SYNCHROTRON SOURCES

Steven L. Hulbert
National Synchrotron Light Source, Brookhaven National Laboratory
Upton, New York

Gwyn P. Williams
Thomas Jefferson National Accelerator Facility
Newport News, Virginia

55.1 INTRODUCTION

Synchrotron radiation is a bright, broadband, polarized, pulsed source of electromagnetic radiation extending from the far infrared to the hard x-ray region. Brightness, defined as flux per unit area per unit solid angle, is normally a more important quantity than flux or intensity, particularly in throughput-limited applications which utilize only a small fraction of the transverse phase space of the emitted radiation or a small energy bandwidth, or both.

It is well known from classical theory of electricity and magnetism that accelerating charges emit electromagnetic radiation. In the case of synchrotron radiation, relativistic electrons are accelerated in a circular orbit and emit electromagnetic radiation in a broad spectral range. The visible portion of this spectrum was first observed on April 24, 1947 at General Electric’s Schenectady facility by Floyd Haber, a machinist working with the synchrotron team, although the first theoretical predictions were by Liénard in the latter part of the 1800s. An excellent early history with references is presented by Blewett and a history covering the development of the utilization of synchrotron radiation is presented by Hartman.

Synchrotron radiation covers the entire electromagnetic spectrum from the far infrared (or THz) and infrared, through the visible, ultraviolet, and x-ray regions and into the very hard x-ray range up to energies of 100 kilovolts and above. If the charged particles are of low mass, such as electrons, and if they are traveling relativistically, the emitted radiation is very intense and highly collimated, with opening angles, which depend inversely on the energy of the particle, on the order of 1 milliradian. In electron storage rings there are two distinct types of sources of synchrotron radiation: dipole (bending) magnets and insertion devices. Insertion devices are further classified as either wigglers, which act like a sequence of bending magnets with alternating polarities, or undulators, which are also multiperiod alternating magnet systems but in which the beam deflections are small resulting in coherent interference of the emitted light.

In typical storage rings used as synchrotron radiation sources, several bunches of up to \( \sim 10^{12} \) electrons circulate in vacuum, guided by magnetic fields. The bunches are typically several 10s of centimeters long, so that the light is pulsed, being on for a few 10s to a few 100s of picoseconds, and off for several 10s to a few 100s of nanoseconds depending on the particular machine and the radio-frequency cavity which restores the energy lost to synchrotron radiation. The revolution time for a ring of circumference 30 m is 100 ns, so that each bunch of \( \sim 10^{12} \) electrons is seen 10^7 times per second, giving a
current of ~1 A (assuming single bunch filling). In linacs that are used as light sources the electrons are also in bunches, but these are usually much shorter resulting in pulses of < 1 ps.

The most important characteristic of accelerators built specifically as synchrotron radiation sources is that they have a magnetic focusing system which is designed to concentrate the electrons into bunches of very small transverse cross section and to keep the electron transverse velocities small. The combination of high intensity with small opening angles and small source dimensions results in the very high brightness.

The first synchrotron radiation sources to be used were operated parasitically on existing high-energy physics or accelerator development programs. These were not optimized for brightness, and were usually accelerators rather than storage rings, meaning that the electron beams were constantly being injected, accelerated, and extracted. Owing to the successful use of these sources for scientific programs, a second generation of dedicated storage rings was built starting in the early 1980s. In the mid 1990s, a third generation of sources was built, this time based largely on insertion devices, especially undulators of various types. A fourth generation of accelerator-based photon sources is now coming on line, based on what is called multiparticle coherent emission, in which coherence along the path of the electrons, or longitudinal coherence, plays the major role. This is achieved by microbunching the electrons on a length scale comparable to or smaller than the scale of the wavelengths emitted. The emission is then proportional to the square of the number of electrons N which, if N is $10^{12}$, can be a very large enhancement. These sources can approach the theoretical diffraction limit of source emittance (the product of solid angle and area).

### 55.2 THEORY OF SYNCHROTRON RADIATION EMISSION

#### General

The theory describing synchrotron radiation emission is based on classical electrodynamics and can be found in the works of Tomboulian and Hartman, Schwinger, Jackson, Winick, Hofmann, Krinsky, Perlman, and Watson, and Kim. A quantum description, presented by Sokolov and Ternov, is quantitatively equivalent.

Here we present a phenomenological description in order to highlight the general concepts involved. Electrons in circular motion radiate in a dipole pattern as shown schematically in Fig. 1a.

![Figure 1](image_url)

**FIGURE 1** Conceptual representation of the radiation pattern from a charged particle undergoing circular acceleration at (a) nonrelativistic and (b) relativistic velocities.
As the electron energies increase and the particles start traveling at relativistic velocities, this dipole pattern appears different to an observer in the rest frame of the laboratory. Special relativity tells us that angles $\theta_t$ in a transmitting object are related to those in the receiving frame $\theta_r$ by

$$\tan \theta_r = \frac{\sin \theta_t}{\gamma \cos \theta_t - \beta}$$

with $\gamma$, the ratio of the mass of the electron to its rest mass, being given by $E/m_ec^2$, $E$ being the electron energy, $m_e$ the electron rest mass, and $c$ the velocity of light. $\beta$ is the ratio of electron velocity $v$ to the velocity of light $c$. Since $\beta = 1$ for electrons travelling at relativistic energies, the peak of the dipole emission pattern in the particle frame, $\theta_t = 90^\circ$, transforms to $\theta_r = \tan^{-1} \gamma^{-1}$ in the laboratory frame as shown in Fig. 1b. Thus $\gamma^{-1}$ is a typical opening angle of the radiation in the laboratory frame. For an electron viewed in passing by an observer, as shown in Fig. 2, the duration of the pulse produced by a particle under circular motion of radius $r$ will be $r/gc$ in the particle frame, or $r/gc \times 1/\gamma^2$ in the laboratory frame owing to time dilation. The Fourier transform of this function will contain frequency components up to the reciprocal of this time interval. For a storage ring with a radius of 2 meters and $\gamma = 1000$, corresponding to a stored electron beam energy of ~500 MeV, the time interval is $10^{-17}$ seconds, which corresponds to light of wavelength 30 Å.

**Bending Magnet Radiation**

For an electron storage ring, the relationship between the electron beam energy $E$, bending radius $\rho$, and field $B$ is

$$\rho = \frac{E}{\gamma c B} = \frac{E[GeV]}{0.300B[T]}$$

**FIGURE 2** Illustration of the derivation of the spectrum emitted by a charged particle in a storage ring.
the ratio of the mass of the electron to its rest mass is given by $\gamma = E/m_e c^2 = E/0.511 \text{ MeV} = 1957 [E \text{ GeV}]$ and $\lambda_\gamma$, which is defined as the wavelength for which half the power is emitted above and below, is

$$\lambda_\gamma = \frac{3\gamma^3}{2\rho} = 5.59 \AA \sqrt{\frac{\rho [m]}{E^2 [\text{GeV}^2]}} = 18.6 \AA \sqrt{\frac{B[T]}{E [\text{GeV}^2]}}$$

The critical frequency and photon energy are

$$\omega = \frac{2\pi c}{\lambda_\gamma} = \frac{3c\gamma^3}{2\rho}$$

$$\epsilon_i [eV] = \hbar \omega_i (eV) = 665.5E^2 [\text{GeV}^2]B[T]$$

The angular distribution of synchrotron radiation flux emitted by electrons moving through a bending magnet with a circular trajectory in the horizontal plane is given by

$$\frac{dI_e}{d\theta d\psi} = \frac{3\alpha}{4\pi^2} \frac{\Delta \omega}{\omega} \left( \frac{\omega}{\omega_0} \right)^2 \left( 1 + \gamma^2 \psi^2 \right) \left[ K_{\frac{3}{2}}^2 (\xi) + \frac{\gamma^2 \psi^2}{1 + \gamma^2 \psi^2} K_{1/3}^2 (\xi) \right]$$

$$= 1.326 \times 10^{13} \text{photon/sec/mrad}/0.1\% \text{bandwidth} \times E^2 [\text{GeV}^2]I[A] (1 + \gamma^2 \psi^2)^{3/2}$$

where $\psi$ is the observation angle in the vertical plane, $\theta$ the observation angle in the vertical plane, $\alpha$ the fine structure constant (1/137), $\omega$ the light frequency, $I$ the beam current, and $\xi = (\omega/2\omega_0) (1 + \gamma^2 \psi^2)^{5/2}$. The subscripted $K$s are modified Bessel functions of the second kind. The $K_{2/3}$ term represents light linearly polarized parallel to the electron orbit plane, while the $K_{1/3}$ term represents light linearly polarized perpendicular to the orbit plane.

If one integrates over all vertical angles, then the total intensity is

$$\frac{dI_e}{d\theta} = \sqrt{\frac{3\alpha}{2\pi}} \frac{\Delta \omega}{\omega} \frac{I}{\omega_0} \int K_{5/3} (y) dy$$

$$= 2.457 \times 10^{13} \text{photon/sec/mrad}/0.1\% \text{bandwidth} \times E [\text{GeV}]I[A] \frac{\omega}{\omega_0} \int K_{5/3} (y) dy$$

The Bessel functions can be computed easily using algorithms of Kostroum:12

$$K_\nu (x) = h \left[ \frac{e^{-x}}{2} + \sum_{r=1} K_\nu (x) \cosh (vrh) \right]$$

and

$$\int K_\nu (\eta) d\eta = h \left[ \frac{e^{-x}}{2} + \sum_{r=1} \cosh (vrh) \frac{\cosh (vrh)}{\cosh (r)} \right]$$

for all $x$ and for any fractional order $\nu$, where $h$ is some suitable interval such as 0.5. In evaluating the series, the sum is terminated when the $r$th term is small, <10^{-5}, for example.
In Fig. 3 we plot the universal function

\[ G_1 \left( \frac{\omega}{c \omega_l} \right) = \frac{\omega}{c \omega_l} \int \frac{K_{3/2}(y)dy}{c \omega_l} \]

from Eq. (6), so that the photon energy dependence of the flux from a given ring can be calculated readily. It is found that the emission falls off exponentially as \( e^{-l/\lambda} \) for wavelengths shorter than \( \lambda_o \), but only as \( l^{-1/3} \) at longer wavelengths.

The vertical angular distribution is more complicated. For a given ring and wavelength, there is a characteristic natural opening angle for the emitted light. The opening angle increases with increasing wavelength. If we define \( \psi \) as the vertical angle relative to the orbital plane, and if the vertical angular distribution of the emitted flux is assumed to be Gaussian in shape, then the \textit{rms} divergence \( \sigma_\psi \) is defined as \( 1/\sqrt{2\pi} \) times the ratio of Eqs. (5) and (6) evaluated at \( y = 0 \):

\[ \sigma_\psi = \sqrt{\frac{2\pi}{3}} \left( \frac{\omega}{\omega_l} \right)^{-1/3} \int \frac{K_{3/2}(y)dy}{c \omega_l K_{2/3}^2(\omega/2\omega)} \]  

(9)

In reality, the distribution is not Gaussian, especially in view of the fact that the distribution for the vertically polarized component vanishes in the horizontal plane (\( \psi = 0 \)). However, \( \sigma_\psi \) defined by Eq. (9) is still a simple and useful measure of the angular divergence. The photon energy \( (\omega) \) dependence of the electron-energy-independent quantity \( \gamma \sigma_\psi \) is plotted in Fig. 4. At \( \omega = \omega_l, \sigma_\psi = 0.647/\gamma \). The asymptotic values of \( \sigma_\psi \) can be obtained from the asymptotic values of the Bessel functions and are

\[ \sigma_\psi = 1.07 \left( \frac{\omega}{\omega_l} \right)^{-1/3} \quad \omega \ll \omega_l \]

(10)

and

\[ \sigma_\psi = 0.58 \left( \frac{\omega}{\omega_l} \right)^{-1/2} \quad \omega \gg \omega_l \]

(11)

In Fig. 5 we show examples of the normalized vertical angular distributions of both parallel and perpendicularly polarized synchrotron radiation for a selection of wavelengths.
Circular Polarization and Aperturing for Magnetic Circular Dichroism

Circularly polarized radiation is a valuable tool for the study of electronic, magnetic, and geometric structure of a wide variety of materials. The dichroic response in the soft x-ray spectral region (100 to 1500eV) is especially important because in this energy range almost every element has a strong dipole transition from a sharp core level to its lowest unoccupied state.\(^{13}\)

The production of bright sources of circularly polarized soft x-rays is therefore a topic of keen interest, and is a problem which has seen a multitude of solutions, from special insertion devices
SYNCHROTRON SOURCES

(crossed undulators, helical undulators, elliptically polarized undulators/wigglers) to optical devices (multiple-bounce reflectors/multilayers and quarter-wave plates). However, standard bending magnet synchrotron radiation sources are good sources of elliptically polarized soft x-rays when viewed from either above or below the orbital plane.

As discussed by Chen, a practical solution involves acceptance of a finite vertical angular range, $\psi_{\text{off}} - \Delta \psi/2 < \psi < \psi_{\text{off}} + \Delta \psi/2$ centered about any vertical offset angle $\psi = \psi_{\text{off}}$ or, equivalently, about $\psi = -\psi_{\text{off}}$. This slice of bending magnet radiation exhibits a circular polarization

$$P_c = \frac{2A_hA_v}{(A_h^2 + A_v^2)}$$

(12)

where $A_h = K_{2/1}(\xi)$ and $A_v = \gamma \psi/(1 + \gamma^2 \psi^2)^{1/2} K_{1/1}(\xi)$ are proportional to the square-roots of the horizontally and vertically polarized components of bending magnet flux [Eq. (5)], i.e., $A_h$ and $A_v$ are proportional to the horizontal and vertical components of the electric field, respectively. $P_c$ depends on the vertical angle $\psi$, electron energy $\gamma$ and, through $\xi$, the emitted photon energy $\omega/\omega_c$. In Fig. 6 we plot values of $P_c$ vs $\gamma \psi$ and $\omega/\omega_c$ for $\gamma = 1565$ ($E = 0.8\text{GeV}$) and $\rho = 1.91\text{m}$ ($h\nu_{\text{crit}} = 594\text{ eV}$).

Magnetic circular dichroism (MCD) measures the normalized difference of the absorption of right circular and left circular light. Assuming no systematic error, the signal to noise ratio in such a measurement defines a figure of merit.

$$\text{MCD figure of merit} = \left( \frac{\text{average circular polarization}}{\text{flux fraction}} \right)^{1/2}$$

(13)

where

$$\text{average circular polarization} = \frac{\int_{\psi_{\text{off}} - \Delta \psi/2}^{\psi_{\text{off}} + \Delta \psi/2} P_c(\psi) \frac{dF}{d\psi} d\psi}{\int_{\psi_{\text{off}} - \Delta \psi/2}^{\psi_{\text{off}} + \Delta \psi/2} \frac{dF}{d\psi} d\psi}$$

(14)

FIGURE 6 $P_c$ versus $\gamma \psi$ versus $\omega/\omega_c$. 
and the fraction of the total (vertically-integrated) flux emitted into the vertical slice $\psi = \psi_{\text{off}} \pm \Delta \psi/2$ is

$$\text{flux fraction} = \frac{1}{dF_{\text{bm}}(\omega)/d\theta} \int_{\psi_{\text{off}} - \Delta \psi/2}^{\psi_{\text{off}} + \Delta \psi/2} dF_{\text{bm}}(\omega)/d \psi d\psi$$

(15)

Here $dF_{\text{bm}}(\omega)/d\theta$ is the angular dependence of the bending magnetic flux from Eq. (5) and $dF_{\text{bm}}(\omega)/d\psi$ is the vertically integrated flux from Eq. (6). For a 0.8 GeV storage ring (e.g., the VUV ring at the National Synchrotron Light Source (NSLS), Upton, NY USA), the choices of $\psi$ and $\Delta \psi$ that maximize the MCD figure of merit are 0.5 mrad and 0.66 mrad, respectively. This yields a flux fraction ~0.3, a circular polarization ~0.65 and a figure of merit ~0.35.

**Bending Magnet Power**

Integration of $\hbar \omega (d^2 F_{\text{bm}}(\omega)/d\theta d\psi)$ from Eq. (5) over all frequencies $\omega$ yields the angular distribution of power radiated by a bending magnet:

$$\frac{d^2 P_{\text{bm}}}{d\theta d\psi} = \int_0^{\hbar \omega} \frac{d^2 F_{\text{bm}}(\omega)}{d\theta d\psi} d\omega = \frac{1}{c^2 \pi \rho^2} \left[ \frac{1}{7} \gamma^5 \right] F(\gamma \psi)$$

$$= 18.082 \, \text{W/mrad}^2 \times \frac{E^3 [\text{GeV}^3] |I| |A|}{\rho^2 m} F(\gamma \psi)$$

(16)

which is independent of the horizontal angle $\theta$ as required by symmetry, and the vertical angular dependence is contained in the factor

$$F(\gamma \psi) = \frac{1}{(1 + \gamma^2 \psi^2)^{\frac{1}{2}}} \left[ 1 + \frac{5}{7} \frac{\gamma^2 \psi^2}{(1 + \gamma^2 \psi^2)^{\frac{1}{2}}} \right]$$

(17)

The first term in $F(\gamma \psi)$ represents the component of the bending magnet radiation parallel to the orbital plane, while the second represents the perpendicular polarization component. $F(\gamma \psi)$ and its polarization components are plotted versus $\gamma \psi$ in Fig. 7. Note that the area under the $F_{\text{parallel}}$ curve is approximately seven times greater than that for $F_{\text{perpendicular}}$.

![FIGURE 7](image_url) Vertical angle dependence of bending magnet power, $F(\gamma \psi)$, versus $\gamma \psi$. 
Integrating Eq. 17 over the out-of-orbital-plane (vertical) angle $\psi$ yields the total power radiated per unit in-orbital-plane (horizontal) angle $\theta$:

$$\frac{dP_{\text{tot}}}{d\theta} = \frac{I}{e} \frac{h c \alpha^2}{3\pi \rho} = 14.08 \text{ W/mrad} \times \frac{E^4[\text{GeV}^4]}{\rho[m]}$$

(18)

For example, a 1.0-GeV storage ring with 2 m radius bends generates 7.04 W/mrad/Amp of stored current. By contrast, a 2.5 GeV machine with 7-m radius bends generates 78.6 W/mrad/A and a 7 GeV machine with 39 m radius bends generates 867 W/mrad/A.

**Bending Magnet Brightness**

Thus far we have calculated the emitted flux in photons per second per milliradian$^2$ of solid angle. In order to calculate the brightness we need to include the source size. In these calculations we calculate the central (or maximum) brightness, for which we use the natural opening angle to define both the horizontal and vertical angles. Using vertical angles larger than this will not increase the flux as there is no emission. Using larger horizontal angles will increase the flux proportionately as all horizontal angles are filled with light, but owing to the curvature of the electron trajectory, the average brightness will actually be less. The brightness expression$^{15,16}$ is

$$B_{\text{bm}} = \frac{E^4 F_{\text{bm}}}{(d\theta \, d\psi)_{\psi=0}}$$

(19)

where

$$\sum_x = \epsilon_x \beta_x + \eta_x \sigma^2_x + \sigma^2_x$$

and

$$\sum_y = \epsilon_y \beta_y + \sigma^2_y + \frac{\epsilon^2_y + \epsilon_y \gamma \sigma^2_y}{\sigma^2_y}$$

(20)

$\epsilon_x$ and $\epsilon_y$ are the electron beam emittances in the horizontal and vertical directions respectively, $\beta_x$ and $\beta_y$ are the electron beam beta functions in the horizontal and vertical planes, $\eta$ is the dispersion function in the horizontal plane, and $\sigma$ is the rms value of the relative energy spread. All the electron beam parameters are properties of a particular storage ring. The diffraction-limited source size is $\sigma = 4\pi \sigma_y$. The effective source sizes ($\sum_x$ and $\sum_y$) are photon energy dependent via the natural opening angle $\sigma_y$ and the diffraction limited source size $\sigma$.

## 55.3 INSERTION DEVICES (UNDULATORS AND WIGGLERS)

**General**

Insertion devices are periodic magnetic structures installed in straight sections of storage rings, as illustrated in Fig. 8, in which the vertical magnetic field varies approximately sinusoidally along the axis of the undulator. The resulting motion of the electrons is also approximately sinusoidal, but in the horizontal plane. One can understand the nature of the spectra emitted from these devices by again studying the electric field as a function of time, and this is shown in Fig. 9. This shows that the electric field and hence its Fourier transform, the spectrum, depend critically on the magnitude of the beam deflection in the device. At one extreme, when the magnetic fields are high, as in Fig. 9a, the deflection is large and the electric field is a series of pulses similar to those obtained from a
dipole. Such a device is termed a “wiggler.” The Fourier transform for the wiggler is $N$ times that from a single dipole. At the other extreme, as in Fig. 9b, the deflection of the electron beam is such that the electric field as a function of time is sinusoidal, and the Fourier transform is then a single peak with a width proportional to the inverse of the length of the wavetrain $L'$ according to $\lambda^2/\Delta \lambda = L'$. $L'$ is obtained by dividing the real length of the device $L$ by $\gamma^2$ because of relativistic effects. Thus for a meter long device emitting at a wavelength $\lambda = 10$ nm in a machine of energy 0.5 GeV ($\gamma \sim 1000$), corresponding to, for example, 1 cm period length and magnetic field strength $B_0 = 1.5$ T, we get $\lambda^2/\Delta \lambda = 10^{-6}$ meters, and $\lambda/\Delta \lambda = 100$. Note that $\Delta \lambda/\lambda \sim 1/N$ as expected. Interference occurs in an undulator since the electric field from one part of the electron path is added coherently to that from adjacent parts.
**Formal Treatment**

We assume that the motion of an electron in an insertion device is sinusoidal, and that we have a magnetic field in the vertical ($y$) direction varying periodically along the $z$ direction, with

$$B_y = -B_0 \sin(2\pi \frac{z}{\lambda_u}) \quad 0 \leq z \leq N\lambda_u$$  \hspace{1cm} (21)

where $B_0$ is the peak magnetic field, $\lambda_u$ is the period length, and $N$ the number of periods. By integrating the equation of motion, the electron transverse velocity $c\beta_y$ is found to be

$$\beta_y = \frac{K}{\gamma} \cos(2\pi \frac{z}{\lambda_u})$$  \hspace{1cm} (22)

where

$$K = eB_0\lambda_u \quad \text{and} \quad \epsilon c = 0.934\lambda_u (cm)B_0[T]$$  \hspace{1cm} (23)

is a dimensionless parameter which is proportional to the deflection of the electron beam. The maximum slope of the electron trajectory is $\delta = (K/\gamma)$. In terms of $\delta$, we define an undulator as a device in which $\delta \leq \gamma^{-1}$, which corresponds to $K \leq 1$. When $K$ is large, the device is called a wiggler. In most insertion devices the field can be changed either electromagnetically or mechanically, and in some cases $K$ can vary between the two extremes of undulator and wiggler operation.

**Wigglers**

For the wiggler, the flux distribution is given by $2N$ (where $N$ is the number of magnetic periods) times the appropriate bending magnet formulae in Eqs. (5) and (6). However, $\rho$ or $\beta$ must be taken at the point in the path of the electron which is tangent to the direction of observation. For a horizontal angle $\theta$,

$$\epsilon_\epsilon(\theta) = \epsilon_{\text{max}} \sqrt{1 - (\theta/\delta)^2}$$  \hspace{1cm} (24)

where

$$\epsilon_{\text{max}}[keV] = 0.665E^3[GeV^2]B_0[T]$$  \hspace{1cm} (25)

from Eq. (4). Integration over $\theta$, which is usually performed numerically, gives the wiggler flux.

The calculation of the brightness of wigglers needs to take into account the depth-of-field effects, i.e., the contribution to the apparent source size from different poles. The expression for the brightness of wigglers is

$$B_W = \frac{d^2F_W}{d\theta d\psi} \sum_{n=-N}^{N} \frac{1}{2\pi} \times \exp \left[ -\frac{1}{2} \left( \frac{\chi^2}{\sigma_x^2 + \sigma_{rn}^2} \right) \right] \times \exp \left[ -\frac{1}{2} \left( \frac{\chi^2}{\sigma_y^2 + \sigma_{r\psi}^2} \right) \right] \left( \frac{\epsilon_+^2}{\sigma_x^2 + \sigma_{rn}^2} \right)^{3/2}$$  \hspace{1cm} (26)

where $\sigma_{rn} = \lambda_u[n \pm (1/4)]$, $\lambda_u$ is the wiggler period, and $\sigma_\psi$ is identical to Eq. (9), but evaluated, in the wiggler case, as the instantaneous radius at the tangent to the straight-ahead ($\theta = \psi = 0$) direction (i.e., minimum $\rho$, maximum $\epsilon$), $\sigma_x = \sqrt{\epsilon \beta_x}$ and $\sigma_y = \sqrt{\epsilon \beta_y}$ are the rms transverse beam sizes,
while \( \sigma_x^* = \sqrt{\epsilon_x/\beta_x} \) and \( \sigma_y^* = \sqrt{\epsilon_y/\beta_y} \) are the angular divergences of the electron beam in the horizontal and vertical directions respectively. The exponential factor in Eq. (26) arises because wigglers have two source points separated by \( 2x_o \), where

\[
x_o = \frac{K \lambda_w}{\gamma 2\pi}
\]  

(27)

The summations in Eq. (26) must be performed for each photon energy because \( \sigma_y \) is photon-energy dependent.

**Undulators**

The interference which occurs in an undulator, i.e., when \( K \) is moderate (\( K \leq 1 \)), produces sharp peaks in the forward direction at a fundamental (\( n = 1 \)) and all odd harmonics (\( n = 3,5,7 \ldots \)) as shown for a zero emittance (\( \epsilon = 0 \)) electron beam in Fig. 10a (dotted line). In the \( \epsilon = 0 \) case, the even harmonics (\( n = 2,4,6,\ldots \)) peak off-axis and do not appear in the forward direction. For real (\( \epsilon \neq 0 \)) electron beams, the spectral shape, angular distribution, and peak brightness are strongly dependent on the emittance and energy spread of the electron beam as well as the period and magnitude of the insertion device field.

In general, the effect of electron beam emittance is to cause all harmonics to appear in the forward direction (solid line in Fig. 10a). The effect of angle integration on the spectrum in Fig. 10a is shown in Fig. 10b, a spectrum which is independent of electron beam emittance except for the presence of “noise” in the zero emittance case. The effect of electron beam emittance on the angular distribution of the fundamental, second, and third harmonics of this device is shown in Fig. 10c, which also nicely demonstrates the dependence on harmonic number.

**FIGURE 10** Spectral output and angular distribution of the emission from the early-2000s vintage NSLS In-Vacuum UNdulator (IVUN) for \( K = 0.75 \). (a) spectral output in the forward direction, with (solid line) and without (dotted line) the effect of electron beam emittance; (b) angle-integrated spectral output with (solid line) and without (faint solid line) the effect of electron beam emittance, and the decomposition into harmonics (\( n = 1,2,3,4 \)) (dotted lines); (c) angular distribution of the first three harmonics (\( n = 1,2,3 \)), with and without the effect of electron beam emittance. Emittance values used: 94-nm horizontal, 0.1-nm vertical.
The peak wavelengths of the emitted radiation $\lambda_n$ are given by

$$\lambda_n = \frac{\lambda_u}{2n\gamma^2} \left( 1 + \frac{K^2}{2} + \gamma^2 \theta^2 \right) \tag{28}$$

where $\lambda_u$ is the undulator period length. They soften as the square of the deviation angle $\theta$ away from the forward direction.

Of main interest is the intense central cone of radiation. An approximate formula for flux integrated over the central cone is (for the odd harmonics)

$$F_z(K,\omega) = \pi \alpha N \frac{\Delta \omega}{\omega} I_n(Q_n(K)) \quad n = 1, 3, 5$$

$$= 1.431 \times 10^{14} \text{photons/sec/0.1% bandwidth} \times I[A] Q_n(K) \tag{29}$$

where

$$Q_n(K) = \left( 1 + \frac{K^2}{2} \right) F_z(K) \quad n = 1, 3, 5 \tag{30}$$
and

\[ F_u(K) = \frac{K n^2}{(1 + K^2/2)^2} \left( I_{\alpha} - I_{\alpha+1/2} \right) \left( I_{\alpha-1/2} - I_{\alpha+1/2} \right) \]

(31)

Here \( I_{\alpha} \), \( n = 1, 3, 5 \) are the integer Bessel functions of the first kind: \( J_0, J_1, J_2, J_3 \).

To calculate the undulator flux angular distribution and spectral output into arbitrary solid angle, one can use freely available codes such as Urgent\(^1\) (R. P. Walker and B. Diviacco). To include magnetic field errors (e.g., measured values), use Ur\(^1\) (R. J. Dejus and A. Luccio), SRW\(^1\) (O. Chubar and P. Elleaume), or Spectra\(^1\) (T. Tanaka and H. Kitamura).

The brightness of an undulator \( B_u \) is approximated by dividing the central cone flux by the effective angular divergence, \( \Sigma_x(\Sigma_y) \), and by the effective source size, \( \Sigma_x(\Sigma_y) \), in the horizontal (vertical) directions. These are given by convolution of the Gaussian distributions of the electron beam and the diffraction limited photon beam, in both angle and space:

\[ \Sigma_x = \sqrt{\sigma_x^2 + \sigma_r^2} \quad \Sigma_y = \sqrt{\sigma_y^2 + \sigma_r^2} \]

(32)

Thus, \( B_u \) is given by

\[ B_u = \frac{F_u}{(2\pi)^3 \Sigma_x \Sigma_y} \]

(34)

The diffraction-limited emittance of a photon beam is the minimum value in the inequality

\[ \epsilon = \sigma_x \sigma_y \geq \frac{\lambda}{2} = \frac{\lambda}{4\pi} \]

(35)

where \( \epsilon \) is the photon emittance and \( \lambda \) is the wavelength, in direct analogy to the Heisenberg uncertainty principle in nonrelativistic quantum mechanics. The space versus angle separation of this minimum emittance is energy and harmonic dependent.\(^2\) For the exact harmonic frequency in the forward direction, given by Eq. (28) with \( \theta = 0 \), there appears to be consensus that \( \sigma_x \) and \( \sigma_y \) are given by

\[ \sigma_r = \frac{\lambda}{2L} \quad \text{and} \quad \sigma_r = \frac{\sqrt{2\lambda L}}{4\pi} \]

(36)

On the other hand, at the peak of the angle-integrated undulator spectrum, which lies a factor of \([1 - (1/nN)]\) below the exact harmonic energy, \( \sigma_x \) and \( \sigma_y \) are given by

\[ \sigma_r = \frac{\lambda}{L} \quad \text{and} \quad \sigma_r = \frac{\sqrt{\lambda L}}{4\pi} \]

(37)

It is clear from Eqs. (32) and (33) that the choice of expression for \( \sigma_x \) and \( \sigma_y \) can have a non-negligible effect on the undulator brightness value especially for small beam size and opening angle. Lacking a specific functional form for \( \sigma_x \) and \( \sigma_y \) as a function of photon energy, we generally use Eq. (32) in evaluating the expression for undulator peak spectral brightness from Eq. (34).
Insertion Device Power

The Schwinger\(^3\) formula for the distribution of radiated power from an electron in a sinusoidal trajectory, which applies with reasonable approximation to undulators and, to a lesser extent, wigglers, reduces\(^2\) to

\[
d\frac{dP}{d\Omega d\psi} = P_{\text{total}} \frac{21\gamma^2}{16\pi K} G(K) f_K(\gamma\theta,\gamma\psi)
\]  

(38)

where the total (angle-integrated) radiated power is

\[
P_{\text{total}} = \frac{N Z_0 I 2\pi c e}{\lambda_0} \gamma^2 K^2 = 633.0 \text{ W} \times E^2 [\text{GeV}^2] B_0^2 [\text{T}^2] L [m] I [\text{A}]
\]

(39)

where \(N\) is the number of undulator or wiggler periods, \(Z_0\) is the vacuum impedance (377 Ω), \(I\) is the storage ring current, \(e\) is the electronic charge, \(c\) is the speed of light, \(L = N\lambda_0\) is the length of the insertion device,

\[
G(K) = \frac{K}{(1 + K^2)^{3/2}} \left( K^4 + \frac{24}{7} K^4 + 4 K^4 + \frac{16}{7} \right)
\]

(40)

and

\[
f_K(\gamma\theta,\gamma\psi) = \frac{16 K}{7\pi G(K)} \int_0^\pi d\alpha \left( \frac{1}{D^3} - \frac{4(\gamma\theta - K \cos \alpha)^3}{D^3} \right) \sin^2 \alpha
\]

(41)

where

\[
D = 1 + \gamma^2 \psi^2 + (\gamma\theta - K \cos \alpha)^2
\]

(42)

The integral in the expression for \(f_K\) is best evaluated numerically.

For \(K > 1\), which includes all wigglers and much of the useful range of undulators, an approximate formula for the angle dependence of the radiated power is

\[
f_K(\gamma\theta,\gamma\psi) = \sqrt{1 - \left( \frac{\theta}{K} \right)^2} F(\gamma\psi)
\]

(43)

where \(F(\gamma\psi)\) is the bending magnet formula from Eq. (17). This form clearly indicates the strong weakening of insertion device power as \(\theta\) increases, vanishing at \(\theta = \pm K/\gamma\).

Since \(f_K(0,0)\) is normalized to unity, the radiated power density in the forward direction (i.e., along the undulator axis) is

\[
d\frac{d^2P}{d\theta d\psi}(\theta = 0, \psi = 0) = P_{\text{total}} \frac{21\gamma^2}{16\pi K} G(K) = 10.84 \text{ W/mrad}^2 \times B_0 [\text{T}] E^4 [\text{GeV}] I [\text{A}] N G(K)
\]

(44)

Polarization of Undulators and Wigglers

The polarization properties of the light emitted by wigglers, is similar to that of dipoles. For both sources the radiation is elliptically polarized when observed at some angle away from the orbital plane as given by Eq. (5). For radiation from planar undulators, however, the polarization is always linear. The polarization direction, which is in the horizontal plane when observed from that plane,
rotates in a complicated way at other directions of observation. A comprehensive analysis of the polarization from undulators has been carried out by Kitamura. 23 The linear polarization of the undulator radiation is due to the symmetry of the electron trajectory within each period. The polarization can in fact be controlled by a deliberate breaking of this symmetry. Circularly polarized radiation can be produced by a helical undulator, in which the series of dipole magnets is arranged such that each period is rotated by a fixed angle with respect to the previous one. To generate variable polarization, one can use a pair of planar undulators oriented at right angles to each other. The amplitude of the radiation from these so-called crossed undulators is a linear superposition of two parts, one linearly polarized along the x direction and another linearly polarized along the y direction, x and y being orthogonal to the electron beam direction. By varying the relative phase of the two amplitudes by means of a variable-field magnet between the undulators, it is possible to modulate the polarization in an arbitrary way. The polarization can be linear and switched between two mutually perpendicular directions, or it can be switched between left and right circularly polarized. For this device to work, it is necessary to use a monochromator with a sufficiently small band-pass, so that the wave trains from the two undulators are stretched and overlap. Also the angular divergence of the electron beam should be sufficiently small or the fluctuation in relative phase will limit the achievable degree of polarization. A planar undulator whose pole boundaries are tilted away from a right angle with respect to the axial direction can be used as a helical undulator if the electron trajectory lies a certain distance above or below the midplane of the device.

Transverse Spatial Coherence

As shown by Kim 24 and utilized in the brightness formulae given above, in wave optics the phase-space area of a radiation beam is given by the ratio of flux \( F_0 \) to brightness \( B_0 \). A diffraction limited photon beam (no electron size or angular divergence contribution) occupies the minimum possible phase-space area. From Eqs. (32 to 37) this area is

\[
(2\pi \sigma_x \sigma_y)^2 = (2\pi \epsilon)^2 = \left(\frac{\lambda}{2}\right)^2
\]  

Thus, the phase space occupied by a single Gaussian mode radiation beam is \( (\lambda/2)^2 \), and such a beam is referred to as completely transversely coherent. It then follows that the transversely coherent flux of a radiation beam is

\[
F_{\text{coherent}} = \left(\frac{\lambda}{2}\right)^2 B_0
\]  

and the degree of transverse spatial coherence is

\[
\frac{F_{\text{coherent}}}{F_0} = \left(\frac{\lambda}{2}\right)^2 \frac{B_0}{F_0}
\]  

Conversely, the number of Gaussian modes occupied by a beam is

\[
\frac{F_0}{F_{\text{coherent}}} = \frac{F_0}{B_0(\lambda/2)^2}
\]  

Transverse spatial coherence is the quantity which determines the throughput of phase sensitive devices such as Fresnel zone plates used for x-ray microscopy.
55.4 COHERENCE OF SYNCHROTRON RADIATION EMISSION IN THE LONG WAVELENGTH LIMIT

We now discuss coherent effects which depend upon the phase coherence between electric fields emitted by different electrons within a bunch. These effects naturally become stronger as the emission wavelength increases to values greater than the bunch length, the so-called long wavelength limit, which is generally in the far-infrared range or beyond for standard storage ring parameters. In the evolution from 3rd to 4th generation sources, the bunch length decreases, thereby pushing the coherent emission to shorter wavelengths, into the UV, VUV, and beyond. One way to generate shorter bunches is with single-pass devices such as linacs, wherein the electrons can be closer packed since relaxation processes which occur in storage rings may not have had time to develop.

In this section, we confine our discussion to coherent synchrotron radiation emission in the long wavelength limit. We describe two types of coherence: (i) longitudinal, or temporal (sometimes called multiparticle or super-radiant) coherence; and (ii) transverse or lateral coherence.

**Temporal (Longitudinal/Multiparticle/Super-radiant) Coherence**

Under normal circumstances in a storage ring, the bunch length is considerably longer than the emitted wavelength, and since the electrons are randomly distributed, there is no phase correlation between the emitted electric fields from different electrons. For \( N_e \) particles emitting, then, the amplitude of the electric field is generated from the statistical noise, which is proportional to \( N_e^{1/2} \). This implies that the power, which goes like \( E^2 \), is proportional to \( N_e \).

In situations in which the wavelength is much longer than the bunch length, the phase differences between the electric fields of the electrons are small compared to the wavelength, and the field is \( N_e \) times that of a single electron. In this regime, the intensity is \( N_e^2 \) times that of a single electron, or \( N_e \) times greater than the incoherent process. This effect is very large because \( N_e \) is generally quite large: the number of electrons per bunch can easily be in the nanoCoulomb range, i.e., containing \( 10^{10} \) to \( 10^{11} \) electrons.

Long-wavelength coherent emission from bunches of relativistic charged particles was first described theoretically by Nodvick and Saxon\(^25\) in 1954. For the situation of multiple electrons, the more general expression for the flux emitted by an electron bunch as a function of frequency (\( \omega \)) and solid angle (\( \Omega \)) is derived by extending the expressions derived earlier for a single electron [Eq. (5)], to a system of \( N_e \) particles, thus:

\[
\frac{d^2 F_{\text{em}}(\omega)}{d\Omega d\omega} = 1.326 \times 10^{13} \text{photons/sec/mrad}^2/\%\text{bandwidth} \times [N_e[1 - f(\omega)] + N_e^2 f(\omega)]
\times E^2 |\text{GeV}^2| |A(1 + \gamma^2 \psi^2)^2 \left( \frac{\omega}{\omega_0} \right)^2 \left[ K^2_{213}(\xi) + \frac{\gamma^3 \psi^2}{1 + \gamma^2 \psi^2} K^2_{113}(\xi) \right]]
\]

(49)

where the term \( N_e^2 f(\omega) \) represents the coherent enhancement and includes the form factor \( f(\omega) \), which is the Fourier transform of the normalized longitudinal particle distribution within the bunch, i.e.,

\[
f(\omega) = \left| \int_{-\infty}^{\infty} e^{i \omega z} S(z) dz \right|^2
\]

(50)

where \( S(z) \) is the distribution function for particles in the bunch, measured relative to the bunch center.
If we assume that the electron bunch has a longitudinal Gaussian particle distribution of width $\sigma_z$, the form factor will be

$$f(\omega) = e^{\left(\frac{\omega \sigma_z}{\omega_o}\right)^2} = e^{-4\pi^2 \sigma_z^2 \omega^2 / \omega_o^2}$$  \hspace{1cm} (51)$$

where $\lambda$ is the wavelength of the light at frequency $\omega$.

The long-wavelength coherent enhancement is shown graphically for a chosen set of beam parameters in Fig. 11.

### Transverse, or Lateral, Coherence

Transverse coherence was defined and described earlier in this chapter in the context of emission from undulators. In the long wavelength limit, even the emission from bending magnets becomes transversely coherent. In this section, we derive the interesting result that, in the long wavelength limit, the flux emitted into the natural opening angle (transverse coherent limit) is constant for all wavelengths, independent of storage ring parameters such as electron energy and bend radius.

1. Natural opening angle for synchrotron radiation for $\omega < \omega_o$:

$$\theta_{rms} = 0.8282 \left(\frac{\lambda(m)}{2\pi \rho(m)}\right)^{1/3} \text{ radians}$$  \hspace{1cm} (52)$$

2. For $\omega < \omega_o$, the universal function

$$G_1\left(\frac{\omega}{\omega_o}\right) = \frac{\omega}{\omega_o} \int_{-\infty}^{\infty} K_{3/2}(y)dy$$
from Eq. (6) and Fig. 3 can be approximated by

\[
G_1 \left( \frac{\omega}{\omega_c} \right) = \frac{1.33 \times 8 \times \pi}{9 \times \sqrt{3}} \left( \frac{\omega}{\omega_c} \right)^{\frac{1}{3}} = 2.144 \left( \frac{\omega}{\omega_c} \right)^{\frac{1}{3}}
\]

3. Therefore, the formula for angle integrated flux \( F \) for \( \omega < \omega_c \) into horizontal collection angle \( \theta \) is

\[
F = 5.27 \times 10^{16} \text{ photons/sec/0.1% bandwidth} \times \theta [\text{rad}] \times E (\text{GeV}) \times I (\text{A}) \left( \frac{\omega}{\omega_c} \right)^{\frac{1}{3}}
\]

4. Critical wavelength \( \lambda_c \) is given by:

\[
\lambda_c = 5.59 \times 10^{-10} \left( \frac{\rho [m]}{E [\text{GeV}^3]} \right)\frac{2\pi c [m/sec]}{\omega_c [\text{sec}^{-1}]}
\]

so that

\[
\omega_c [\text{sec}^{-1}] = \frac{2\pi c [m/sec] E [\text{GeV}^3]}{5.59 \times 10^{-10} \rho [m]}
\]

5. Substituting Eqs. (52) and (55) into Eq. (53) and using \( 4 \times \theta_{rms} \) for the horizontal collection angle \( \theta \), we obtain:

\[
F = 5.374 \times 10^{16} \text{ photons/sec/0.1% bandwidth} \times E (\text{GeV}) \times I (\text{A})
\]

\[
\times \left( \frac{5.59 \times 10^{-10} \rho [m] [\text{sec}^{-1}]}{2\pi c [m/sec] E [\text{GeV}^3]} \right)^{\frac{1}{3}} \times 4 \times 0.828 \left( \frac{2\pi c [m/sec]}{2\pi \rho [m] \omega [\text{sec}^{-1}]} \right)^{\frac{1}{3}}
\]

or

\[
F = 7.94 \times 10^{13} \text{ photons/sec/0.1% bandwidth} \times I (\text{A})
\]

Figure 12 shows a flux plot based on an exact calculation for a synchrotron storage ring source with 3-GeV electron energy and 5-meter bend radius. This plot agrees with Eq. (57) from \( -2 \times 10^{-4} \omega_c \).
to $\omega$. The curve turns downward at the low-energy end owing to the finite size of the ring vacuum chamber. At the upper energy end it also turns downward owing to a failure of the approximation in Eq. (53) at and above $\omega_c$.

For consistency as well as interest, another route to the result in Eq. (57) is via the undulator flux formula given by Eq. (29), with $K = 1, n = 1,$ and $N = 1.$ In this case, intended to approximate a bending magnet, $F(K) = 0.37$ and one obtains, from Eq. (29), a flux value of $F = 7.99 \times 10^{13}$ photons/sec/0.1%bandwidth $\times I[A]$, in remarkable agreement with Eq. (57).

55.5 CONCLUSION

We have attempted to compile the formulae needed to calculate the flux, brightness, polarization (linear and circular), and power produced by the three standard storage ring synchrotron radiation sources: bending magnets, wigglers and undulators. Where necessary, these formulae have contained reference to the emittance ($\epsilon$) of the electron beam, as well as to the electron beam size ($\sigma$) and its divergence ($\sigma'$). For all three types of sources, the source phase space area, i.e., the spatial and angular extent of the effective (real) source, is a convolution of its electron and photon components. For a more detailed description of these properties, see Ref. 26 and references therein.

55.6 REFERENCES


