Decays of Non-strange Negative Parity Baryons in the $1/N_c$ Expansion *

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Abstract

The decays of non-strange negative parity baryons via the emission of single $\pi$ and $\eta$ mesons are analyzed in the framework of the $1/N_c$ expansion. A basis of spin-flavor operators for the partial wave amplitudes is established to order $1/N_c$ and the unknown effective coefficients are determined by fitting to the S- and D-wave partial widths as provided by the Particle Data Group. A set of relations between widths that result at the leading order, i.e. order $N_c^0$, is given and tested with the available data. Up to a few exceptions, a good description of the partial decays widths is already obtained at that order. Because of the rather large errors in the empirical input data the next to leading order fit fails to pin down with satisfactory accuracy the subleading effective coefficients. The hierarchy expected from the $1/N_c$ expansion is reflected in the results.

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* This work is dedicated to the memory of Professor Luis Masperi.
I. INTRODUCTION

The $1/N_c$ expansion has proven to be a very useful tool for analyzing the baryon sector. This success is mostly a consequence of the emergent contracted spin-flavor symmetry in the large $N_c$ limit \cite{1, 2}. In the sector of ground state baryons (identified for $N_c = 3$ with the spin $1/2$ octet and spin $3/2$ decuplet in the case of three flavors), that symmetry gives rise to several important relations that hold at different orders in the $1/N_c$ expansion \cite{3–5}. The domain of excited baryons (baryon resonances) has also been explored in the framework of the $1/N_c$ expansion \cite{6–14} with very promising results. The analyses carried out so far have been constrained to states that belong to a definite spin-flavor and orbital multiplet, i.e. the possibility of mixing of different such multiplets (so called configuration mixing) has been disregarded. One good reason for this approximation is that the resonance data is not sufficient for a full fledged $1/N_c$ analysis that includes those effects. It is also very likely that such effects are small for dynamical reasons. Indeed, it has been shown that the only configuration mixings that are not suppressed by $1/N_c$ factors involve couplings to the orbital degrees of freedom \cite{15}. The $1/N_c$ analyses of excited baryon masses have shown that orbital angular momentum couplings turn out to be very small \cite{8, 9}, which is in agreement with older results in the quark model \cite{16}. This strongly suggests that a similar suppression, which is not a consequence of the $1/N_c$ expansion but rather of QCD dynamics, also takes place in configuration mixings. Thus, disregarding configuration mixing is likely to be a good approximation for the purpose of phenomenology. Within such a framework, a few analyses of excited baryon strong decays have been carried out, namely the decays of the negative parity SU(6) 70-plet \cite{10} and of the Roper 56-plet \cite{11}. In the case of interest in the present work, namely the 70-plet, the analysis in Ref. \cite{10} used an incomplete basis of operators at sub-leading order in $1/N_c$. One of the motivations of the present work is to provide a complete analysis to $\mathcal{O}(1/N_c)$ for the decays of the non-strange members of the 70-plet (i.e., the mixed symmetry 20-plet of SU(4)) into ground state baryons plus a pion or an eta meson. In particular, a complete basis of effective operators that provide the various S- and D-wave amplitudes is furnished. It should be advanced that the lack of a complete basis of operators in \cite{10} does not affect in a significant way the conclusions of that work. As discussed later in this work, the fact that the input partial widths have large errors implies a rather uncertain determination of sub-leading effects.
The negative parity 70-plet is the experimentally best established and known excited baryon multiplet. In particular the S- and D-wave partial decay widths have been determined from different data analyses [17] with varying degrees of certainty. In all, these available widths provide sufficient input for the analysis at $O(1/N_c)$ pursued in this work. To make the analysis more conclusive, however, higher precision in the inputs would be required.

This paper is organized as follows: section II contains the framework for calculating the decays, section III provides the basis of effective operators, section IV presents the results, and finally the conclusions are given in section V.

II. FRAMEWORK FOR DECAYS

In the application of the $1/N_c$ expansion to excited baryons the assumption is made that there is an approximate spin-flavor symmetry. This assumption is rigorous in the large $N_c$ limit for the ground state baryons. For excited states, however, the symmetry is broken at order $N_c^0$. This has been found to be the case for the masses of the negative parity baryons, where it was shown [6, 8] that effective mass operators of spin-orbit type produce such a breaking. In the large $N_c$ limit a different scheme becomes rigorously valid [13, 14]. In the real world with $N_c = 3$ there is, however, plenty of evidence that the dominating spin-flavor breaking in masses are the hyperfine effects that are order $1/N_c$, while the spin-orbit effects are substantially smaller [8, 9]. Thus, a scheme based on an approximate spin-flavor symmetry is very convenient for phenomenological purposes.

The excited baryons are therefore classified in multiplets of the $O(3) \times SU(2N_f)$ group. $O(3)$ corresponds to spatial rotations and $SU(2N_f)$ is the spin-flavor group where $N_f$ is the number of flavors being considered, equal to two in the present work. The ground state baryons, namely the $N$ and $\Delta$ states, belong to the $(1, 20_S)$ representation, where the $20_S$ is the totally symmetric representation of $SU(4)$. The negative parity baryons considered here belong instead to the $(3, 20_{MS})$ representation, where $20_{MS}$ is the mixed symmetric representation of $SU(4)$. For general $N_c$ the spin-flavor representations involve, in the Young tableaux language, $N_c$ boxes and are identified with the totally symmetric and the mixed-symmetric representations of type $(N_c - 1, 1)$ for ground and excited negative parity states respectively. Since in the mixed symmetric spin-flavor representation one box of the Young tableaux is distinguished, such a box associated with the “excited quark” in the baryon. In
a similar fashion, and without any loss of generality it is possible to distinguish one box in
the ground state multiplet as well. This is a very convenient procedure that has been used
repeatedly in previous works. The spin and isospin quantum numbers of the distinguished
box will be denoted with lower cases, and the corresponding quantum numbers of the rest
of $N_c - 1$ boxes (which are in a totally symmetric representation of $SU(4)$ and form the
so called “core” of the large $N_c$ baryon) after they are coupled to eigenstates of spin and
isospin will be denoted by $S_c$ and $I_c$ respectively. Notice that $S_c = I_c$ for totally symmetric
representations of $SU(4)$. For a given core state, the coupling of the excited quark gives
eigenstates of spin and isospin:

$$
| S, S_3; I, I_3; S_c \rangle = \sum_{s_3, i_3} \langle S, S_3 - s_3; S_3 | S, S_3 \rangle \langle I = S_c, I_3 - i_3; \frac{1}{2}, i_3 | I, I_3 \rangle
\times | S_c, S_3 - s_3; I = S_c, I_3 - i_3; \frac{1}{2}, s_3; \frac{1}{2}, i_3 \rangle
$$

(1)

These states are not in an irreducible representation of $SU(4)$, as they are not in an irre-
ducible representation of the permutation group. The totally symmetric states are given
by:

$$
| S, S_3; I, I_3; S_c = S + \eta \rangle
$$

(2)

where

$$
C_S(S, \pm \frac{1}{2}) = \sqrt{\frac{(2S + 1 \mp 1)(N_c + 1 \pm (2S + 1))}{2N_c(2S + 1)}},
$$

(3)

while the mixed symmetric (MS) states $(N_c - 1, 1)$ are given by:

$$
| S, S_3; I, I_3; S_c = S + \eta \rangle
$$

(4)

where

$$
C_{MS}(I, S, \pm \frac{1}{2}) = \begin{cases}
1 & \text{if } I = S \pm 1 \\
0 & \text{if } I = S \mp 1 \\
\pm \sqrt{(2S + 1 \mp 1)(N_c + 1 \pm (2S + 1))} & \text{if } I = S
\end{cases}
$$

(5)

Finally, upon coupling the orbital degrees of freedom, the excited baryons in the $(3, MS)$
representation are given by:

$$
| J, J_3; I, I_3; S \rangle_{MS} = \sum_m \langle 1, m; S, J_3 - m | J, J_3 \rangle \langle 1, m | S, J_3 - m; I, I_3 \rangle_{MS}.
$$

(6)
For $N_c = 3$ the states are displayed in Table I along with their quantum numbers, masses, decay widths and branching ratios.

Note that there are two sets of $N^*$ states each consisting of two states with the same spin and isospin. The physical states are admixtures of such states, and are given by:

$$
\begin{pmatrix}
N^*_J \\
N^*_J'
\end{pmatrix}
= \begin{pmatrix}
\cos \theta_{2J} & \sin \theta_{2J} \\
-\sin \theta_{2J} & \cos \theta_{2J}
\end{pmatrix}
\begin{pmatrix}
2N^*_J \\
4N^*_J'
\end{pmatrix},
$$

(7)

where $J = \frac{1}{2}$ and $\frac{3}{2}$, $N^*_J$($N^*_J'$) are mass eigenstates, and the two mixing angles can be constrained to be in the interval $[0, \pi)$. Here the notation $^{2S+1}N^*_J$ has been used.

The possible strong decays with emission of a single $\pi$ or $\eta$ meson (including those in the $\eta-$channel that turn out to be kinematically forbidden) are shown in Table I. Only S- and D-wave decays are considered [18]. The effective amplitudes for the emission of a pseudoscalar meson have the most general form:

$$
M[^{\ell_P,I_P}(\vec{k}_P)] = (-1)^{\ell_P} \sqrt{2M_{B^*}} Y^*_{\ell_P,m_{\ell_P}}(\vec{k}_P) \sum_{\mu,\alpha} \langle \ell_P,m_{\ell_P};I_P,I_{P3} | P[^{\ell_P,I_P}]_{[\mu,\alpha]} | 0 \rangle \times \nonumber
$$

$$
S\langle S;S_3;I_3 | B[^{\ell_P,I_P}]_{[-\mu,-\alpha]} | J^*,J^*_{3};I^*,I^*_{3};S^* \rangle_{\text{MS}},
$$

(8)

where $\ell_P$ and $I_P$ are the orbital angular momentum and isospin of the pseudoscalar meson, and $\mu$ and $\alpha$ the corresponding projections. $P[^{\ell_P,I_P}]_{[\mu,\alpha]}$ is a mesonic operator that creates the final state pseudoscalar and $B[^{\ell_P,I_P}]_{[-\mu,-\alpha]}$ is a baryonic operator that transforms the initial excited baryon into a ground state baryon. Here, the partial wave is projected onto a meson momentum eigenstate with momentum $\vec{k}_P$. The factor $\sqrt{2M_{B^*}}$, where $M_{B^*}$ is the mass of the excited baryon, has been added for convenience. Since as shown later, the dynamics of the decay can be encoded in effective coefficients, the mesonic operator matrix elements can be chosen to be:

$$
\langle \ell_P,m_{\ell_P};I_P,I_{P3} | P[^{\ell_P,I_P}]_{[\mu,\alpha]} | 0 \rangle = \sqrt{(2\ell_P + 1)(2I_P + 1)} \delta_{\mu,m_{\ell_P}} \delta_{\alpha,I_{P3}},
$$

(9)

The baryonic operator admits an expansion in $1/N_c$ and has the general form:

$$
B[^{\ell_P,I_P}]_{[\mu,\alpha]} = \left(\frac{\vec{k}_P}{\Lambda}\right)^{\ell_P} \sum_q C_q[^{\ell_P,I_P}] (\vec{k}_P) \left(B[^{\ell_P,I_P}]_{[\mu,\alpha]}ight)_q,
$$

(10)

where

$$
\left(B[^{\ell_P,I_P}]_{[\mu,\alpha]}ight)_q = \sum_m \langle 1,m;j_z,j_z | \ell_P,\mu \rangle \xi^4_m \left(G[^{j_z,I_P}]_{[j_z,\alpha]}ight)_q,
$$

(11)
and the factor \( \left( \frac{k_P}{\Lambda} \right)^F \) is included to take into account the chief meson momentum dependence of the partial wave. The scale \( \Lambda \) is chosen in what follows to be 200 MeV. Here, \( \xi_{im} \) is an operator that produces a transition from the triplet to the singlet \( O(3) \) state, and \( \left( G^{[j,IP]}_{jz,\alpha} \right)_q \) is a spin-flavor operator that produces the transition from the mixed-symmetric to the symmetric \( SU(4) \) representation. The label \( j \) denotes the spin of the spin-flavor operator, and as it is clear, its isospin coincides with the isospin of the emitted meson. As it was already mentioned, the dynamics of the decay is encoded in the effective dimensionless coefficients \( C^{[IP]}_{q}(k_P) \).

The terms in the right hand side of Eq.(10) are ordered in powers of \( 1/N_c \). As it has been explained in earlier publications [3], the order in \( 1/N_c \) is determined by the spin-flavor operator. For an \( n \)-body operator, this order is given by

\[
\nu = n - 1 - \kappa,
\]

where \( \kappa \) is equal to zero for incoherent operators and can be equal to one or even larger for coherent operators. More details can be found in the following section where a basis of operators \( G \) is explicitly built.

With the definition of effective operators used in this work, all coefficients \( C^{[IP]}_{q}(k_P) \) in Eq.(10) are of zeroth order in \( N_c \). The leading order of the decay amplitude is in fact \( N_c^0 \) [15]. At this point it is important to comment on the momentum dependence of the coefficients. The spin-flavor breakings in the masses, of both excited and ground state baryons, give rise to different values of the momenta \( k_P \). In the expansion where the spin-flavor breaking is treated as small, \( k_P = k_P^0 + \delta k_P \), where \( k_P^0 \) is the spin-flavor symmetry limit value, and \( \delta k_P \) is taken as small and expanded upon. Only the momentum dependence of the coefficients associated with leading order operators should be included if one takes \( \delta k_P \) to be of the same order as \( 1/N_c \) corrections. The goal would be to treat these corrections in a model independent fashion. In an operator analysis, they require the consideration of reducible effective operators that are the product of a mass operator that gives the spin-flavor breaking mass shifts (e.g. of the ground state baryons) times the leading order operator to which the coefficient is associated with. These corrections should be considered to be of the same order as the subleading in \( 1/N_c \) corrections given by irreducible operators that are analyzed below in full detail. In principle, this could be achieved with no difficulties in a world where \( N_c \) is large. The problem is that for \( N_c = 3 \) the number of independent
amplitudes for the decays here under consideration is essentially the same as the number of irreducible operators that appear in the analysis to order \(1/N_c\). There is, therefore, no room to separate the momentum dependence effects in the coefficients from the \(1/N_c\) effects due to irreducible operators. Thus, one can adopt the scheme followed in this paper where the only momentum dependence assigned to the coefficients is the explicitly shown factor \((k_P/\Lambda)^{\ell_P}\) that takes into account the chief momentum dependence of the corresponding partial wave, and the rest of the dependence is then encoded in the coefficients of the sub-leading operators. The other possibility is to model the momentum dependence, as it was done in [11], with a profile function motivated for instance by a quark model, in such a way that less of the momentum dependence is absorbed by subleading operators. Both ways of proceeding are equally valid.

Using the standard definition for the decay width and averaging over the initial- and summing over the corresponding final-baryon spins and isospins, the decay width for each \([\ell_P, I_P]\) channel is given by

\[
\Gamma_{[\ell_P, I_P]} = f_{ps} \left| \sum_q C_q^{[\ell_P, I_P]} B_q(\ell_P, I_P, S, I, J^*, I^*, S^*) \right|^2 / \sqrt{(2J^* + 1)(2I^* + 1)},
\]

where the phase space factor \(f_{ps}\) is

\[
f_{ps} = \frac{k_P^{1+2\ell_P} M_{B^*}}{8\pi^2 A^2 \Lambda^{2\ell_P} M_B}
\]

and \(B_q(\ell_P, I_P, S, I, J^*, I^*, S^*)\) are reduced matrix elements defined via the Wigner-Eckart theorem as follows:

\[
B_q(\ell_P, I_P, S, I, J^*, I^*, S^*) = (-1)^{J^*+J^*+S+I_P-I^*+I} \sqrt{(2J^* + 1)(2I^* + 1)}
\]

\[
\times \quad \langle \ell_P, m_P; J^*, J^*_3 \mid S, S_3 \rangle \langle I_P, I_P^3; I^*, I^*_3 \mid I = S, I_3 \rangle B_q(\ell_P, I_P, S, I, J^*, I^*, S^*). \tag{15}
\]

These reduced matrix elements can be easily calculated in terms of the reduced matrix elements of the spin-flavor operators, namely

\[
B_q(\ell_P, I_P, S, I, J^*, I^*, S^*) = (-1)^{J^*+J^*+S+I_P-I^*+I} \sqrt{(2J^* + 1)(2I^* + 1)} \left\{ \begin{array}{c} J^* \ S^* \ 1 \\ j \ \ell_P \ S \end{array} \right\}
\]

\[
\times \quad S < S; I \mid \langle G^{[j, \ell_P]} \rangle_q \mid S^*; I^* >_{\text{MS}}, \tag{16}
\]

where, due to the fact that the unknown dynamics can be included in the effective coefficients, the operators \(\xi_m^1\) can be chosen such that their matrix elements are simply given
by
\[ \langle 0 | \xi_{m'}^1 | 1 m \rangle = -\sqrt{3} \delta_{m m'}. \]  

(17)

Note that in the present case, where \( \ell_P \) can be 0 or 2 only, Eq.(16) implies that the spin-flavor operators can carry spin \( j \) that can be 1, 2 or 3.

### III. BASIS OF OPERATORS

The construction of a basis of spin-flavor operators follows similar lines as in previous works on baryon masses. The spin-flavor operators considered in the present paper must connect a mixed-symmetric with a symmetric representation. Generators of the spin-flavor group acting on the states obviously do not produce such connection. However, generators restricted to act on the excited quark or on the core of \( N_c - 1 \) quarks can do this. The spin flavor operators can, therefore, be represented by products of generators of the spin-flavor group restricted to act either on the excited or on the core states. In the following the generators acting on the core are denoted by \( S_c, G_c, T_c \), and the ones acting on the excited quark by \( s, g, t \). The generators \( G_c \) are known to be coherent operators, while all the rest are incoherent. In order to build a basis of operators for the present problem one has to consider products of such generators with the appropriate couplings of spins and isospins. The \( n \)-bodyness (\( nB \)) of an operator is given by the number of such factors, and the level of coherence of the operator is determined by how many factors \( G_c \) appear in the product. It should be noticed that in the physical case where \( N_c = 3 \), only operators of at most 3B have to be considered. Still, to order \( 1/N_c \) there is a rather long list of operators of given spin \( j \) and isospin \( I_P \). This list can be drastically shortened by applying several reduction rules. The first rule is that the product of two or more generators acting on the excited quark can always be reduced to the identity operator or to a linear combination of such generators. The second set of rules can be easily derived for products of operators whose matrix elements are taken between a mixed-symmetric and a symmetric representation. These reduction rules are as follows for 1B up to 3B operators (here \( \lambda \) represents generators acting on the excited quark and \( \Lambda_c \) represent generators acting on the core):

\[ \lambda = -\Lambda_c \]
\[ (\Lambda_c)_1(\Lambda_c)_2 = -\lambda_1(\Lambda_c)_2 - \lambda_2(\Lambda_c)_1 + 1B \text{ operators.} \]

(18)
Therefore, only the following types of operators should be considered

\[
\begin{align*}
1B & \quad \lambda \\
2B & \quad \frac{1}{N_c} \lambda_1 (\Lambda_c)_2 \\
3B & \quad \frac{1}{N_c^2} \lambda_1 (\Lambda_c)_2 (\Lambda_c)_3
\end{align*}
\]  

(19)

It is convenient to make explicit the transformation properties of each basic operator under spin \( j \) and isospin \( t \). In what follows the notation \( O^{[j,t]} \) is used to indicate that the operator \( O \) has spin \( j \) and isospin \( t \). It is easy to see that

\[
\begin{align*}
\lambda^{[j,t]} & = s^{[1,0]}, \quad t^{[0,1]}, \quad g^{[1,1]} \\
(\Lambda_c)^{[j,t]} & = (S_c)^{[1,0]}, \quad (T_c)^{[0,1]}, \quad (G_c)^{[1,1]}
\end{align*}
\]  

(20)

For decays in the \( \eta^- \)-channels the spin-flavor operators transform as \([j,0]\) while for decays in the pion channels they should transform as \([j,1]\), where in both cases \( j = 1, 2, 3 \). Knowing the transformation properties of each basic operator given in Eq.(20) it is easy to construct products of the forms given in Eqs.(19) with the desired spin and isospin. An example of a 2B operator that transforms as \([2,1]\) is

\[
(gG_c)^{[2,1]}_{[\mu,\alpha]} = < 1, m; 1, m'|2, \mu > < 1, a; 1, a'|1, \alpha > g_{ma} (G_c)_{m'a'},
\]  

(21)

where the spin and isospin projections (e.g. \( m, m', a, \) etc) are considered to be spherical.

Thus, the 1B operators that contribute to a given \([j,t]\) are just those \( \lambda^{[j,t]} \) given in Eq.(20) which have the proper spin and isospin quantum numbers. Similarly, the possible 2B operators are given by the products \( \lambda_1^{[j_1,t_1]} (\Lambda_c)^{[j_2,t_2]}_2 \) coupled to the required \([j,t]\) by means of the conventional Clebsch-Gordan (CG) coefficients.

In order to construct all possible 3B operators, it is convenient to note that there are additional reduction rules for the products \((\Lambda_c)_2(\Lambda_c)_3\). These rules apply to matrix elements of products of generators between states in the symmetric representation [2]. The starting point is to consider all possible products of two core operators. Since one is interested in keeping contributions of at most order \( 1/N_c \), and because these products will appear only in 3B operators, at least one of the operators must be a \( G_c \). Thus, the list of 2B core operators to be considered is:

\[
(T_c G_c)^{[1,j]}, \quad (G_c T_c)^{[1,j]}, \quad (S_c G_c)^{[j,1]}, \quad (G_c S_c)^{[j,1]}, \quad (G_c G_c)^{[j,t]},
\]  

(22)
where \( j, t = 0, 1, 2 \). Using transformation properties of the CG coefficients and dropping all the combinations leading to commutators (because the basic operators are generators of the algebra, the commutators are proportional to another basic operator), only the following products remain

\[
\{\{T_c, G_c\}\}^{[1,0]}, \quad \{\{T_c, G_c\}\}^{[1,1]}, \quad \{\{T_c, G_c\}\}^{[1,2]}, \quad \{\{S_c, G_c\}\}^{[0,1]}, \quad \{\{S_c, G_c\}\}^{[1,1]}, \quad \{\{S_c, G_c\}\}^{[2,1]}, \quad \{G_c G_c\}^{[0,0]}, \quad \{G_c G_c\}^{[0,2]}, \quad \{G_c G_c\}^{[1,1]}, \quad \{G_c G_c\}^{[2,0]}, \quad \{G_c G_c\}^{[2,2]},
\]

where \( \{\{T_c, G_c\}\}^{[1,1]} \) denotes \( T_c G_c^{[1,1]} - (G_c T_c)^{[1,1]} = \langle 1a1b|1d\rangle (T_c^a G_c^{mb} - G_c^{ma} T_c^b) = \langle 1a1b|1d\rangle \{G_c^{ma}, T_c^b\} \), etc. Using now the reduction relations [2], it can be shown that several of these eleven operators can be eliminated. The final list of independent products of two core operators turns out to be

\[
\{\{T_c, G_c\}\}^{[2,2]}, \quad \{\{S_c, G_c\}\}^{[1,1]}, \quad \{\{S_c, G_c\}\}^{[2,1]}, \quad \{G_c G_c\}^{[2,2]}
\]

By coupling any of these operators with one of the excited core operators \( \lambda_1^{[j_1,t_1]} \) (see Eq.(20)) to the required \([j, t]\) all the possible 3B operators are obtained.

Using this scheme to couple products of generators it is straightforward to construct lists of operators with spin 1, 2 and 3 and isospin 0 and 1. Further reductions result from the fact that not all the resulting operators are linearly independent up to order \( 1/N_c \). The determination of the final set of independent operators for each particular decay channel is more laborious. This is achieved by coupling the resulting spin-flavor operators with the \( \xi^{[1,0]} \) orbital transition operator to the corresponding total spin and isospin and by explicitly calculating all the relevant matrix elements. In this way operators that are linearly dependent at the corresponding order in the \( 1/N_c \) expansion can be eliminated. The resulting basis of independent operators \( \{O_{[m_p,I_p]}^{[l_p,I_p]}\}_q \) is shown in Table II, where for simplicity the corresponding spin and isospin projections have been omitted. Their reduced matrix elements are given in Tables III through VI. In the bottom rows of these tables normalization coefficients \( \alpha_q^{[l_p,I_p]} \) are displayed. These coefficients are used to define normalized basis operators such that, for \( N_c = 3 \), their largest reduced matrix element is equal to one for order \( N_c^0 \) operators and equal to 1/3 for order 1/\( N_c \) operators. Thus,

\[
\left( B_{[m_p,I_p]}^{[l_p,I_p]} \right)_q = \alpha_q^{[l_p,I_p]} \left( O_{[m_p,I_p]}^{[l_p,I_p]} \right)_q
\]

furnishes the list of basis operators normalized according to the \( 1/N_c \) power counting.
IV. RESULTS

The different S- and D-wave partial widths used in the analysis are the ones provided by the Particle Data Group [17]. The values for the widths and branching ratios are taken as the ones indicated there as “our estimate”, while the errors are determined from the corresponding ranges. The total widths and branching fractions are given in Table I, while in Table VIII the partial widths calculated from those values are explicitly displayed. The entries in Table I indicated as unknown reflect channels for which no width is provided by the Particle Data Group or where the authors consider that the input is unreliable, such as in the $\pi\Delta$ decay modes of the $N(1700)$ and $N(1675)$ and the D-wave $\eta-$ decay modes. At this point it is important to stress the marginal precision of the data for the purposes of this work. This work performs an analysis at order $1/N_c$, which means that the theoretical error is order $1/N_c^2$. This implies that amplitudes are affected by a theoretical uncertainty at the level of 10%. Thus, in order to pin down the coefficients of the subleading operators, the widths provided by the data should not be affected by errors larger than about 20%. As shown in Tables I and VIII, the experimental errors are in most entries 30% or larger. In consequence, the determination of the subleading effective coefficients is affected by large errors as the results below show.

Before presenting the results of the fits, it is convenient to derive some parameter independent relations that can be obtained to leading order. These relations serve as a test of the leading order approximation. Since at this order there are only four coefficients and two angles to be fitted, and there are a total of twenty partial widths (excluding all D-wave $\eta$ channels but including kinematically forbidden $\eta$-channel decays), there are fourteen independent parameter free relations that can be derived. These relations are more conveniently written in terms of reduced widths, i.e., widths where the phase space factor $f_{sp}$ (see Eq.(14)) has been removed and denoted here by $\tilde{\Gamma}$. Considering the S-wave decays in the $\pi$ mode, there are six decays and three parameters in the fit. Thus, three parameter free relations must follow. These relations and the corresponding comparison with experimental values
\[ \tilde{\Gamma}_{N(1535)\to \pi N} + \tilde{\Gamma}_{N(1650)\to \pi N} = \tilde{\Gamma}_{N(1520)\to \pi N} + \tilde{\Gamma}_{N(1700)\to \pi N} = \tilde{\Gamma}_{N(1675)\to \pi N} \quad \text{Th.} \]

\[ \begin{array}{cccc}
1 & : & 1 & : 0.17 : 0.42
\end{array} \quad \text{Exp.} \]

Within the experimental errors the relations are satisfied. They can be used to give a leading order prediction for the unknown S-wave width: \( \Gamma_{N(1700)\to \pi \Delta} = 160 \pm 40 \text{ MeV} \). The following expression for the \( \theta_1 \) mixing angle also follows:

\[ \frac{\tilde{\Gamma}_{N(1535)\to \pi N} - \tilde{\Gamma}_{N(1650)\to \pi N}}{\tilde{\Gamma}_{N(1535)\to \pi N} + \tilde{\Gamma}_{N(1650)\to \pi N}} = \frac{1}{3} \left[ \cos(2\theta_1) - \sqrt{8} \sin(2\theta_1) \right] . \quad \text{(29)} \]

Using the empirical values the angle that results from this relation is: \( \theta_1 = 1.62 \pm 0.12 \) or \( 0.29 \pm 0.11 \). One of these angles, namely the latter, turns out to be close to the angle obtained in the next to leading order fit.

In a similar fashion, relations involving D-wave decays in the \( \pi \) mode can be obtained. There are in this case four parameters to fit and eleven partial widths. Thus, seven parameter free relations can be obtained. In general these relations are quadratic and/or involve some of the unknown decay widths. However, the following testable three linear relations can be obtained

\[ 2 \tilde{\Gamma}_{\Delta(1620)\to \pi \Delta} + \tilde{\Gamma}_{\Delta(1700)\to \pi \Delta} = 8 \tilde{\Gamma}_{\Delta(1700)\to \pi N} + 15 \tilde{\Gamma}_{N(1675)\to \pi N} \quad \text{Exp.} \]

\[ 5.24 \pm 1.95 \quad = \quad 2.19 \pm 0.62 \quad \text{(30)} \]

\[ \frac{2}{9} \left( \tilde{\Gamma}_{\Delta(1535)\to \pi \Delta} + \tilde{\Gamma}_{N(1650)\to \pi \Delta} \right) + \frac{20}{3} \tilde{\Gamma}_{\Delta(1620)\to \pi \Delta} = 16 \tilde{\Gamma}_{\Delta(1700)\to \pi N} + 15 \tilde{\Gamma}_{N(1675)\to \pi N} \quad \text{Exp.} \]

\[ 16.87 \pm 6.46 \quad = \quad 6.49 \pm 1.65 \quad \text{(31)} \]

\[ \frac{1}{36} \left( \tilde{\Gamma}_{\Delta(1520)\to \pi N} + \tilde{\Gamma}_{N(1700)\to \pi N} \right) + \frac{5}{12} \tilde{\Gamma}_{\Delta(1620)\to \pi \Delta} = \tilde{\Gamma}_{\Delta(1700)\to \pi N} + \tilde{\Gamma}_{N(1675)\to \pi N} \quad \text{Exp.} \]

\[ 1.07 \pm 0.40 \quad = \quad 0.42 \pm 0.11 \quad \text{(32)} \]

These relations are not well satisfied by the empirical data. Thus, one can anticipate the need for next to leading corrections in order to have a correct description of some D-wave decays. In the case of S-wave \( \eta^- \) mode decays, there are two parameters to fit (since there is no dependence on the mixing angle \( \theta_3 \)), and three possible decays. However, one of the
decays is kinematically forbidden, namely the $\Delta(1700) \to \eta \Delta$. It is also possible to express
the mixing angle $\theta_1$ in terms of the reduced S-wave widths:

$$\frac{\tilde{\Gamma}_{N(1535) \to \eta N} - \tilde{\Gamma}_{N(1650) \to \eta N}}{\tilde{\Gamma}_{N(1535) \to \eta N} + \tilde{\Gamma}_{N(1650) \to \eta N}} = -\frac{1}{3} \left[ \cos(2\theta_1) - \sqrt{3} \sin(2\theta_1) \right].$$

(33)

Using the empirical values the angle that results from this relation is: $\theta_1 = 1.26 \pm 0.14$
or $0.65 \pm 0.14$. Although these results seem not to be far from the results derived from
Eq.(29), the leading order fit discussed below indicates a poor description of the ratio of the
$N(1650) \to \eta N$ to $N(1535) \to \eta N$ widths. This is due to the high sensitivity of this ratio to
the mixing angle $\theta_1$. Indeed, it is necessary to include some $1/N_c$ effects as discussed below
to arrive at a good description of the $\eta$-modes together with the other modes, and in this
case the resulting angle is $0.39 \pm 0.11$.

In Table VII the results of several fits are displayed. In these fits the decay amplitudes are
expanded keeping only the terms that correspond to the order in $1/N_c$ of the fit. Similarly,
when performing the fits the errors have been taken to be equal or larger than the expected
accuracy of the fit (30% to the LO fits and about 10% for the NLO ones).

The first LO fit only considers the S-wave $\pi$-modes. As expected, the values for the
mixing angle $\theta_1$ resulting from this fit turn out to be equal to the ones obtained through
the relation given by Eq.(29). Notice that $\theta_3$ also has a two fold ambiguity at this order.
The second leading order fit includes the D-waves and $\eta$-modes. The angles remain within
errors equal to the ones from the first fit. Table VIII shows that the $N(1535) \to \eta N$ width
results to be a factor four smaller than the empirical one (this having, however, a rather
generous error). In the D-waves, several widths involving decays with a $\Delta$ in the final state
are also too small. The S-wave $\pi$-modes are well fitted and there is no real need for NLO
improvement. In the D-wave decays in the $\pi$ channel there are two leading order operators
that contribute. The fit shows that the 1B operator has a coefficient whose magnitude is a
factor two to three larger than that of the coefficient of the 2B operator. The 1B D-wave
operator as well as the 1B S-wave operator $O^{[0,1]}_1$ stem from the 1B coupling of the pion via
the axial current. Such a coupling naturally occurs as the dominant coupling in the chiral
quark model [19]. Thus, the result of the D-wave fit might suggest that such a mechanism
is dominant over other mechanisms that give rise to the 2B operator $O^{[2,1]}_6$.

The NLO fit involves a rather large number of effective constants. In addition to the four
effective constants and the two mixing angles that appear at LO, there are ten new effective
constants, three of them in the S-wave pion channels, six in the D-wave pion channels, and one in the S-wave eta channels. Since there are only sixteen data available, some operators must be discarded for the purpose of the fit. It is reasonable to choose to neglect the 3B operators and the subleading S-wave operator for \( \eta \) emission. The NLO fit has been carried out by demanding that the LO coefficients are not vastly different from their values obtained in the LO fit. This demand is reasonable if the assumption is made that the \( 1/N_c \) expansion makes sense. The fact that the NLO coefficients do not have unnaturally large values with respect to the scale set by the LO fit indicates the consistency of the assumption. This clearly is no proof, however, that the \( 1/N_c \) expansion is working. As mentioned earlier, the chief limitation here is due to the magnitude of the errors in the inputs. This leads to results for the NLO coefficients being affected with rather large uncertainties. Indeed, no clear NLO effects can be pinned down, as most NLO coefficients are no more than one standard deviation from zero. Because the number of coefficients is approximately equal to the number of inputs, there are important correlations between them. For instance, the S-wave NLO coefficients are very correlated with each other and with the angle \( \theta_3 \). For the S-waves the LO fit is already excellent, and therefore nothing significantly new is obtained by including the NLO corrections. Correlations are smaller for the D-wave coefficients. As mentioned before, here the LO fit has room for improvement, and thus the NLO results are more significant than in the case of the S-waves. Still, no clear pattern concerning the NLO corrections is observed. One interesting point, however, is that without any significant change in the value of \( \theta_1 \) the \( \eta \)-modes are now well described. The reason for this is that in the LO fit the matrix elements of the operator \( O_{1}^{[0,0]} \) were taken to zeroth order in \( 1/N_c \), while in the NLO fit the \( 1/N_c \) terms are included. These subleading corrections enhance the amplitude for the \( 2N_{1/2}^* \) and suppress the amplitude for the \( 4N_{1/2}^* \). This along with an increment in the coefficient brings the fit in line with the empirical widths. One important point is that at NLO the two fold ambiguity in \( \theta_1 \) that results at LO is eliminated. The smaller mixing angle turns out to be selected. The angle \( \theta_3 \) remains ambiguous and close to the values obtained in the LO fits. It should be noticed that the present values of both mixing angles are somewhat different from the values \( \theta_1 = 0.61, \theta_3 = 3.04 \) obtained in other analyses [10, 16, 20]. Finally, a clear manifestation of the lack of precision in the data is the ratio between the errors affecting the coefficients of NLO versus those of LO operators. In a situation where the data would have precision of the order of NNLO corrections, that ratio
should be \( \simeq N_c \). In most cases the ratios turn out to be much larger than that, as Table VII shows.

V. CONCLUSIONS

In this work the method of analysis of excited baryon decays in the \( 1/N_c \) expansion has been presented. It is limited to the situation where configuration mixings are neglected, and therefore the baryon states are taken to belong to a single multiplet of \( O(3) \times SU(4) \). The application to the decays of the negative parity baryons illustrates the method. A basis of effective operators, in which the S- and D-wave amplitudes are expanded, was constructed to order \( 1/N_c \). All dynamical effects are then encoded in the effective coefficients that enter in that expansion.

The application to the decays addressed here shows that a consistent description within the \( 1/N_c \) expansion is possible. Indeed, up to the relatively poor determination of \( 1/N_c \) corrections that results from the magnitude of the errors in the input widths, these corrections are of natural size. A few clear cut observations can be made. The most important one is that the S-wave \( \pi \)- and \( \eta \)-channels are well described by the leading order operators (one for each channel) provided one includes the contributions subleading in \( 1/N_c \) in the matrix elements for the \( \eta \)-decays. The mixing angle \( \theta_1 \) is then determined by these channels up to a twofold ambiguity, which is lifted when all channels are analyzed at NLO. The angle \( \theta_3 \) is also determined up to a two fold ambiguity at LO. The ambiguity remains when the NLO is considered. The LO results also indicate the dominance of the 1B effective operators that have the structure that would result from a chiral quark model [19]. This is explicitly seen in the D-wave channels where the LO 2B operator turns out to be suppressed with respect to the 1B one. The subleading operators are shown to be relevant to fine tune the S-wave decays and improve the D-wave decays. Because of the rather large error bars in and significant correlations between the resulting effective coefficients, no clear conclusions about the physics driving the \( 1/N_c \) corrections can be made. The mixing angles \( \theta_1 = 0.39 \pm 0.11 \) and \( \theta_3 = (2.82, 2.38) \pm 0.11 \) that result at NLO are similar to the ones determined at LO. They are, however, somewhat different from the angles \( \theta_1 = 0.61 \) and \( \theta_3 = 3.04 \) obtained in other analyses [10, 16, 20].
ACKNOWLEDGEMENTS

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   J. L. Goity, C. L. Schat and N. N. Scoccola, Phys.Rev. D 66, 114014 (2002);


[18] There is, in addition, the pion G-wave decay \( N_{5/2}^* \rightarrow \pi \Delta_{3/2} \) which is not considered here.


TABLE I: Negative parity non-strange baryons and their decay widths and branching ratios from the PDG. Channels not explicitly indicated are forbidden.

<table>
<thead>
<tr>
<th>State</th>
<th>Notation</th>
<th>Mass  [MeV]</th>
<th>Total width [MeV]</th>
<th>Branching ratios [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
<td>S-wave</td>
</tr>
<tr>
<td>$N(1535)$</td>
<td>$N_{1/2}^*$</td>
<td>1538 ± 18</td>
<td>150 ± 50</td>
<td>$\pi N : 45 \pm 10$</td>
</tr>
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<td></td>
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</tr>
<tr>
<td>$N(1520)$</td>
<td>$N_{3/2}^*$</td>
<td>1523 ± 8</td>
<td>122 ± 13</td>
<td>$\pi \Delta : 8.5 \pm 3.5$</td>
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</tr>
<tr>
<td>$N(1650)$</td>
<td>$N_{1/2}^{*'}$</td>
<td>1660 ± 20</td>
<td>167 ± 23</td>
<td>$\pi N : 72.5 \pm 17.5$</td>
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</tr>
<tr>
<td>$N(1700)$</td>
<td>$N_{3/2}^{*'}$</td>
<td>1700 ± 50</td>
<td>100 ± 50</td>
<td>$\pi \Delta : \text{unknown}$</td>
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</tr>
<tr>
<td>$N(1675)$</td>
<td>$N_{5/2}^*$</td>
<td>1678 ± 8</td>
<td>160 ± 20</td>
<td>$\pi N : 45 \pm 5$</td>
</tr>
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<td></td>
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</tr>
<tr>
<td>$\Delta(1620)$</td>
<td>$\Delta_{1/2}^*$</td>
<td>1645 ± 30</td>
<td>150 ± 30</td>
<td>$\pi N : 25 \pm 5$</td>
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</tr>
<tr>
<td>$\Delta(1700)$</td>
<td>$\Delta_{3/2}^*$</td>
<td>1720 ± 50</td>
<td>300 ± 100</td>
<td>$\pi \Delta : 37.5 \pm 12.5$</td>
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<td></td>
<td></td>
<td></td>
<td>$\eta \Delta : \text{kinematically forbidden}$</td>
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</tr>
<tr>
<td>$n$-bodyness</td>
<td>Name</td>
<td>Operator</td>
<td>Order in $1/N_c$</td>
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<tr>
<td>1B</td>
<td>$O_{1}^{[0,1]}$</td>
<td>$(\xi g)^{[0,1]}$</td>
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<tr>
<td>Pion</td>
<td>$O_{2}^{[0,1]}$</td>
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<td>1</td>
<td></td>
</tr>
<tr>
<td>S wave</td>
<td>2B</td>
<td>$O_{3}^{[0,1]}$</td>
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<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{1}{N_c} \left( \xi (t S_c)^{[1,1]} \right)^{[0,1]}_{[0,a]}$</td>
<td>1</td>
<td></td>
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<tr>
<td></td>
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<td>$O_{4}^{[0,1]}$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{1}{N_c} \left( \xi (g S_c)^{[1,1]} \right)^{[0,1]}_{[0,a]}$</td>
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</tr>
<tr>
<td>1B</td>
<td>$O_{1}^{[2,1]}$</td>
<td>$(\xi g)^{[2,1]}_{[i,a]}$</td>
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<tr>
<td>Pion</td>
<td>$O_{2}^{[2,1]}$</td>
<td>$\frac{1}{N_c} \left( \xi (s T_c)^{[1,1]} \right)^{[2,1]}_{[i,a]}$</td>
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<tr>
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<tr>
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<tr>
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<td>$O_{7}^{[2,1]}$</td>
<td>$\frac{1}{N_c} \left( \xi \left( {S_c, G_c}^{[2,1]} \right)^{[2,1]}<em>{[i,a]} \right)^{[2,1]}</em>{[i,a]}$</td>
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<td></td>
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<td>$O_2^{[0,1]}$</td>
<td>$O_3^{[0,1]}$</td>
<td>$O_4^{[0,1]}$</td>
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<td>$-\frac{1}{2N_c}$</td>
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<td>$-\frac{1}{6}$</td>
<td>$\frac{2}{3N_c}$</td>
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<td>$O_3^{[2,1]}$</td>
<td>$O_4^{[2,1]}$</td>
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<td>$\frac{1}{3\sqrt{5}N_c}$</td>
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<td>$\frac{1}{3\sqrt{5}N_c}$</td>
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<td>$^4N^*_3/2 \to N$</td>
<td>$-\frac{1}{6\sqrt{5}}$</td>
<td>$\frac{1}{6\sqrt{5}N_c}$</td>
<td>$-\frac{1}{3\sqrt{5}N_c}$</td>
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<tr>
<td>$^4N^*_3/2 \to \Delta$</td>
<td>$-\frac{1}{3\sqrt{5}}$</td>
<td>$\frac{1}{3\sqrt{5}N_c}$</td>
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<tr>
<td>$^5N^*/2 \to N$</td>
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<td>$\frac{1}{4\sqrt{5}N_c}$</td>
<td>$\frac{2}{8\sqrt{5}N_c}$</td>
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</tr>
<tr>
<td>$^5N^*/2 \to \Delta$</td>
<td>$\frac{1}{2\sqrt{5}}$</td>
<td>$-\frac{1}{2\sqrt{5}N_c}$</td>
<td>$\frac{1}{\sqrt{5}N_c}$</td>
<td>0</td>
</tr>
<tr>
<td>$^5\Delta^*_1/2 \to \Delta$</td>
<td>$\frac{1}{6\sqrt{5}}$</td>
<td>$\frac{1}{3\sqrt{5}N_c}$</td>
<td>$-\frac{1}{6\sqrt{5}N_c}$</td>
<td>$\frac{1}{4\sqrt{15}N_c}$</td>
</tr>
<tr>
<td>$^5\Delta^*_3/2 \to N$</td>
<td>$-\frac{1}{6\sqrt{5}}$</td>
<td>$\frac{1}{6\sqrt{5}N_c}$</td>
<td>$\frac{1}{3\sqrt{5}N_c}$</td>
<td>0</td>
</tr>
<tr>
<td>$^5\Delta^*_3/2 \to \Delta$</td>
<td>$\frac{1}{6\sqrt{5}}$</td>
<td>$\frac{1}{3\sqrt{5}N_c}$</td>
<td>$-\frac{1}{6\sqrt{5}N_c}$</td>
<td>$\frac{1}{4\sqrt{15}N_c}$</td>
</tr>
<tr>
<td>$\alpha^{[2,1]}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{3\sqrt{5}}$</td>
<td>$\frac{1}{2\sqrt{5}}$</td>
<td>$\frac{2}{4\sqrt{5}}$</td>
</tr>
</tbody>
</table>

**Table IV: Reduced matrix elements of pion D wave operators**
TABLE V: Reduced matrix elements of eta S wave operators

<table>
<thead>
<tr>
<th>Eta S waves</th>
<th>$O_1^{[0,0]}$</th>
<th>$O_2^{[0,0]}$</th>
<th>Overall factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^2N_{1/2}^* \rightarrow N$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{\sqrt{(N_c+3)(N_c-1)}}{\sqrt{3}N_c}$</td>
</tr>
<tr>
<td>$^4N_{1/2}^* \rightarrow N$</td>
<td>1</td>
<td>$-\frac{3}{2\sqrt{2}N_c}$</td>
<td>$-\frac{\sqrt{2}N_c-1}{3N_c}$</td>
</tr>
<tr>
<td>$\Delta_{1/2}^* \rightarrow \Delta$</td>
<td>1</td>
<td>$\frac{3}{2\sqrt{2}N_c}$</td>
<td>$\frac{\sqrt{2}N_c+5}{3N_c}$</td>
</tr>
</tbody>
</table>

| $\alpha^{[0,0]}$ | $\frac{3}{4}$ | $\frac{1}{\sqrt{2}}$ | |

TABLE VI: Reduced matrix elements of eta D wave operators

<table>
<thead>
<tr>
<th>Eta D waves</th>
<th>$O_1^{[2,0]}$</th>
<th>$O_2^{[2,0]}$</th>
<th>$O_3^{[2,0]}$</th>
<th>Overall factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^2N_{3/2}^* \rightarrow N$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$5\frac{\sqrt{(N_c+3)(N_c-1)}}{\sqrt{15}N_c}$</td>
</tr>
<tr>
<td>$^4N_{3/2}^* \rightarrow N$</td>
<td>1</td>
<td>$-\frac{3}{2\sqrt{2}N_c}$</td>
<td>$\frac{3\sqrt{3}}{2\sqrt{2}N_c}$</td>
<td>$-\frac{\sqrt{N_c-1}}{3N_c}$</td>
</tr>
<tr>
<td>$N_{5/2}^* \rightarrow N$</td>
<td>1</td>
<td>$-\frac{3}{2\sqrt{2}N_c}$</td>
<td>$-\frac{1}{2\sqrt{2}N_c}$</td>
<td>$-\sqrt{3}\frac{N_c-1}{N_c}$</td>
</tr>
<tr>
<td>$\Delta_{1/2}^* \rightarrow \Delta$</td>
<td>1</td>
<td>$\frac{3}{2\sqrt{2}N_c}$</td>
<td>$\frac{3\sqrt{3}}{2\sqrt{2}N_c}$</td>
<td>$5\frac{N_c+5}{15N_c}$</td>
</tr>
<tr>
<td>$\Delta_{3/2}^* \rightarrow \Delta$</td>
<td>1</td>
<td>$\frac{3}{2\sqrt{2}N_c}$</td>
<td>$-\frac{\sqrt{3}}{2\sqrt{2}N_c}$</td>
<td>$5\frac{N_c+5}{15N_c}$</td>
</tr>
</tbody>
</table>

| $\alpha^{[2,0]}$ | $\frac{3}{2\sqrt{10}}$ | $\frac{1}{\sqrt{5}}$ | $-\frac{3}{5}$ |
TABLE VII: Fit parameters. Fit #1: Pion S-waves LO. In this case there is four-fold ambiguity for the angles \( \theta_1, \theta_3 \) given by the two values shown for each angle. Fit #2: Pion S and D-waves, eta S-waves, LO. In this case there is two-fold ambiguity for the angle \( \theta_1 \). For the angle \( \theta_3 \) there is an almost two-fold ambiguity given by the two values indicated in parenthesis and which only differ in the two slightly different values of \( C^{[2,1]}_6 \). Fit #3: Pion S and D-waves, eta S-waves, NLO, no 3-body operators. No degeneracy in \( \theta_1 \) but almost two-fold ambiguity in \( \theta_3 \) given by the two values indicated in parenthesis. Values of coefficients which differ in the corresponding fits are indicated in parenthesis.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>#1 LO</th>
<th>#2 LO</th>
<th>#3 NLO</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C^{[0,1]}_1 )</td>
<td>31 ± 3</td>
<td>31 ± 3</td>
<td>23 ± 3</td>
</tr>
<tr>
<td>( C^{[0,1]}_2 )</td>
<td>-</td>
<td>-</td>
<td>( (7.4, 32.5) ) ± ( (27, 41) )</td>
</tr>
<tr>
<td>( C^{[0,1]}_3 )</td>
<td>-</td>
<td>-</td>
<td>( (20.7, 26.8) ) ± ( (12, 14) )</td>
</tr>
<tr>
<td>( C^{[0,1]}_4 )</td>
<td>-</td>
<td>-</td>
<td>( (26.3, 66.8) ) ± ( (39, 65) )</td>
</tr>
<tr>
<td>( C^{[2,1]}_1 )</td>
<td>-</td>
<td>4.6 ± 0.5</td>
<td>3.4 ± 0.3</td>
</tr>
<tr>
<td>( C^{[2,1]}_2 )</td>
<td>-</td>
<td>-</td>
<td>( -4.5 ) ± ( 2.4 )</td>
</tr>
<tr>
<td>( C^{[2,1]}_3 )</td>
<td>-</td>
<td>-</td>
<td>( (-0.01, 0.08) ) ± ( 2 )</td>
</tr>
<tr>
<td>( C^{[2,1]}_4 )</td>
<td>-</td>
<td>-</td>
<td>( 5.7 ) ± ( 4.0 )</td>
</tr>
<tr>
<td>( C^{[2,1]}_5 )</td>
<td>-</td>
<td>-</td>
<td>( 3.0 ) ± ( 2.2 )</td>
</tr>
<tr>
<td>( C^{[2,1]}_6 )</td>
<td>- ( (-1.86, -2.25) ) ± ( 0.4 )</td>
<td>( -1.73 ) ± ( 0.26 )</td>
<td></td>
</tr>
<tr>
<td>( C^{[2,1]}_7 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( C^{[2,1]}_8 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( C^{[0,0]}_1 )</td>
<td>-</td>
<td>11 ± 4</td>
<td>17 ± 4</td>
</tr>
<tr>
<td>( C^{[0,0]}_2 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>1.62 ± 0.12</td>
<td>1.56 ± 0.15</td>
<td>0.39 ± 0.11</td>
</tr>
<tr>
<td>( 0.29 ) ± 0.11</td>
<td>0.35 ± 0.14</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>( \theta_3 )</td>
<td>3.01 ± 0.07</td>
<td>( (3.00, 2.44) ) ± ( 0.07 )</td>
<td>( (2.82, 2.38) ) ± ( 0.11 )</td>
</tr>
<tr>
<td>( 2.44 ) ± ( 0.06 )</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>( \chi^2_{dof} )</td>
<td>0.25</td>
<td>1.5</td>
<td>0.9</td>
</tr>
<tr>
<td>( dof )</td>
<td>2</td>
<td>10</td>
<td>3</td>
</tr>
</tbody>
</table>
TABLE VIII: Partial widths resulting from the different fits in Table I. Values indicated in parenthesis correspond to the cases in which *almost* degenerate values of $\theta_3$ lead to different partial widths.

<table>
<thead>
<tr>
<th>Decay</th>
<th>Emp. Width</th>
<th>#1 LO</th>
<th>#2 LO</th>
<th>#3 NLO</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[MeV]</td>
<td>[MeV]</td>
<td>[MeV]</td>
<td>[MeV]</td>
</tr>
<tr>
<td>$N(1535) \rightarrow \pi N$</td>
<td>68 ± 27</td>
<td>74</td>
<td>62</td>
<td>(58,68)</td>
</tr>
<tr>
<td>$N(1520) \rightarrow \pi \Delta$</td>
<td>10 ± 4</td>
<td>10</td>
<td>9.7</td>
<td>9.5</td>
</tr>
<tr>
<td>$\pi S$ $N(1650) \rightarrow \pi N$</td>
<td>121 ± 40</td>
<td>132</td>
<td>144</td>
<td>122</td>
</tr>
<tr>
<td>wave $N(1700) \rightarrow \pi \Delta$</td>
<td>unknown</td>
<td>175</td>
<td>175</td>
<td>(259,156)</td>
</tr>
<tr>
<td>$\Delta(1620) \rightarrow \pi N$</td>
<td>38 ± 13</td>
<td>35</td>
<td>35</td>
<td>38</td>
</tr>
<tr>
<td>$\Delta(1700) \rightarrow \pi \Delta$</td>
<td>112 ± 53</td>
<td>81</td>
<td>81</td>
<td>(135,112)</td>
</tr>
<tr>
<td>$N(1535) \rightarrow \pi \Delta$</td>
<td>1 ± 1</td>
<td>0.1</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>$N(1520) \rightarrow \pi N$</td>
<td>67 ± 9</td>
<td>70</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>$N(1520) \rightarrow \pi \Delta$</td>
<td>15 ± 3</td>
<td>2.8</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>$N(1650) \rightarrow \pi \Delta$</td>
<td>7 ± 5</td>
<td>0.12</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>$\pi D$ $N(1700) \rightarrow \pi N$</td>
<td>10 ± 7</td>
<td>10</td>
<td>(11,9)</td>
<td></td>
</tr>
<tr>
<td>wave $N(1700) \rightarrow \pi \Delta$</td>
<td>unknown</td>
<td>4</td>
<td>(4,9)</td>
<td></td>
</tr>
<tr>
<td>$N(1675) \rightarrow \pi N$</td>
<td>72 ± 12</td>
<td>85</td>
<td>76</td>
<td></td>
</tr>
<tr>
<td>$N(1675) \rightarrow \pi \Delta$</td>
<td>unknown</td>
<td>45</td>
<td>79</td>
<td></td>
</tr>
<tr>
<td>$\Delta(1620) \rightarrow \pi \Delta$</td>
<td>68 ± 26</td>
<td>30</td>
<td>87</td>
<td></td>
</tr>
<tr>
<td>$\Delta(1700) \rightarrow \pi N$</td>
<td>45 ± 21</td>
<td>49</td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>$\Delta(1700) \rightarrow \pi \Delta$</td>
<td>12 ± 10</td>
<td>15</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>$\eta S$ $N(1535) \rightarrow \eta N$</td>
<td>64 ± 28</td>
<td>17</td>
<td>(57,61)</td>
<td></td>
</tr>
<tr>
<td>wave $N(1650) \rightarrow \eta N$</td>
<td>11 ± 6</td>
<td>14</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>