Diffractive model of the high-frequency impedance

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High frequency diffraction can be described by iterations based on an approximate formulation of the boundary conditions. The method formulated is analogous to the Born series of scattering theory. It is used to study the interaction of short bunches with the beam environment in terms of the impedances. The impedances of typical elements of an accelerator structure are obtained. The cross-talk between elements, the impedance of a periodic array, and the effect of a taper are discussed. The method can be applied to a cavity of an arbitrary shape.

I. INTRODUCTION

Stability of a particle in an accelerator in the relativistic case, where direct interaction between particles is negligible, depends mostly on the interaction with EM wakefields, generated by a bunch of particles through all variations of the beam pipe cross-section. The wakefields are responsible for such effects as additional heat load, change of energy spread, emittance growth, and collective instabilities.

The wakefield can be considered as a linear response of the system to the beam current. In the linear approximation the Fourier frequency harmonics of the field and of the current are proportional and for a given frequency the coefficient of proportionality, which is the impedance, depends only on the geometry and the wall resistivity. The wake potential,

related to the impedance by the Fourier transformation, describes the bunch-environment interaction in the time domain. The impedance and the wake potential are important characteristics of an accelerator.

Impedances can be calculated numerically. The powerful numeric codes such as URMEL, TBCI, MAFIA, and some others are available¹. Nevertheless, analytic estimations can be very useful, especially for very short bunches with the bunch length of a few picoseconds and a frequency content up to hundreds of gigahertz, where numeric calculations are limited by an increasing number of mesh points.

In this paper, we study the high-frequency behavior of the impedance using simple models based on an approximate formulation of boundary conditions in diffraction theory. This approach is very close to the diffractive model discussed by M. Sands and K. Bane² and B. Palmer³.

In the beginning we formulate the systematic iterative approach based on an approximate formulation of the boundary conditions. Then the impedance is estimated in the first approximation for two typical cases: an abrupt change of the beam pipe radius (the case referred to later as a step) and a pill-box cavity. These two cases are of interest because most of the impedance generating elements may be approximated by one of them. After that the model is used to discuss problems which do not yet have well established solutions: the cross-talk between impedance generating elements, the impedance of a periodic array of cavities, and the effect of tapering. The analytic results are compared whenever it is possible with the results obtained by more rigorous methods and with calculations with TBCI.

II. THE METHOD

The field of a free particle with a charge e moving with velocity v in the z direction is the solution of the wave equation

$$\Delta \vec{E}_\omega + k^2 \vec{E}_\omega = 4\pi \nabla \rho_\omega - \frac{4\pi i \omega}{c^2} \vec{j}_\omega \quad (1)$$

with appropriate boundary conditions on the conductive walls. Here $k = \omega/c$ and the Fourier components of the current $\vec{j}_\omega = \hat{z}j_\omega$ and density ρ_ω are

$$j_\omega = \int dt j(t) e^{i\omega t} = e\rho(\vec{r}_\perp) e^{ikz}, \quad \rho_\omega = j_\omega/v, \quad \int \rho(\vec{r}_\perp) d^2r_\perp = 1. \quad (2)$$

For a point-like particle in free space the solution is

$$\begin{aligned} E_{\omega,r}^{part} &= \frac{2e\tau}{v} K_1(\tau r) e^{ikz/\beta}, \\ E_{\omega,z}^{part} &= \frac{-2ie\tau^2}{\omega} K_0(\tau r) e^{ikz/\beta}, \\ B_{\omega}^{part} &= \frac{2e\tau}{c} K_1(\tau r) e^{ikz/\beta}, \end{aligned} \quad (3)$$

where $\beta = v/c$ and $\tau = k/\beta\gamma$. This field propagates synchronously with the particle.

For an ultrarelativistic particle $E_{\omega,z}^{part} = 0$. $E_{\omega,r}^{part}$ is exponentially small for $r > \gamma/k$, and for $r < \gamma/k$

$$E_{\omega,r}^{part} = \frac{2e}{vr} e^{ikz/\beta}, \quad B_{\omega,r}^{part} = \frac{2e}{cr} e^{ikz/\beta}. \quad (4)$$

For a particle in a perfectly conductive straight pipe with the radius a the solution of the wave equation gives

$$\begin{aligned} E_{\omega,r} &= \frac{2e\tau}{v} [K_1(\tau r) + I_1(\tau r) \frac{K_0(\tau a)}{I_0(\tau a)}] e^{ikz/\beta}, \\ E_{\omega,z} &= \frac{-2ie\tau^2}{\omega} [K_0(\tau r) - I_0(\tau r) \frac{K_0(\tau a)}{I_0(\tau a)}] e^{ikz/\beta}. \end{aligned} \quad (5)$$

The first terms here are the same as for a free particle. The second terms give the field of the image current in the beam pipe wall. They are of the order of γ^{-2} for $\gamma \rightarrow \infty$, and the field is the same as for a free particle.

For an arbitrary cavity the field is the sum of the synchronous field Eq.(3) and the radiated field. The radiated field satisfies the homogeneous wave equation and its value in the cavity is defined by the fields on the inner surface of the cavity⁴:

$$\vec{E}_\omega = \int dS' [ikG(\vec{n}' \times \vec{B}_\omega) + (\vec{n}' \vec{E}_\omega) \nabla' G + \vec{E}_\omega (\vec{n}' \vec{\nabla}') G - \vec{n}' (\vec{E}_\omega \vec{\nabla}' G)] \quad (6)$$

$$\vec{B}_\omega = \int dS' [-ikG(\vec{n}' \times \vec{E}_\omega) + (\vec{n}' \vec{B}_\omega) \nabla' G + \vec{B}_\omega (\vec{n}' \vec{\nabla}') G - \vec{n}' (\vec{B}_\omega \vec{\nabla}' G)]$$

where $\vec{n}$ is the unit vector normal to the surface pointed inside of the volume and

$$G(\vec{r}, \vec{r}') = \frac{e^{ikR}}{4\pi R}, \quad R = |\vec{r} - \vec{r}'| = \sqrt{(z - z')^2 + r^2 + r'^2 - 2rr' \cos \phi}$$

is Green's function of the wave equation

$$(\Delta + k^2)G = -\delta(\vec{r} - \vec{r}')$$

It can be as well represented for $k > 0$ in the form

$$G = \frac{i}{8\pi} \int_{-\infty}^{\infty} dq e^{iq(z-z')} [J_0(rs) H_0^{(1)}(r's) + 2 \sum_{m=1}^{\infty} J_m(rs) H_m^{(1)}(r's) \cos m\phi] \theta(r' - r) + (r \leftrightarrow r') \quad (7)$$

where $s = \sqrt{k^2 - q^2}$ for $|q| < k$, and $s = +i\sqrt{q^2 - k^2}$ for $|q| > k$. G_k for $k < 0$ is given by $G_{-k} = G_k^*$.

For simplicity we consider the azimuthally symmetric structures with the axis of symmetry in the z -direction although the method can be formulated in a general case.

The integral equation given by Eq.(6) can be solved for an arbitrary cavity only by iterations. The fields at the boundaries of the first iteration define all successive iterations and the right choice of the fields is crucial for the method. In this paper we formulate the boundary conditions for the first iteration and study this conjecture for a number of cases.

Before we specify the fields on the surface of the cavity of the arbitrary shape $r = a(z)$, let us consider the situation in the straight pipe. The electric field of a particle in the ultrarelativistic case has only a radial (i.e. the normal to the beam pipe wall) component which induces an image current in the wall. However, as it was noticed above, the fields of the image current are small for $\gamma \rightarrow \infty$. The large normal component of the electric field of the particle does not induce the normal component of the electric field E_n , and

the tangential (azimuthal) component of the magnetic field B_t , which are zero on the pipe walls. The reason for this is that the image current in the pipe has only a tangential component.

The basic conjecture we make is that the normal component of the radiated field is zero in the high frequency limit on any conductive boundary surface

$$E_n = 0, \quad r = a(z) - 0. \quad (8)$$

At the same time, the sum of the tangential component of the synchronous field of a particle E_t^{part} , and the tangential component of the field of the image current E_t on the metallic surface has to be zero. Therefore,

$$E_t = -E_t^{\text{part}}, \quad r = a(z) - 0 \quad (9)$$

and the components of the field on the conductive boundaries are

$$E_r = -E_t^{\text{part}} \sin \alpha, \quad E_z = -E_t^{\text{part}} \cos \alpha, \quad (10)$$

where

$$E_t^{\text{part}} = (2e/cr)e^{ikz} \sin \alpha. \quad (11)$$

The angle α is the angle between the unit tangential vector $\vec{t}$ to the surface and the z -axis, $\tan \alpha = da(z)/dz$.

The magnetic field in the azimuthally symmetric case has only azimuthal component

$$\vec{B}_\omega = \hat{\phi} B_\omega. \quad (12)$$

To find the boundary condition for the magnetic field we notice that the radiated field may be found as a solution of the Maxwell equations with the appropriate image current j_ω and image charge ρ_ω rather than as a solution of the homogeneous wave equation with boundary conditions. It follows from the Maxwell equations that ρ_ω and j_ω are related by the continuity equation

$$-i\omega\rho_\omega + \text{div } j_\omega = 0. \quad (13)$$

The total field outside of the cavity is zero. Hence, the radiated field outside of the cavity $r > a(z)$ is

$$\vec{E}_\omega = -\vec{E}_\omega^{part}, \quad \vec{B}_\omega = -\vec{B}_\omega^{part}, \quad r > a(z). \quad (14)$$

We assume that there is only the surface image charge. Then the assumption Eq.(9), Eq.(14), and $\text{div } \vec{E}_\omega = 4\pi\rho_\omega$ define the image charge

$$\rho_\omega = -\frac{e}{v} \frac{\delta(r - a(z))}{2\pi a(z)} \cos^2 \alpha \quad e^{ikz/\beta}. \quad (15)$$

We assume that the image current is the surface current with the tangential component only: $\vec{j}_\omega = \vec{t}j_\omega$. Then, Eq.(13) gives

$$j_\omega = -\frac{e}{\beta} \frac{\delta(r - a(z))}{2\pi a(z)} \cos \alpha \quad e^{ikz/\beta}. \quad (16)$$

The Maxwell equation

$$\nabla \times \vec{B}_\omega = -ik\vec{E}_\omega + \frac{4\pi}{c} \vec{j}_\omega$$

and Eq.(16) give the relation between the azimuthal components on the boundary inside and outside of the cavity

$$(B_\omega)_{in} = \frac{2e}{va(z)} e^{ikz/\beta} + (B_\omega)_{out}. \quad (17)$$

The field outside of the cavity is given by Eq.(14) and Eq.(4)

$$(B_\omega)_{out} = -\vec{B}_\omega^{part}.$$

Finally, the field on the inner surface of the cavity is

$$B_\omega = 0, \quad r = a(z) - 0. \quad (18)$$

Eqs.(8),(9) and (18) specify the fields on the boundary for the first iteration.

In some cases the boundary conditions for the magnetic field may have a different form. One of them, where there is the abrupt change of the beam pipe radius, is discussed

in the next section. Another example is the semi-infinite pipe at $z < 0$ inside the infinite pipe of the larger radius. The total radial electric field between two pipes is $E_{\omega,r} = 0$ for $z < 0$. Hence, the field of the image current in the walls of the narrow pipe must cancel the field of the particles

$$E_{\omega,r} = -E_{\omega,r}^{part}$$

for $z < 0$. That gives the boundary condition at $z = 0$ for the radiated field in the region $z > 0$. Notice, that this condition is the same as that for the conductive wall Eq.(9). Therefore we can expect the same impedance in that case as for the abrupt change of the pipe radius. This is indeed the case as it has been found by an exact solution of the problem⁵.

There is no image current on the beam pipe opening, and in the first approximation we take $\vec{E} = 0$ on the opening. That means that calculating the radiated field in the cavity in the first approximation we assume unperturbed incident wave neglecting the wave diffracted in the tube. Hence, the surface integral in Eq.(6) is taken only over the metallic boundaries. The fields in the cavity found in the first approximation defines the radiated field on the boundaries, including beam pipe openings. They may be taken as the fields on the boundary for the next iterations. The series obtained in this way are analogous to Born's series of scattering theory. The parameter of expansion of the series is the ratio of the amplitude of the tangential component of the radiated field to the tangential component of the field of a particle Eq. (4). Notice, that the results of the first approximation are enough to estimate this parameter and to find the amplitude of the diffracted waves in the beam pipes.

Eq. (6), and Eqs. (8,11,18) give the E_z component in the cavity as the surface integral over the metallic walls of the cavity

$$E_{\omega z} = - \int dS' \frac{\partial G}{\partial r'} E_t(r', z'), \quad E_t = -E_t^{part}. \quad (19)$$

Let us check Eq.(19) for the straight pipe. In this case $a(z) = a$, $dS = 2\pi a dz$, and E_t^{part}

is E_z^{part} defined by Eq.(3). Eq.(19) and Eq.(7) give

$$E_{\omega,z} = \frac{2ie\tau^2}{\omega} I_0(\tau r) \frac{K_0(\tau a)}{I_0(\tau a)} e^{ikz}$$

in perfect agreement with Eq.(5).

III. A CAVITY AND A STEP

Consider a pill box cavity of the length g , and the radius b with attached tubes with the radius a . The boundary condition for B_ω on the flanges at $z = 0$, and $z = g$ of the pill-box cavity is different in this case from that of Eq.(18) although Eq.(17) is still valid. Let us consider the boundary at $z = 0$. The radiated field $(B_\omega)_{out} = 0$ for $z < 0$, because the field of a particle in the tube is screened by the image current in the tube walls. Therefore, Eq.(17) gives

$$(B_\omega)_{in} = \frac{2e}{va(z)} e^{ikz/\beta}$$

However, B_ω on the flanges for the geometry under consideration does not give any contribution to E_z and Eq.(19) is still valid. The image current in the wall $r = b$ gives the negligible small contribution of the order of $(1/\gamma)^2$. Hence, Eq.(19) gives $E_{\omega,z}(r, z)$ for $r < a$

$$E_{\omega z} = \frac{i}{8\pi} \int_{-\infty}^{\infty} dq \int_a^b 2\pi r' dr' e^{iqz} [E_{\omega,r}^{part}(r', 0) - E_{\omega,r}^{part}(r', g) e^{-iqg}] J_0(rs) \frac{\partial H_0^{(1)}(sr')}{\partial r'}$$

or

$$E_{\omega z} = \frac{ie}{2c} \int_{-\infty}^{\infty} dq e^{iqz} [1 - e^{i(k-q)g}] J_0(rs) [H_0^{(1)}(bs) - H_0^{(1)}(as)] \quad (20)$$

where $s = \sqrt{k^2 - q^2}$. The longitudinal impedance is defined as^{6,7}

$$Z_l(\omega, r) = -\frac{1}{e} \int dz e^{-ikz/\beta} E_{\omega,z}(r, z) \quad (21)$$

The result of integration actually does not depend on r^8 . It is convenient to choose the offset equal to the beam pipe radius $r = a$ because $E_\omega(a, z) = 0$ in the tubes. In this case the range of the integration in Eq.(21) is $0 < z < g$. That gives for $k > 0$

$$Z_l(k) = \frac{2}{c} \int_{-\infty}^{\infty} \frac{dq}{k - q} J_0(as) [H_0^{(1)}(as) - H_0^{(1)}(bs)] \sin^2 \frac{g}{2} (k - q) \quad (22)$$

The real part $Re Z_l(k)$ is given by q in the range $-k < q < k$:

$$Re Z_l(k) = \frac{2}{c} \int_{-k}^k \frac{dq}{k-q} J_0(as) [J_0(as) - J_0(bs)] \sin^2 \frac{g}{2} (k-q) \quad (23)$$

For estimation we can divide the integral into two parts: the first, over small $k-q < 1/ka^2$, $s < 1/a$, and the second over $k-q > 1/ka^2$, $s > 1/a$. The first integral gives a negligible contribution on the order of $(g/ka^2)^2 \ll 1$. In the second integral, the term $J_0(bs)$ may be omitted for $b \gg a$. The factor $J_0^2(as)$ may be replaced by its average value $\pi/(as)$. With the variables $x = k-q$, $1/(ka^2) < x < 2k$, the integral takes the form

$$\frac{2}{\pi c a \sqrt{2k}} \int \frac{dx}{x^{\frac{3}{2}}} \sin^2 \left(\frac{gx}{2} \right).$$

The integral converges rapidly for small $x \ll 1$, as well as for large $x \gg 1$. Therefore, the interval of integration can be expanded to the limits $0 < x < \infty$. That gives

$$Re Z_l(k) = \frac{Z_0}{2\pi a} \sqrt{\frac{g}{\pi k}} \quad (24)$$

where $Z_0 = 4\pi/c \simeq 377$ ohms, and the numeric coefficient is defined according to numeric integration of Eq. (23). Notice that the impedance does not depend on the cavity radius.

For $kb^2 \gg g \gg ka^2$

$$Re Z_l(k) = \frac{Z_0}{2\pi} \ln \left(\frac{g}{ka^2} \right). \quad (25)$$

If g is large, the cavity radius b becomes important. For $g \gg kb^2$

$$Re Z_l(k) = \frac{Z_0}{\pi} \ln \left(\frac{b}{a} \right). \quad (26)$$

We call this case "a step".

Eq.(23) shows that there is continuous transition from one case to another. The results of the numeric integration of Eq.(23) are shown in Fig. 1 together with the estimate Eqs.(24), (26). According to the simulations the transition from the cavity regime to the step regime occurs at

$$p = \frac{g}{k(b-a)^2} \approx 1. \quad (27)$$

The difference between the results for a cavity and the results for a step corresponds to two different regimes of diffraction. For small $z \ll kr^2$, i.e., for Fresnel diffraction, the typical transverse deflection of the diffracted wave increases with z as $r \sim \sqrt{\frac{2z}{k}}$, and the angle of diffraction is $\theta \sim \frac{1}{\sqrt{kz}}$. For large z the Fraunhofer diffraction takes place for which $r \sim z/ka$, $\theta \sim 1/ka$. For a step all waves with $k < 1/\sigma$ are diffracted in the Fraunhofer regime giving Eq. (26). For a cavity Fraunhofer diffraction takes place only for long waves with $k < g/a^2 < 1/\sigma$ giving Eq. (24). The criterion (27) corresponds to the condition that all waves with $k < 1/\sigma$ within the bunch spectrum are deflected in the Fraunhofer diffraction regime. A similar expression can be obtained in the approximation of the geometric optics requiring that a wave generated by a head of a bunch at $z = 0$, $r = a$ may catch up with the tail of the bunch within the cavity after reflection from the wall $r = b$.

Eq.(24) was obtained by Lawson¹¹ for a single particle and by G.Dome¹² who conjectured that the radial field in the cavity at $r = a$ is the same as that in the close pill-box cavity. This assumption was clarified in the paper¹³ where rigorous matching was carried out and the correction to Eq.(24) was estimated. Eq.(24) was confirmed also by numeric calculations¹⁴ and in diffractive models^{2,3}.

The impedance of a step, Eq. (26), was studied by numeric solution of the truncated exact system of equations¹⁶. It is shown in this paper that the high frequencies impedance is approximately independent of frequency. It is substantially different from zero only for a particle entering a wider pipe, whereas the impedance is almost zero for a particle entering a narrower pipe. For small frequencies the impedance in the last case is negative giving some total energy gain. This corresponds to attraction of a particle by the image charge in the wall of the cavity. Impedances of the two steps are different because the waves generated in two cases propagate in opposite directions. The impedances are different, however, only by a constant which corresponds to the different energy in the synchronous components of the field of a particle in the pipes of different radii.

The real part of the impedance gives the energy loss. For the Gaussian bunch with rms length σ the loss factor is^{6,7}

$$k_l = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} Z_l(\omega) e^{-k^2 \sigma^2} \quad (28)$$

If $g\sigma/a^2 \ll 1$ the loss is given mostly by the high-frequency part of the impedance. Eq.(24) and Eq.(28) give

$$k_l = \frac{\Gamma(1/4)}{\pi a} \sqrt{\frac{g}{\pi \sigma}}, \quad \frac{\Gamma(1/4)}{\pi} = 1.154 \quad (29)$$

If $g\sigma/a^2 \gg 1$ but $\sigma \ll a$ the main contribution is given by the frequencies $1 \ll ka \ll a/\sigma$. Eq.(25) gives

$$k_l = \frac{1}{\sigma\sqrt{\pi}} \ln\left(\frac{g\sigma}{a^2}\right) \quad (30)$$

For a step g has to be replaced by kb^2 . It gives the V. Balakin and A. Novokhatsky results¹⁷

$$k_l = \frac{2}{\sigma\sqrt{\pi}} \ln\left(\frac{b}{a}\right) \quad (31)$$

The transition from one regime to another takes place at^{9,10}

$$p = \frac{2g\sigma}{(b-a)^2} \approx 1. \quad (32)$$

We compared⁹ Eq. (29-31) with the code TBCI using Eq.(29) for small $p < 1$ and the formula

$$k_l = \frac{2}{\sigma\sqrt{\pi}} \ln\left\{1 + \frac{b-a}{a} \sqrt{\frac{2g\sigma}{2g\sigma + (b-a)^2}}\right\} \quad (32)$$

which is written to interpolate Eq. (30) and Eq.(31) for $p > 1$. For example, the dependence of k_l on the cavity radius b (see Fig. 2) shows the transition from the cavity regime to the step regime in good agreement with the estimate of Eq. (32). Numeric agreement is within 10% accuracy and sometimes even better. For example : TBCI gives $k_l = 5.547 V/pC$ for the parameters $a = 1$ cm, $b = 10$ cm, $g = 9.0$ cm, and $\sigma = 0.1$ cm. Eq.(29) gives $k_l = 5.56 V/pC$.

Let us now check how small the tangential component of the radiated field is which we neglected in the first approximation in comparison with the field of a particle E_i^{part} . The r - component of the radiated field can be estimated from $div \vec{E} = 0$ and Eq.(19) where the surface integral has taken over the surfaces $z = 0$ and $z = g$, $a < r < b$. Neglecting terms of order of $1/k$, we obtain

$$E_{\omega,r}(r,0) = \frac{2e}{c} \sum_n \int_0^{2\pi} d\phi \left[\frac{z - z_n}{r - a \cos \phi} \frac{e^{ikR(a)}}{4\pi R(a)} - (a \rightarrow b) \right] \quad (34)$$

where $R^2(a) = r^2 + a^2 - 2ra \cos \phi + (z - z_n)^2$, $n = 1, 2$, and the sum describes two waves radiated from the surfaces $z_1 = 0$ and $z_2 = g$. It is easy to see that the wave radiated from the surface $z = 0$ does not contribute to E_r at $z = 0$. Thus, only the wave scattered from another surface changes the boundary condition of the first approximation which is similar to the situation in the scattering theory. The phase of this wave is proportional to the wave vector k and is large for $kg \gg 1$ giving, on the average, a small correction in the high frequency limit. $E_{\omega,r}(r,0)$ has a singularity at $r = a$ as it should be. The singularity in Eq. (34) is only logarithmic, and it is not essential in the integral Eq. (19).

IV. CROSS-TALK BETWEEN CAVITIES

The wakefield generated in one element of an accelerator propagates into the adjacent elements downstream of the system and interferes there with the local wakefields. Even if the impedances of all elements considered as independent are known it is not clear in general how the interference affects the impedance of the whole system. We consider two examples: two adjacent cavities and, in the next section, the periodic structure as another extreme case.

Let us consider two cavities with the lengths g_1 and g_2 , $g_2 < g_1$, and the radii b and d , $d > b$, correspondingly. The beam pipe radius is a . The short cavity encircles the large cavity and placed symmetrically: $g = 2g_1 + g_2$. The radius of the system changes with z as follows

$$r = a, \text{ for } z < 0; \quad r = b, \text{ for } 0 < z < g_1; \quad r = d, \text{ for } g_1 < z < g_1 + g_2;$$

$$r = b, \text{ for } g_1 + g_2 < z < g; \quad r = a, \text{ for } z > g$$

According to Eq. (19), $E_{\omega,z}$ in the cavities is given by the integrals over the surfaces $z = 0$, $a < r < b$; $z = g_1$, $b < r < d$; $z = g_1 + g_2$, $b < r < d$; and $z = g$, $a < r < b$. Calculating the impedance according to Eq. (21) with the offset $r = a$ we obtain:

$$Z_l(\omega) = Z_l(a, b, g) + \delta Z_l \quad (35)$$

where the first term is the impedance Eq. (22) of the cavity with the radius b and the length g with the beam pipe of the radius a , and the second term gives a correction due to the second cavity. The last is given by the integral Eq. (21) over the waves $E_{\omega,z}(z, a)$ generated at $z = g_1$, and $z = g_1 + g_2$:

$$E_{\omega,z}(z, a) = -\frac{ie}{2c} \int_b^{d_{eff}(z)} dr' \int dq e^{iqz} J_0(as) \frac{\partial H_0^{(1)}(s, r')}{\partial r'} [e^{i(k-q)g_1} - e^{i(k-q)(g_1+g_2)}] \quad (36)$$

In the integral over dr' the upper limit depends on z because, say, at large z the surface $z = g_1$ is partially screened by the surface at $z = g_1 + g_2$. That screening is actually the second order effect caused by the surface current generated on the surface $z = g_1 + g_2$ by the wave coming from the surface $z = g_1$. Nevertheless, the screening changes the diffracted wave drastically and we take it into account in the first order iteration. Thus, we assume that the wave generated by a boundary is not zero only outside of the shade caused by other metallic boundaries. For example, the first term in Eq. (36) is not zero only for $z > g_1$ and $r < d_{eff}$, where

$$d_{eff} = \min \left(d; \quad b + g_2 \frac{b - a}{z - (g_1 + g_2)} \right) \quad (37)$$

Similarly, the wave from the surface $z = g_1 + g_2$ contributes only at $z < g_1 + g_2$. Eq. (36) and Eq. (21) give the real part of the correction:

$$\text{Re } \delta Z_l / Z_0 = \frac{1}{4\pi} \int_0^{g_1+g_2} dz \int_0^{2k} dq \sin(qz) J_0(as) [J_0(bs) - J_0(d_{eff}s)] \quad (38)$$

where $s = \sqrt{k^2 - (k - q)^2}$. The main contribution is given by $q \ll k$. Hence $s \simeq \sqrt{2kq}$ and the integration over dq can be extended up to infinity.

The integral Eq. (28) for the loss factor can be simplified if a bunch with the spectrum $\sqrt{\pi}/2 e^{-k\sigma}$ is used instead of a Gaussian bunch with the spectrum $e^{-k^2\sigma^2}$. Both distributions are normalized in the same way. Notice, that¹⁸

$$\int_0^\infty dk e^{-k\sigma} J_0(a\sqrt{2kq}) J_0(b\sqrt{2kq}) = \frac{1}{\sigma} e^{-\frac{q}{2\sigma}(b-a)^2} \tilde{I}_0\left(\frac{q}{\sigma}ab\right)$$

where $\tilde{I}_0(x) = e^{-x} I_0(x)$, and $I_0(x)$ is the modified Bessel function of the first kind. That gives the energy loss factor

$$k_l = k_l^0 + \delta k_l \quad (39)$$

where the first term is the loss for a cavity with parameters (a, b, g) :

$$k_l^0 = \frac{1}{\sigma\sqrt{\pi}} \int_0^\infty \frac{dx}{x} \sin^2\left(\frac{\sigma}{2}gx\right) [\tilde{I}_0(xa^2) - e^{-\frac{\sigma}{2}(b-a)^2} \tilde{I}_0(xba)], \quad (40)$$

and δk_l describes the effect of the short cavity. For long g_2 ,

$$g_2 > \frac{g(d-b)}{d+b-2a},$$

$$\delta k_l = \frac{1}{\sigma\sqrt{\pi}} \int_0^\infty \frac{dx}{x} \sin^2\left(\frac{\sigma}{2}g_{eff}x\right) [e^{-\frac{\sigma}{2}(b-a)^2} \tilde{I}_0(xba) - e^{-\frac{\sigma}{2}(d-a)^2} \tilde{I}_0(xda)], \quad (41)$$

where $g_{eff} = \min(g_1 + g_2, g_2 \frac{d-a}{d-b})$. Otherwise, δk_l is the sum of Eq. (41) and $\delta \tilde{k}_l$,

$$\delta \tilde{k}_l = \frac{1}{2\sqrt{\pi}} \int_{g_{eff}}^{g_1+g_2} dz \int_0^\infty dx \sin(\sigma zx) [e^{-\frac{\sigma}{2}(b-a)^2} \tilde{I}_0(xba) - e^{-\frac{\sigma}{2}(d(z)-a)^2} \tilde{I}_0(xd(z)a)] \quad (42)$$

where

$$d(z) = b + g_2 \frac{b-a}{z-g_2}.$$

The loss factor calculated according to Eqs. (39-42) is compared with that given by TBCI in Fig. 3, where k_l is given as a function of the gap g_2 of the short cavity. The agreement is within 15% accuracy, and much better for $g_2 = 0$ or $g_2 = g$ where the structure degenerates into a single cavity.

For small g_2 impedance of the structure is the same as for a single cavity with parameters a, b, g . In that sense impedance does not depend on the shape of a cavity²¹. However, it is not a general statement as it is clear from Fig. 3.

V. PERIODIC ARRAY

For an array of cavities where the number of cavities is large, the cross-talk between cavities can be enhanced and that can drastically change the impedance. Let us consider a periodic array of diaphragms with the iris radius a separated by the distance L . For simplicity we assume that the outer iris radius is infinite. For a single cavity in this case $\text{Re } Z_l(k) \propto 1/\sqrt{k}$.

The radiated field at an iris located at z is the sum of the field Eq. (9) and the wave $f(k, r)$ diffracted at the irises upstream in the array:

$$E_{\omega, r}(z, r) = -\frac{2e}{cr} e^{ikz} \theta(r - a) + \frac{2e}{c} f(k, r) e^{ikz} \theta(a - r). \quad (43)$$

Here $\theta(x)$ is a step function. Eq. (19) with Eq. (43) used for E_t yields the field between the irises $0 < z < L$:

$$E_{\omega, z}(z, r) = \frac{ie}{2c} \int_{-\infty}^{\infty} dq e^{iqz} [1 - e^{i(k-q)L}] \int_0^{\infty} dr' [\theta(r' - a) - r' f(k, r') \theta(a - r')] \\ \frac{\partial}{\partial r'} [J_0(rs) H_0^{(1)}(r's) \theta(r' - r) + J_0(r's) H_0^{(1)}(rs) \theta(r - r')] \quad (44)$$

where $s = \sqrt{k^2 - q^2}$. The impedance per cell is

$$Z_l(k) = \frac{2}{c} \int_{-\infty}^{\infty} \frac{dq}{k - q} \sin^2 \frac{L}{2} (k - q) J_0(as) H_0^{(1)}(as) [1 + af(k, a)] \\ - \frac{2}{c} \int_{-\infty}^{\infty} \frac{dq}{k - q} \sin^2 \frac{L}{2} (k - q) H_0^{(1)}(as) \int_0^a dr' J_0(sr') \frac{\partial}{\partial r'} r' f(k, r') \quad (45)$$

The first term has the same structure as that for a single cavity except the additional factor $[1 + af(k, a)]$. We will see that this factor is of the order of $1/k$ whereas the second

term in Eq. (45) is small. This modifies the high frequency behavior of the impedance of a periodic array to $Re Z_l(k) \propto k^{-3/2}$.

The equation for the function $f(k, r)$ can be obtained from the condition of periodicity, i.e., at $z = L$, $r < a$ the radial component of the field $E_{\omega, r}(L, r)$ has to be given again by Eq. (43). The $E_{\omega, r}(z, r)$ is given by $div \vec{E} = 0$ and Eq. (44). At $z = L$ this yields

$$E_{\omega, r}(L, r) = -\frac{e}{2c} \int_{-\infty}^{\infty} q dq e^{iqL} \int_0^{\infty} dr' [\theta(r' - a) - r' f(r') \theta(a - r')] \\ [J_1(rs) H_1^{(1)}(r's) \theta(r' - r) + J_1(r's) H_1^{(1)}(rs) \theta(r - r')] \quad (46)$$

Thus, the condition of periodicity gives the integral equation for $f(k, r)$:

$$f(k, r) = \Phi(r) + \int_0^r r' dr' \Psi(r', r) f(k, r') + \int_r^a r' dr' \Psi(r, r') f(k, r') \quad (47)$$

where

$$\Psi(r, r') = \frac{1}{4} \int_{-\infty}^{\infty} q dq e^{-i(k-q)L} J_1(rs) H_1^{(1)}(r's), \quad (48)$$

and

$$\Phi(r) = - \int_a^b dr' \Psi(r, r'). \quad (49)$$

Notice, that $f(k, 0) = 0$.

The function $\Psi(r, r')$ describes the Fresnel diffraction. The estimate gives:

$$\Psi(r, r') \simeq \Psi_0(r, r'), \\ \Psi_0(r, r') = \frac{1}{2} \sqrt{\frac{k}{2\pi L r r'}} e^{i \frac{k}{2L} (r-r')^2 - i \frac{\pi}{4}} \quad (50)$$

for $r' - r \gg \sqrt{2L/k}$, and

$$\Phi(r) \simeq \frac{-iL}{k} \frac{1}{a-r} \Psi_0(r, a) \quad (51)$$

for $a - r \gg \sqrt{2L/k}$. Eq. (46) can be written in this approximation as

$$f(k, r) = \Phi(r) + \int_0^a r' dr' \Psi_0(r, r') f(k, r') \quad (52)$$

The first term describes the diffraction at a single iris. The function oscillates rapidly and therefore negligible for $(a-r) > \sqrt{2L/k}$. The solution by iterations gives the n th iteration

$$f_n(k, r) \propto e^{\frac{ik}{2L(n+1)}(r-r')^2}.$$

Hence, the function gets broader with n as $(r' - r) \simeq \sqrt{(2L/k)(n+1)}$. The function $f(k, r)$ becomes smooth after $n \simeq ka^2/L$ iterations. Iterations describe diffraction at the consecutive irises in the array. B. Palmer noticed that this gives the estimate of the number of cavities sufficient for the array of cavities to be described by the impedance of the periodic structure.

The function $\Psi(r, r')$ is a sharp function of $r - r'$ and for $ka^2/L \gg 1$ represents the δ function

$$\lim(k \rightarrow \infty) \Psi(r, r') = \frac{1}{r} \delta(r - r')$$

and Eq.(52) gives in this limit $af(a) = -1$, or $1 + af(a) \sim (1/k)$. The radiated field $f(k, r)$ at the iris increases from $f = 0$ at $r = 0$ to $f \approx -1/a$ at $r = a$ and decreases as $-1/r$ for $r > a$. Recall that for a single cavity $E_{\omega, r}$ has the discontinuity at $r = a$ changing from zero to $-1/a$. For numeric simulations it is convenient to introduce the function

$$\frac{\partial}{\partial r} r f(r) = -\Lambda \sqrt{\frac{r}{a}} F(a - r) \quad (53)$$

where

$$\Lambda = 1 + af(a). \quad (54)$$

Eqs. (52-54) give the integral equation

$$F(a - r) = \Psi_0(a, r) + \int_0^a dr' F(a - r') \Psi_0(r, r') \quad (55)$$

We solved Eq.(54) numerically for different frequencies k . The typical behavior of the function $rf(r)$ is depicted in Fig.4. The parameter $\Lambda(k)$ has been found from these calculations and its dependence on $ka^2/2L$ is shown in Fig.5. It wobbles around

$$\Lambda \approx \frac{4\pi L}{ka^2} \quad (56)$$

The same behavior of $\Lambda(k)$ has been noticed before, see the reference¹⁹. The function $F(a - r)$ oscillates rapidly, see Fig. 6. Thus the last term in Eq. (45) is small. The remaining term has the same structure as that for a single cavity, but has the additional factor $\Lambda \propto 1/k$. Hence, the impedance of the periodic array decreases with frequencies as $k^{-3/2}$. For the real part we get

$$\text{Re}Z(\omega) = Z_0 \frac{2}{\sqrt{\pi}} \left(\frac{L}{ka^2} \right)^{3/2} \quad (57)$$

whereas it goes down as $k^{-1/2}$ for a single cavity. The $k^{-3/2}$ behavior is the main result of the optical resonator model¹⁵.

The difference in the ω dependence of the impedances obtained with the optical resonator model and for a single cavity has been explained in the reference¹⁹. The impedance of the array with an arbitrary number M of periods is given in closed form by an approximate solution of the exact system of equations obtained by field matching. It is shown that for $M \ll ka^2/L$ the impedance per period is the same as that for a single cavity. The length $ML \simeq ka^2$ is a typical coherent length of waves radiated with the Fraunhofer angle k/a along the axis. The same expression for $M \gg (ka^2/L)^{3/2}$ gives the impedance of the optical resonator model. The transition from the regime of a single cavity to the regime of a periodic array takes place for M in the interval $ka^2/L \ll M \ll (ka^2/L)^{3/2}$. For an accelerator with fixed length $ML > a^2/\sigma$ the impedance behaves asymptotically as $k^{-3/2}$. The $k^{-3/2}$ behavior of the impedance has been obtained also by B. Gluckstern²².

VI. TAPER

If a jump of a beam pipe radius is not abrupt, but tapered, the losses become smaller. There are no analytic results available on the effect of a taper today. From the results of the previous discussion we can expect that taper affects the waves with small k diffracted into the large angles giving the main contribution to the impedance. The taper may be effective if the taper angle (which is $\pi/2$ for a step) is comparable or less than the Fresnel

diffraction angle. This is indeed the case. As it is shown below, the loss decreases with the increasing length of the taper until the loss reaches a plateau at $g_1 \simeq (b-a)^2/\sigma$ remaining almost constant with the further increase of g_1 . It follows, in particular, that for short cavities in the "cavity regime", $g\sigma/2(b-a)^2 \ll 1$, a taper can not be efficient.

Thus, studying the effect of a taper we consider a very long cavity with the length g , $0 < z < g$, and with attached tubes. The radii of the tubes and the cavity are a and b correspondingly. We taper the cavity so that the radius is changed linearly from $r = a$ to $r = b$ within the length $0 < z < g_1$ where g_1 is small in comparison with the total length $g_1 \ll g$. If g is sufficiently large the results are independent of g . Actually, to simplify consideration, we consider a symmetric cavity tapered at both ends.

Eq. (11) and Eq. (19) give

$$E_{\omega,z}(z,r) = \frac{ie}{2c} \int_a^b dr' \int_{-\infty}^{\infty} dq J_0(as) e^{iqz} \frac{\partial H_0^{(1)}(r's)}{\partial r'} \{ e^{i(k-q)[r'-a] \cot \alpha} - e^{i(k-q)[g-(r'-a) \cot \alpha]} \} \quad (58)$$

where $s = \sqrt{k^2 - q^2}$, and $\cot \alpha = g_1/(b-a)$.

The real part of the impedance Eq. (21) takes the form

$$Re Z_l(k) = -\frac{Z_0 k}{2\pi} \int_a^b dr' \int_{-k}^k \frac{dq}{k-q} J_0(as) \frac{\partial J_0(r's)}{\partial r'} \{ \sin^2 \left[\frac{k-q}{2} (r'-a) \cot \alpha \right] - \sin^2 \left[\frac{k-q}{2} (r'-a) \cot \alpha \right] \} \quad (59)$$

For $g \gg g_1$ the first term is simply the impedance of the pill box cavity of the length g with attached tubes. The loss of the long cavity is given with good accuracy by Eq. (31). The second term is the impedance of the cavity with the length $g_{eff} = g_1 y$, $0 < y < 1$, and describes the effect of the taper. The main contribution is given by $y \simeq 1$ or $g_{eff} \simeq g_1$. Therefore, the effect of the taper can be described subtracting the loss of the cavity with the length g_1 given by Eq. (29) and Eq. (33), from the loss Eq. (31) of the nontapered cavity with the length g . Figure 7 shows the result of the loss calculated in this way (solid

lines) in comparison with TBCI results as a function of the length g_1 of the taper. The agreement is surprisingly good. For a very long taper the loss goes to zero. But if a cavity is tapered only from one side, the taper can reduce the loss two times at best. The calculations with TBCI confirm this statement.

IV. CONCLUSION

The main results of this paper are two-fold. First, the handy formulas describing the high frequency longitudinal impedance of the typical elements have been rederived. We also discussed more difficult problems for which answers are not well established or unknown, such as the cross-talk between different elements of the system, transition to a periodic array, and the effect of taper. We compared these results with the results of TBCI whenever it was possible. The agreement of the results confirms that the conjecture of Eqs. (9),(18) for the boundary conditions of the first approximation and demonstrates that the method may be used for a cavity of arbitrary shape. Secondly, we formulated the iterative method defining corrections to these results. In zero approximation the field is given by Eq.(19). In the next approximation the field in the cavity from Eq.(19) gives the boundary conditions for the next iteration. The series obtained in this way is similar to the Born series of theory of scattering. This approach can be used for calculations of the transverse impedance and can be generalized for calculations of the high-frequency impedances of a cavity without cylindrical symmetry. This method gives the iterative solution to problems of the short wave diffraction and might be useful in applications not related to accelerator physics.

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