Relativistic aspects of few-body physics

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After a brief review of alternative approaches to the relativistic treatment of few-body problems, I discuss (i) recent progress in the study of NN and NNN systems, (ii) a new method for solving the Bethe-Salpeter equation in Minkowski space, (iii) new relativistic treatment of spin 3/2 fields, and (iv) construction of conserved electromagnetic currents in a relativistic optical model.

1. OVERVIEW

1.1. Why use relativistic dynamics?

Relativistic calculations, technically and conceptually difficult, are clearly needed at very high energies. However, for energies less than the nucleon mass (characteristic of few-nucleon phenomena) it is natural to ask “why should we use a relativistic theory?” I believe that the principal reasons fall into four categories:

- Intellectual: to preserve an exact (Poincaré) symmetry of nature,
- Practical: to calculate boosts and other relativistic kinematical effects consistently to all orders in $(v/c)^2$, useful even when these effects are not large,
- Predictable: to use field theory for guidance in the construction of 3 (and many) body forces, and the corresponding interaction currents, from 2 body forces, and
- Conceptual: to understand the origin of many seemingly unrelated effects that emerge when a single relativistic interaction is reduced to nonrelativistic form. Examples of this are the automatic emergence of the Thomas precession and the gyro-magnetic ratio from the Dirac equation, the emergence of the large $L \cdot S$ interaction from the scalar-vector theory of nucleon-nucleus scattering, and (hopefully) an economical explanation of the complicated structure of the NN interaction in terms of a covariant one boson exchange model.

1.2. The requirement of cluster separability

Cluster separability is the physical requirement that the internal interactions *within* any cluster of particles be independent of those within another cluster in the limit when

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the distance between the two clusters approaches infinity. At infinite separation, the clusters behave as independent systems. Clearly, without such an assumption it would be impossible to isolate systems from each other, and the behavior of any cluster of particles would depend on the behavior of all other particles in the universe. The way in which this requirement is imposed defines the different relativistic “schools” of dynamics.

1.3. Schools of dynamics

There are many different relativistic approaches, most of which belong to one of two main types, or “schools,” referred to as *Hamiltonian dynamics* or *Field dynamics*.

Hamiltonian dynamics (*HD* for short), reviewed by Keister and Polyzou [1], is a true quantum mechanics using a Hilbert space of physical states with positive definite norm. The Poincaré transformations are unitary transformations on the Hilbert space. A convenient way to construct these transformations in closed form was developed by Bakamjian and Thomas [2] but this approach does not preserve cluster separability for systems of more than two particles. Imposing both covariance and cluster separability is difficult and a major disadvantage of *HD*.

Field dynamics (*FD*) uses the language of field theory to construct manifestly covariant equations and interactions that are also (if one is careful) cluster separable. Covariance and cluster separability are satisfied simultaneously through the use of off mass-shell particles (with four-momentum p unconstrained by the mass-shell condition $p^2 = m^2$). The completeness relation for off mass-shell particles requires that the sum over mass-shell particles of positive energy $E(p) = \sqrt{m^2 + \mathbf{p}^2} > 0$ be extended to include particles of negative energy $-E(p)$ (usually reinterpreted as antiparticles moving backward in time). This takes one out of the familiar Hilbert space of positive energy states, leading to a generalization of quantum mechanics where states of negative norm may appear (as in the usual discussion of the Klein-Gordon equation). The principal disadvantage of *FD* is the need to treat negative energy states.

I believe that the reason *HD* has a problem with cluster separability is because it does not conserve four-momentum microscopically; the four-momenta of particles and subsystems in intermediate states does not match the physical four-momenta of these systems. This idea will be developed elsewhere.

2. TWO- AND THREE-NUCLEON CALCULATIONS

2.1. Hamiltonian dynamics

Fits to two body data that determine the NN phase shifts have been obtained using the Schrödinger equation in the center of mass system. In the *HD* framework this includes relativistic effects phenomenologically (except, possibly, those coming from the kinetic energy operator – see below). When extended to the three body sector, calculations in the three body center of mass frame using the Faddeev or AGS formalism require that the two body subamplitudes be boosted from the three body rest frame to the two body rest frame. Several studies of the size of relativistic effects in the three-nucleon sector exist, but there is no complete relativistic solution in the framework of *HD*.

In 1986, Glöckle, Lee, and Coester (GLC) [3] calculated relativistic corrections using the S -wave Malfliet-Tjon potential and found that they reduced the triton binding energy by about 0.2 MeV. Carlson, Pandharipande, and Schiavilla [4] calculated the boost

corrections to first order in $(v/c)^2$ for the realistic Argonne V18 potential, and found a larger boost correction of 0.3 MeV, and a correction from the relativistic kinetic energy operator of 0.1 MeV, for a total reduction in the triton binding energy of 0.4 MeV (as discussed at previous conferences, it may be that the effects are reduced when calculated to all orders in $(v/c)^2$, as suggested by GLC) . Sammarruca and Machleidt [5], using minimal relativity with a variety of potentials, obtained an *increase* in binding of the order of 0.3 MeV. We can conclude that relativistic contributions to the three body binding are of order 0.2 – 0.4 MeV, with some uncertainty in the sign. Recently, using the AV18 potential, relativistic effects in neutron deuteron scattering have been calculated [6]. The effects due to the Wigner rotations were small, but the boost effects were significant at higher energies,

The present status of these calculations is reviewed in a recent paper by Keister and Polyzou [7]. They support the conclusion that the relativistic corrections to the three body binding energy reduce its value. They also question the methods used to translate nonrelativistic fits into relativistic ones. One way to do this is to assume that the only difference (in the cm frame) is the replacement of the nonrelativistic kinetic energy operator with the relativistic one

$$2m + \frac{q^2}{m} \rightarrow 2\sqrt{m^2 + p^2} \quad (1)$$

If we assume $p = q$, this assumption requires a refit of the NN data, and the effect on the three body binding energy was found in Ref. [4] to be of the order of 0.1 MeV (repulsive). An alternative method has been proposed by Kamada and Glockle (KG) [8]. They convert Eq. (1) into an equality and find the relationship between p and q . Then they show that the Schrödinger equation can be transformed into an equation with a relativistic kinetic energy operator and a new (nonlocal) potential. Since the potential is phenomenological, the equivalence of the two equations shows that it is no longer necessary to refit the data; the phase shifts are identical.

However, it is not clear that the energies in (1) should be equated. An interesting coincidence pointed out by Coester some time ago is that the relation $p_{\text{cm}}^2 = \frac{1}{2}mE_{\text{lab}}$ holds both relativistically and nonrelativistically. This could imply, as assumed in Ref. [4], that the cm momenta should be equated,

This issue seems to be unresolved at the moment.

2.2. Field dynamics

An essential feature of FD is the presence of negative energy states connected with the propagation of virtual off-shell particles. This is a feature of both the Bethe-Salpeter equation [9] and the spectator equations [10]. Before either of these field dynamical equations can be applied to the description of few nucleon problems, it is necessary to first find an NN kernel that will fit the data. Attempts to use the Bethe-Salpeter equation to describe the NN data have not been successful so far [11].

The first fits to NN data using the spectator equations were completed in 1991 [12]. At that time we were only able to fit phase shifts, trusting the phases to be a good representation of the data. These fits gave a χ^2 of more than 2/datum, not very good by today's standards but still reasonable enough to begin exploring the implications of a

Figure 1. An off-shell coupling can generate an effective three-body force.

Figure 2. The same off-shell coupling generates triangular two-body forces.

relativistic model. A thorough study of the deuteron form factors showed that the models worked well [13]. When the model was used to calculate the triton binding energy [14] we found that a good result could be obtained without three body forces if we introduced an off-shell coupling for the sigma meson. Today we define the sNN vertex function with off-shell coupling (for scalar mesons with either isospin = 0 or isospin = 1) to be

$$\Lambda_s(p, p') = g_s + \frac{\nu_s}{2m} [2m - \not{p} - \not{p}'] \quad (2)$$

where p and p' are the four-momenta of the outgoing and incoming nucleons. Note that the term proportional to ν_s is zero if the nucleons are on-shell, and that it is not multiplied by g_s . Our latest results show that a good result for the triton binding energy requires $\nu_\sigma \sim -4$ and that this result is also close to the value which gives the *best fit* to the two body data.

The discovery of the importance of the off-shell sigma coupling is an interesting surprise. We understand why this coupling produces needed attraction, and also how its presence simulates a three body force. When an off-shell term sits next to a propagator, the propagator is canceled shrinking the successive exchanges to a point. For example, successive interactions of Λ_s on nucleon line 2 generates the three-body force illustrated in Fig. 1:

$$\begin{aligned} & \frac{\Lambda_{1s}(p_1, p'_1) \otimes \Lambda_{2s}(p_2, k)}{m_s^2 - (p_1 - p'_1)^2} \left[\frac{1}{m - \not{k}} \right] \frac{\Lambda_{2s}(k, p'_2) \otimes \Lambda_{3s}(p_3, p'_3)}{m_s^2 - (p_3 - p'_3)^2} \\ &= \frac{g_s^2}{d_1 d_3} \left(\frac{\nu_s}{2m} [m - \not{k}] \left[\frac{1}{m - \not{k}} \right] g_s + g_s \left[\frac{1}{m - \not{k}} \right] [m - \not{k}] \frac{\nu_s}{2m} \right) = \frac{\nu_s g_s^3}{m d_1 d_3}, \end{aligned} \quad (3)$$

where $d_i = m_s^2 - (p_i - p'_i)^2$ is the inverse of the meson propagator, and the initial and final on-shell nucleon spinors have been suppressed. The effective three-body force has a strength proportional to $\nu_s g_s^3$. The same off-shell coupling generates the two body force triangle shown in figure 2.

In view of the excellent high precision few-body data, and the very accurate nonrelativistic calculations, a precise comparison between relativistic and nonrelativistic theory requires a high precision fit to the NN data, and A. Stadler and I have been preoccupied with this task for several years. This work is nearly complete. At the conference I reported on the state of our work as it was then, but we have already moved well beyond that point and here I very briefly summarize where we are now (October, 2006). The best

fits are now nearly as good as the best results reported by nonrelativistic groups, and quite comparable to the new chiral model fits reported by Machleidt at this conference. Some results are summarized below:

- The most general relativistic kernel is the sum of only eight t -channel exchanges: pseudoscalar, scalar, vector (with both g and f couplings) and axial vector (without f coupling) exchanges, each with both isovector and isoscalar parts. This follows from the fact that the most general on-shell NN kernel can have only 5 spin invariants for each isospin.
- Except for the $\pi^\pm$, π^0 , and η , the bosons being exchanged are regarded as effective particles, with masses and coupling constants not necessarily related to those of physical mesons. In fact, the masses of the vector and axial vector mesons are not well determined by the data, and can be replaced by constant propagators, regulated by a form factor with a mass of about 1 GeV.
- An excellent fit with $\chi^2 \sim 1.16/\text{datum}$ requires only 22 parameters: 8 couplings for the isoscalar and isovector scalar, pseudoscalar, vector, and axial vector interactions, 2 vector meson f -type couplings, 2 masses of the scalar mesons, 6 off-shell couplings for the scalar, pseudoscalar, and vector mesons, and 4 form factor masses for the nucleon, pions, η and scalar mesons, and all spin-1 mesons. Several additional off-shell couplings are possible and may be studied in the future.
- We find models that fit both the NN and the triton binding energy, but it is not yet clear that these models also provide the very best fit to the NN data. If we shut off the Z -diagram contributions (from propagation in negative energy states) in the three-body sector, the triton is less bound by about 0.3 MeV, comparable to results (of different origin) obtained nonrelativistically.

In addition to very good agreement with data, these new models solve several other long standing problems. The meson propagators are covariantly redefined so that all singularities (see the discussion in Ref. [12]) are removed, both the deuteron binding energy and the 1S_0 scattering length are constrained during the fits, and the structure of the interactions will permit a straightforward construction of interaction currents following the methods of Ref. [15].

Once this work is completed, it will serve as a basis for the relativistic study of three body scattering, and the electromagnetic interactions of the two and three nucleon systems.

3. NEW RESULTS FROM FIELD DYNAMICS

During the last year, significant progress has been made on several long standing issues in relativistic field dynamics. In the following subsections, I give my own interpretation of the significance of three of these developments.

Figure 3. The small circles in the upper right corner are the binding energies obtained by CK from their K_2 kernel (upper) and their $K_2 + K_4$ kernel (lower). The background curves are from Ref. [20].

Figure 4. Three kernels referred to in the text.

3.1. Convergence of the Bethe-Salpeter equation

Carbonell and Karmanov (CK) have found a new way to solve the bound state scalar Bethe-Salpeter equation in Minkowski space [16–18]. Their method involves first substituting the Nakanishi representation for the BS amplitude into the BS equation, and then reducing the equation to a singularity free form using a light-cone projection. The Nakanishi representation [19] for a bound state of mass M composed of two identical particles of mass m can be written

$$\Phi(k, P) = \frac{1}{2} \int_{-1}^1 dz \int_0^\infty d\gamma \frac{g(\gamma, z)}{\gamma + m^2 - \frac{1}{4}M^2 - k^2 - zP \cdot k - i\epsilon)^3} \quad (4)$$

where k and P are the relative and total four-momentum of the pair. Substituting this into the BS equation and integrating over the light cone using the projection

$$\int_{-\infty}^{\infty} \Phi(k + \beta\omega, P) d\beta \quad (5)$$

where ω is a light-like vector ($\omega^2 = 0$), gives a singularity free equation for the spectral function $g(\gamma, z)$ that can be solved numerically. They show the power of their method by

solving the scalar equation with a kernel that includes the crossed box; to my knowledge, this is the first time this has been done.

The CK results are shown in figure 3. The background curves are from the work of Nieuwenhuis and Tjon (NT) [20] and compare the *exact* result for the sum of all ladder and crossed ladder diagrams (obtained from the Feynman-Schwinger formalism) with one boson exchange contributions from various equations. In this example, the mass of the exchanged meson is $\mu/m = 0.15$. Two of the contributions given by CK are shown as open circles in the upper right corner of the figure. Their result for the BS ladder sum (obtained with the kernel K_2 shown in figure 4) agrees with the results given by NT (the solid line labeled “BSE” in the figure), and the additional binding provided by the crossed box, K_4 is shown as the lower circle (it lies approximately on top of the BSLT result). The result obtained from the spectator theory using the one-boson exchange kernel K_{2s} is the short dashed line.

These results support my contention that the BS equation converges more slowly (to the exact result for the generalized ladder sum) than any of the relativistic three dimensional equations shown in the figure. In particular, the spectator theory in one boson exchange approximation is already more accurate than the BS theory with a kernel including crossed boxes, and far more tractable.

3.2. Relativistic treatment of spin 3/2 particles

The relativistic treatment of the Δ (and all spin 3/2 particles) has always been a challenging problem. The well known Rarita-Schwinger propagator can be written as the superposition of propagating spin 3/2 and 1/2 parts:

$$\begin{aligned}\Delta_{\mu\nu}(P) &= \frac{1}{M - \not{P} - i\epsilon} \left[g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{2}{3M^2} P_\mu P_\nu - \frac{1}{3M} (\gamma_\mu P_\nu - P_\mu \gamma_\nu) \right] \\ &= \frac{1}{M - \not{P} - i\epsilon} \mathcal{P}_{\mu\nu}^{(3/2)} + \frac{2}{3M^2} (M + \not{P}) \mathcal{P}_{22,\mu\nu}^{(1/2)} + \frac{1}{\sqrt{3}M} \left(\mathcal{P}_{12,\mu\nu}^{(1/2)} + \mathcal{P}_{21,\mu\nu}^{(1/2)} \right)\end{aligned}\quad (6)$$

where $\mathcal{P}_{\mu\nu}^{(3/2)}$ is the relativistic spin 3/2 projection operator

$$\mathcal{P}_{\mu\nu}^{(3/2)} = g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{1}{3P^2} (\not{P} \gamma_\mu P_\nu + P_\mu \gamma_\nu \not{P}) \quad (7)$$

and the other $\mathcal{P}$'s are all spin 1/2 operators

$$\mathcal{P}_{22,\mu\nu}^{(1/2)} = \frac{P_\mu P_\nu}{P^2}; \quad \mathcal{P}_{12,\mu\nu}^{(1/2)} = \frac{P^\rho i \sigma_{\mu\rho} P_\nu}{\sqrt{3}P^2}; \quad \mathcal{P}_{21,\mu\nu}^{(1/2)} = \frac{P_\mu i \sigma_{\rho\nu} P^\rho}{\sqrt{3}P^2} \quad (8)$$

linear or bilinear in P_μ or P_ν . The presence of spin 1/2 pieces in this propagator means that the summation of pion-nucleon bubbles, such as those shown in figure 5, does not reproduce the desired spin 3/2 structure.

Pascalutsa and collaborators [21–24] have developed a new field theory of spin 3/2 particles. They introduce a new strong gauge invariance that has the effect of reducing the number of degrees of freedom of the Rarita-Schwinger field from 8 to 4, the correct number for a spin 3/2 particle. Under this gauge invariance, the coupling Γ_μ of the Δ to the πN channel (for example) must satisfy $P^\mu \Gamma_\mu = 0$, insuring that the spin 1/2 components of the propagator given in Eq. (8) will vanish. This leads to many significant

Figure 5. A sequence of πN self-energy contributions to the dressed Δ propagator.

improvements, and is the necessary first step in the development of a relativistic chiral effective theory of ΔN interactions [23,24]. One consequence is that the sequence of πN bubbles shown in figure 5 can be summed analytically, as was done (without the justification) in Ref. [25].

This work is a major new development, opening the way to the inclusion of high spin baryons in covariant theories of hadronic interactions.

3.3. Current conservation with relativistic optical potentials

Relativistic optical potentials are an important tool for the discussion of electron scattering from nuclei, but it has been unclear how to conserve current with such a formalism. It is known, however, how to build conserved current operators for use with the three-body BS and spectator equations. In a recent paper, Van Orden [26] has used this three-body formalism to study the scattering of three nonidentical spin zero particles in the case when the 23 channel (for example) includes a bound state. By rearranging the three body equations, one can describe the effective scattering of particle 1 from the 23 bound state in terms of a relativistic two-body scattering equation with an effective optical potential (or kernel). This defines the optical potential microscopically. If the microscopic three body current operator is also reduced, one obtains an effective two-body interaction current for use with the optical potential. The interaction current that emerges satisfies a two-body Ward-Takahashi identity involving the optical potential, which guarantees that the matrix elements of the current are conserved. This conserved current contains a substantial number of contributions not included in current relativistic distorted-wave impulse approximation (RDWIA) calculations, and the contributions of these extra terms in various kinematic regions need to be considered carefully.

It is possible to extend this work to systems of more than 3 particles, and to include spin.

4. CONCLUSIONS

There are two fundamentally different ways to treat the relativistic few-body problem: Hamiltonian dynamics and Field dynamics. It is important that both methods be developed so that comparisons can be made, giving us understanding and insight into the limits of nonrelativistic quantum mechanics. *HD*, with its close connection to nonrelativistic quantum mechanics, is more developed than *FD*. New accurate fits to the NN data and the triton binding energy, improved methods for solving the BS equation in Minkowski space, a breakthrough in the theory of elementary high spin baryons, and a new roadmap for the relativistic construction of optical potentials and currents, are significant advances in the *FD* approach.

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