that the Wilson line operators lie in the standard representation of the conformal Möbius group $SL(2, \mathbb{C})$ with conformal spin 0.

The NLO evolution of color dipole in QCD [9] is not expected to be Möbius invariant due to the conformal anomaly leading to dimensional transmutation and running coupling constant. To understand the relation between the high-energy behavior of amplitudes and Möbius invariance of Wilson lines, it is instructive to consider first the conformally invariant $\mathcal{N} = 4$ super Yang-Mills theory. This theory was intensively studied in recent years due to the fact that at large coupling constants it is dual to the IIB string theory in the AdS_5 background. In the leading order, the evolution equation for color dipoles in the $\mathcal{N} = 4$ SYM has the same form as (3); at the NLO level, the contributions from gluino and scalar loops enter the picture.

As we mentioned above, formally the light-like Wilson lines are Möbius invariant. Unfortunately, the light-like Wilson lines are divergent in the longitudinal direction and moreover, it is exactly the evolution equation with respect to this longitudinal cutoff which governs the high-energy behavior of amplitudes. At present, it is not known how to find the conformally invariant cutoff in the longitudinal direction. When we use the non-invariant cutoff we expect, as usual, the invariance to hold in the leading order but to be violated in higher orders in perturbation theory. In our calculation we restrict the longitudinal momentum of the gluons composing Wilson lines, and with this non-invariant cutoff the NLO evolution equation in $\mathcal{N} = 4$ has a non-conformal term. However, if we construct the “composite conformal dipole operator” (order by order in perturbation theory) which mimics the conformal cutoff in the longitudinal direction, the corresponding evolution equation is Möbius invariant. This is similar to the construction of the composite renormalized local operator in the case when the UV cutoff does not respect the symmetries of the bare operator - in this case the symmetry of the UV-regularized operator is preserved order by order in perturbation theory by subtraction of the symmetry-restoring counterterms.

Let us present our result for the NLO evolution of the color dipole in the fundamental representation [10] (hereafter we use notations $z_{ij} \equiv z_i - z_j$)

$$\begin{aligned}
& \frac{d}{d\eta} [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\
&= \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 - \frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} \right] [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\
&+ \frac{\alpha_s^2}{16\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2}{z_{13}^2 z_{34}^2 z_{24}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right. \\
&\times [2 \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_4}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_4}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} \hat{U}_{z_4}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\
&\quad \left. - \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_4}^{\dagger\eta} \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - (z_4 \rightarrow z_3) \right]
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
& [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\
&= \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \frac{\alpha_s}{4\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]
\end{aligned} \tag{6}$$

is the “composite dipole” with the conformal longitudinal cutoff in the next-to-leading order and a is an arbitrary dimensional constant. In fact, $a(\eta) = a e^\eta$ plays the same role for the rapidity evolution as μ^2 for the usual DGLAP evolution: the derivative $\frac{d}{da}$ gives the evolution equation (5). The kernel in the r.h.s. of Eq. (5) is obviously Möbius invariant

since it depends on two four-point conformal ratios $\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}$ and $\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}$. In addition, linear part of Eq. (5) agrees with forward NLO BFKL calculation of Ref. [11] (see the discussion in Refs. [10, 12]).

3 NLO evolution of color dipoles in QCD

Let us present the evolution equation for composite operator (6) in QCD. The resulting equation will not be Möbius invariant because of the running coupling constant so the composite operator (6) is not strictly speaking conformal. We will, however, keep the notation $[\dots]^{\text{conf}}$ as a reminder that this operator was conformal in $\mathcal{N} = 4$ SYM.

To get the evolution equation for “conformal” composite operators (6) in QCD we subtract the scalar and gluino contributions from the evolution equation (5) and add the quark contribution calculated in Refs. [13, 14]. We obtain

$$\begin{aligned}
\frac{d}{d\eta} [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} &= \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 + \frac{\alpha_s}{4\pi} \left[b \left(\ln \frac{z_{12}^2 \mu^2}{4} + 2C \right) \right. \right. \\
&- b \frac{z_{13}^2 - z_{23}^2}{z_{12}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} n_f \left. \right] [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\
&+ \frac{\alpha_s^2}{16\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \left[\left\{ \left(-2 + 2 \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left[\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right) \right. \right. \right. \right. \\
&\quad \left. \left. + \frac{2z_{13}^2 z_{24}^2 - 4z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right) + (z_3 \leftrightarrow z_4) \right\} \\
&\times [(\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_4}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_4}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} \hat{U}_{z_4}^\eta \hat{U}_{z_2}^{\dagger\eta}\}) - (z_4 \rightarrow z_3)] \\
&+ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\} \\
&\quad \times [(\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_4}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_4}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - z_3 \leftrightarrow z_4)] \\
&+ 8n_f \left\{ 2 - \frac{z_{13}^2 z_{24}^2 + z_{23}^2 z_{14}^2 - z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\} \text{tr}\{t^a \hat{U}_{z_1}^\eta t^b \hat{U}_{z_2}^{\dagger\eta}\} \text{tr}\{t^a \hat{U}_{z_3}^\eta t^b (\hat{U}_{z_4}^{\dagger\eta} - \hat{U}_{z_3}^{\dagger\eta})\} \left. \right]
\end{aligned} \tag{7}$$

where $b = \frac{11}{3} N_c - \frac{2}{3} n_f$. We see that the QCD kernel is a sum of the conformal part and the running coupling part proportional to b . One can hope that in all orders in perturbation theory QCD kernel will have a conformal structure modified by running coupling constant.

In conclusion let us discuss the argument of coupling constant in the BK equation (3). It should be emphasized that the NLO result (7) itself does not lead automatically to the argument of coupling constant in Eq. (3). In order to get this argument one can use the renormalon-based approach: first get the quark part of the running coupling constant coming from a bubble chain of quark loops and then make a conjecture that the gluon part of the β -function will follow that pattern. This analysis was performed in Refs. [13, 14] and the result is

$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} &= \frac{\alpha_s((x-y)^2)}{2\pi^2} \int d^2 z [\text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} - N_c \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}] \\
&\times \left[\frac{(x-y)^2}{X^2 Y^2} + \frac{1}{X^2} \left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right] + \dots
\end{aligned} \tag{8}$$

It is easy to see now that the argument of coupling constant is determined by the size of the smallest dipole $\min |x - y|, |x - z|, |y - z|$.

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