SIMPLIFIED HELIUM REFRIGERATOR CYCLE ANALYSIS USING THE ‘CARNOT STEP’

P. Knudsen, V. Ganni

Thomas Jefferson National Accelerator Facility
Newport News, VA, 23606, USA

ABSTRACT

An analysis of the Claude form of an idealized helium liquefier for the minimum input work reveals the ‘Carnot Step’ for helium refrigerator cycles. As the ‘Carnot Step’ for a multi-stage polytropic compression process consists of equal pressure ratio stages; similarly for an idealized helium liquefier the ‘Carnot Step’ consists of equal temperature ratio stages for a given number of expansion stages. This paper presents the analytical basis and some useful equations for the preliminary examination of existing and new Claude helium refrigeration cycles.

KEYWORDS: helium, refrigerator, liquefier, cycle, Claude, Carnot

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INTRODUCTION

It is the goal of the process designer to maximize the system efficiency to accomplish the given process task. From thermodynamics, we know that the optimal cycle design to accomplish a given process task is a reversible process path or cycle. This analysis is intended to provide guidance for designing a helium refrigerator cycle that yields the maximum efficiency for a given number of expansion stages. This paper specifically addresses the Claude form of refrigeration cycle typically used for helium refrigeration systems.

A liquefier is a cycle that is intended purely for a liquefaction load (varying charge). Whereas a refrigerator cycle is purely intended for a refrigeration load (fixed charge). In this paper, a helium refrigerator is a cycle that is required to provide both a liquefaction and refrigeration load (i.e., a ‘mixed’ load). This paper will not address a pure refrigerator. All practical systems must handle component non-idealities (i.e., losses) and the optimum process path will be shown to be the same as that required for a pure liquefier.

FIGURE 1. The Carnot Liquefier [2].

In the following analysis, helium will be the working fluid and is assumed to be an ideal gas with a constant specific heat. Although this assumption may initially appear crude, it is quite adequate for establishing the optimum path from ambient temperature (say 300K) to around 12K. Cycle details below 12K, or 'cold end' issues, will not be specifically addressed (refer to [1]). The principles presented in this paper are indeed applicable for other working fluids, as long as the fluid can be approximated as an ideal gas within the specified temperature range.

IDEAL HELIUM LIQUEFIER

This benchmark that all other cycles (discussed in this paper) will be compared to is the Carnot liquefier depicted below in FIGURE 1 [2]. The gas is isothermally compressed, and then isentropically expanded to yield saturated liquid. The output work from the expander is recovered to reduce the input power required by the compressor.

The required work for the Carnot liquefier is,

$$\dot{W}_{Carnot} = \dot{m}_L \cdot (T_0 \cdot \Delta s - \Delta h) \quad (1)$$

Where, $\dot{W}_{Carnot}$ – liquefier Carnot work [W]

$\dot{m}_L$ – liquefaction rate or flow [g/s]

T_0 – 'zero-grade' exergy temperature [K] = 300K

Δs – entropy change due to isothermal compression [J/g-K]

Δh – enthalpy change from isentropic expansion [J/g]

If the isothermal compression is performed at T_0 (i.e., $T_2 = T_1 = T_0$) and using an assumption of an ideal gas with a constant specific heat, C_p [J/g],

$$\frac{\dot{W}_C}{\dot{m}_L} = T_0 \cdot \Delta s = T_1 \cdot C_p \cdot \ln \left(\frac{T_1}{T_L} \right) = T_1 \cdot C_p \cdot \phi \cdot \ln(\Gamma) \quad (2)$$

$$\frac{\dot{W}_X}{\dot{m}_L} = \Delta h = C_p \cdot (T_1 - T_L) \quad (3)$$

Where, $T_L = T_3$; is the load temperature [K]

$\Gamma = p_2/p_1$; compressor pressure ratio

$\phi = (\gamma - 1)/\gamma$; with γ the ratio of specific heats

$$\text{So,} \quad \dot{W}_{Carnot}^{\circ} = \dot{W}_C^{\circ} + \dot{W}_X^{\circ} = \dot{m}_L \cdot C_p \cdot \left[T_1 \cdot \ln \left(\frac{T_1}{T_L} \right) - (T_1 - T_L) \right] \quad (4)$$

Where, the super-script ‘ $\circ$ ’ signifies that the assumption of an ideal gas with constant C_p was invoked. For the above process,

$$\frac{\dot{W}_C^{\circ}}{\dot{m}_L} = T_0 \cdot C_p \cdot \phi \cdot \ln(\Gamma) \cong T_0 \cdot \Delta s \quad (5)$$

With, $\Delta s = 28.01 \text{ J/g-K}$, the required pressure ratio is $\Gamma = 718,200$. Obviously, this pressure ratio is impractical in real systems.

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The FIGURE 2 below depicts the process of cooling a mass flow rate of $\dot{m}_L$ from a temperature of T_1 to T_{N+1} . Depicted are ‘ N ’ reversible heat pumps (RHP $_i$, with $i = 1$ to N), that remove ‘ $Q_{L,i}$ ’, the amount of heat energy at a ‘low’ temperature T_{i+1} , and in turn dissipate ‘ $Q_{H,i}$ ’, the amount of heat energy to the ambient temperature at T_i . This idealization is the liquefaction process; continuously cooling the mass flow to a successive lower temperature at each stage, with the temperature of the heat energy being transferred varying at each stage.

The required net input work for the idealization presented in FIGURE 2 is equal to the Carnot liquefier if the number of reversible heat pumps (RHP’s) approaches infinity and the assumption of an ideal gas with constant specific heat is valid.

For each RHP $_i$, if the heat exchanger (HX) is perfect, the cooling provided is,

$$q_{L,i} = \dot{m}_L \cdot C_p \cdot (T_i - T_{i+1}) \quad (6)$$

$$\text{The reversible work input required for RHP}_i \text{ is, } \dot{W}_{RHP,i} = q_{L,i} \cdot \left(\frac{T_i}{T_{i+1}} - 1 \right) \quad (7)$$

FIGURE 2. Liquefier Idealization.

$$\text{So,} \quad \dot{W}_{RHP} = \sum_{i=1}^N \dot{W}_{RHP,i} = \dot{m}_L \cdot C_p \cdot \sum_{i=1}^N \left[T_1 \cdot \left(\frac{T_i}{T_{i+1}} - 1 \right) - (T_i - T_{i+1}) \right] \quad (8)$$

$$\text{With,} \quad \Delta T_i = T_i - T_{i+1}$$

$$\text{So,} \quad \dot{W}_{RHP} = \dot{m}_L \cdot C_p \cdot \left\{ \left[T_1 \cdot \sum_{i=1}^N \left(\frac{\Delta T_i}{T_{i+1}} \right) \right] - (T_1 - T_{N+1}) \right\} \quad (9)$$

$$\text{As,} \quad \Delta T_i \rightarrow dT \text{ and } N \rightarrow \infty$$

$$\begin{aligned} \dot{W}_{RHP} &= \dot{m}_L \cdot C_p \cdot \left\{ \left[T_1 \cdot \int_{T_1}^{T_{N+1}} \frac{dT}{T} \right] - (T_1 - T_{N+1}) \right\} \\ \dot{W}_{RHP} &= \dot{m}_L \cdot C_p \cdot \left[T_1 \cdot \ln \left(\frac{T_1}{T_{N+1}} \right) - (T_1 - T_{N+1}) \right] \end{aligned} \quad (10)$$

$$\text{With,} \quad T_{N+1} = T_L \text{ this verifies that } \dot{W}_{RHP} = \dot{W}_{Carnot}^{\circ}$$

This idealization is obviously not a very practical liquefier, but does demonstrate that heat energy is being transferred at a temperature that continuously varies. FIGURE 2 depicts the liquefaction process as a refrigeration 'load' whose temperature starts at 300K and ends at 4.2K (for helium at 1 atm.). Since neither the Carnot liquefier or the idealization in FIGURE 2 leads to a practical liquefier, an ideal Claude liquefier, as presented in FIGURE 3 and FIGURE 4, will next be examined. This cycle operates with an ideal gas, utilizes perfect HX's (i.e., infinite NTU's) and recovers the expander output work to reduce the net compressor input work. FIGURE 3 is the TS diagram for FIGURE 4 (note that the temperature is a logarithmic scale). The cycles in FIGURE 2 and FIGURE 4 accomplish the same process task and both require the same net input work as the Carnot liquefier.

In general for each expander, the isentropic output work is,

$$\dot{W}_{Xs,i} = \dot{m}_{X,i} \cdot C_p \cdot (T_i - T_{s,i+1}) = \dot{m}_{X,i} \cdot C_p \cdot T_{s,i+1} \cdot (\theta_{Xs,i} - 1) \quad (11)$$

FIGURE 3. Temperature-Entropy (TS) Diagram for Ideal Cascade Liquefier.

Where, T_1 – ambient (isothermal compression) temperature [K]
 T_{N+1} – load temperature [K]
 Θ – total temperature ratio (ambient to load temperature ratio)
 N – total number of (ideal) expansion stages

So, *the Carnot step is the same for each expander stage* (i.e., θ_x is the same for each stage) and in this case for the ideal Claude liquefier, it is the same as the expander temperature ratio. This is true even if the expander isentropic efficiencies are less than unity, as long as the expander efficiencies for all stages are equal.

As an example, for a 300K to 4.2K liquefier (e.g., $T_1 = 300\text{K}$, $T_{N+1} = 4.2\text{K}$), with an expander pressure ratio of 16 (e.g., $\Gamma = 16$), the total temperature ratio is, $\Theta = 300/4.2 = 71$ and the temperature ratio for each expander stage is, $\theta_x = \Gamma^\phi = (16)^{0.4} = 3.03$, if the adiabatic efficiency, $\eta_x = 1$. So, the ideal number of expander stages is, $N = \ln(71) / \ln(3.03) = 3.85 \cong 4$.

Referring to FIGURE 4, a simple energy balance shows that each expander flow is the same and equal to the liquefaction flow, e.g., $\dot{m}_{x,i} = \dot{m}_x = \dot{m}_L$. It will be shown later that the expander flow will always be greater than the liquefaction flow to compensate for component losses as well as process and fluid non-idealities (which will not be covered in this paper; refer to [1]). With the expander flow the same for each Carnot step (or stage), the (ideal or isothermal) *compressor work for each stage is also equal* (refer to FIGURE 4). However, the *Carnot work for each stage is not the same* since the expander output work, which is recovered by the ideal Claude and Carnot liquefiers (and used to reduce the compressor input power), is not the same for each Carnot step.

The net work required for this ideal Claude liquefier is,

$$\dot{W}_{ICL} = \dot{W}_C - \sum_{i=1}^N \dot{W}_{x,i} = \dot{m}_x \cdot C_p \cdot \left[N \cdot \phi \cdot T_1 \cdot \ln(\Gamma) - \sum_{i=1}^N (T_i - T_{i+1}) \right] \quad (19)$$

$$\text{Where, } \dot{W}_C = \dot{m}_C \cdot C_p \cdot \phi \cdot T_1 \cdot \ln(\Gamma) ; \text{ isothermal compressor input work [W]} \quad (20)$$

$$\dot{m}_C = \sum_{i=1}^N \dot{m}_{x,i} = N \cdot \dot{m}_x ; \text{ compressor mass flow [g/s]} \quad (21)$$

Note that $\dot{m}_L$ is not included in $\dot{m}_C$ since any availability given to it by the compressor is provided to the load).

With, $\dot{m}_C = N \cdot \dot{m}_x = N \cdot \dot{m}_L$ (for perfect HX's)

$$\sum_{i=1}^N (T_i - T_{i+1}) = T_1 - T_{N+1} \quad \text{with, } T_{N+1} = T_L$$

$$\Theta = \frac{T_1}{T_L} = \frac{T_1}{T_2} \cdot \frac{T_2}{T_3} \cdots \frac{T_i}{T_{i+1}} \cdots \frac{T_N}{T_{N+1}} = (\theta_i)^N = (\theta_x)^N$$

$$\theta_i = \theta_x = \frac{T_i}{T_{i+1}} = \Gamma^\phi = \text{constant}$$

$$\text{So, } \dot{W}_{ICL} = \dot{m}_L \cdot C_p \cdot \left[T_1 \cdot \ln\left(\frac{T_1}{T_L}\right) - (T_1 - T_L) \right] = \dot{W}_{Carnot}^\circ \quad (22)$$

Therefore, the ideal Claude liquefier requires the same net input work as the Carnot liquefier if the assumption of an ideal gas with constant specific heat is valid. This concept (although not literally stated) was first presented in [3].

FIGURE 5. Practical Claude Liquefier.

PRACTICAL CLAUDE LIQUEFIER

Referring to FIGURE 5, if an additional HX is added before each expander, for the same pressure ratio (Γ) fewer expander stages are necessary for the same total temperature ratio (Θ). These HX's in between the expansion stages make it possible to reduce the number of expansion stages. However, this increases the mass flow at each expansion stage (also increasing the total flow into the cold box) and decreases the coefficient of performance, since a portion of the heat energy is being absorbed at a colder temperature (i.e., from T_i' to T_{i+1} , instead of T_i to T_{i+1} in FIGURE 4). The refrigeration is being provided over the temperature span of each expander, which is only a portion of each Carnot step.

Referring to stage ' i ' in FIGURE 5, it is straightforward to see that the difference between the high (h) to low (l) pressure stream at ' i ' is always the liquefaction rate, $\dot{m}_L$. It is important to note that the (h) stream is the maximum capacity stream for HX_i , and the (l) stream is the maximum capacity stream for $HX_{i'}$. This tends to 'pinch' the approach temperatures at ' i ' and ' $i+1$ '. However, due to the unbalanced flow, the temperature difference between the (h) and (l) stream at ' i ' tends to open-up. So, the expander is providing the cooling for this irreversibility (even if the HX's were perfect). The reader should notice that the (h) stream flow at $i = (N+1)$ is $\dot{m}_{h,N+1} = \dot{m}_L + \dot{m}_R$, where $\dot{m}_R$ is the flow required for the refrigeration load. The energy balance is then,

$$\dot{m}_{h,i} \cdot f_i + \dot{m}_L \cdot g_i = \dot{m}_{x,i} \cdot h_i \quad (23)$$

Where,
$$f_i = \left[\alpha_{l,i} \cdot \theta_i \cdot \left(\alpha_{hl,i} - \frac{1}{\theta_{hl,i}} \right) - (\alpha_{hl,i+1} \cdot \theta_{hl,i+1} - 1) \right] \quad (24)$$

$$g_i = \alpha_{l,i} \cdot \left(\frac{\theta_i}{\theta_{hl,i}} \right) + \varepsilon_{k,i} - 1 \quad (25)$$

$$h_i = \alpha_{hl,i} \cdot \theta_{X,i} - \alpha_{hl,i+1} \cdot \theta_{hl,i+1} \quad (26)$$

$$\theta_{hl,i} = T_{h,i} / T_{l,i} \quad (27)$$

$$\alpha_{l,i} = C_{p,l,i} / C_{p,l,i+1} \quad (28)$$

$$\alpha_{hl,i} = C_{p,h,i} / C_{p,l,i+1} \quad (29)$$

$$\alpha_{hl,i} = C_{p,h,i} / C_{p,l,i+1} \quad (30)$$

$$\varepsilon_{k,i} = \dot{m}_{k,i} / \dot{m}_L \quad (31)$$

$$\dot{m}_{k,i} = q_{k,i} / (C_{p,l,i} \cdot T_{l,i+1}) \quad (32)$$

With, $\dot{m}_{h,i}$ – high pressure stream flow at 'i' [g/s]
 $C_{p,l,i}$ – low pressure stream specific heat at 'i' [J/g-K]
 $C_{p,h,i}$ – high pressure stream specific heat at 'i' [J/g-K]
 $C_{p,h,i}$ – expander inlet specific heat at 'i' [J/g-K]
 $q_{k,i}$ – heat leak at stage 'i' into HX_i and $HX_{i'}$ [W]

The problem posed is to determine the values for θ_i and $\dot{m}_{X,i}$ that produce a minimum compressor flow $\dot{m}_C$, given that the number of stages (N), the total temperature ratio [$\Theta (= T_1/T_{N+1} = T_1/T_L)$], expander efficiencies ($\eta_{X,i}$), expander pressure ratio (Γ), required liquefaction flow ($\dot{m}_L$), required refrigeration flow ($\dot{m}_R$), heat leaks ($q_{k,i}$) and stream C_p 's are specified. Observing that,

$$\Theta = \frac{\theta_1 \cdot \theta_2 \cdots \theta_i \cdots \theta_N}{\theta_{hl,2} \cdot \theta_{hl,3} \cdots \theta_{hl,N}} \quad (33)$$

And,
$$\dot{m}_C = \dot{m}_L + \dot{m}_R + \sum_{i=1}^N \dot{m}_{X,i} \quad (34)$$

The general solution is not available in closed form. However, assuming the following:

$$\theta_{hl,i} = \theta_{hl} = \text{constant}, \eta_{X,i} = \eta_X = \text{constant}, \text{ And } \varepsilon_{k,i} = \text{constant}$$

$$C_{p,l,i} = C_{p,l,i+1} = \text{constant}, C_{p,h,i} = C_{p,h,i} \text{ And, } C_{p,h,i} = C_{p,h,i+1} = \text{constant}$$

Then,
$$f_i = \theta_i \cdot \left(\alpha - \frac{1}{\theta_{hl}} \right) - (\alpha \cdot \theta_{hl} - 1) \quad (35)$$

$$g_i = \frac{\theta_i}{\theta_{hl}} + \varepsilon_k - 1 \quad (36)$$

$$h = \alpha \cdot (\theta_X - \theta_{hl}) \quad (37)$$

$$\alpha = C_{p,h} / C_{p,l} \quad (38)$$

$$\varepsilon_k = \dot{m}_k / \dot{m}_L; \text{ where, } \dot{m}_k = \text{constant}$$

A closed form solution is then possible under these conditions. Employing some algebra and making use of Lagrange multipliers, the following result can be arrived at,

$$\theta_i = \theta = \Theta^{1/N} \cdot \theta_{hl}^{(N-1)/N} = \text{constant} \quad (39)$$

$$f_i = f \quad \text{and,} \quad g_i = g$$

$$\dot{m}_{LR} = \dot{m}_L + \dot{m}_R \quad (40)$$

$$v_L = \dot{m}_L / \dot{m}_{LR} \quad (41)$$

$$\beta_{r,0} = \frac{\dot{m}_{X,N}}{\dot{m}_{LR}} = \left(\frac{f + v_L \cdot g}{k} \right) \quad (42)$$

$$\beta_r = \frac{\dot{m}_{X,i}}{\dot{m}_{X,i+1}} = \frac{h}{k} \quad (43)$$

$$\frac{\dot{m}_C}{\dot{m}_{LR}} = 1 + \beta_{r,0} \cdot \left[\frac{\beta_r^N - 1}{\beta_r - 1} \right] \quad (44)$$

$$k = h - f \quad (45)$$

Notice that the expander flow rate increases by a constant (multiplied) factor of β_r , from the coldest to the warmest expander. This is due to the HX irreversibilities and C_p effect propagating from the previous colder Carnot step to the next (warmer one). However, the Carnot step (i.e., temperature ratio), θ , is the same for each stage.

Introducing the quantity, $\tau = \Delta T_{hl,i} / T_{l,i} = \theta_{hl,i} - 1$, which is a good measure of whether the HX's are within practical limits. Typically, it is recommended not to exceed $\tau \geq 0.02$ (or, $\theta_{hl} \geq 1.02$). For the special case of perfect HX's with no heat leak (i.e., $\tau = 0$, $\theta_{hl} = 1$ and $\varepsilon_k = 0$) and equal stream C_p 's (i.e., $\alpha = 1$), the equations above reduce to,

$$\theta_i = \theta = \Theta^{1/N} = \text{constant}$$

$$\beta_{r,0} = \frac{\dot{m}_{X,N}}{\dot{m}_{LR}} = \left[\frac{\theta - 1}{\theta_X - 1} \right] \quad (46)$$

$$\beta_r = \frac{\dot{m}_{X,i}}{\dot{m}_{X,i+1}} = 1 \quad (47)$$

$$\frac{\dot{m}_C}{\dot{m}_{LR}} = 1 + \beta_{r,0} \cdot N \quad (48)$$

In this case, the expander flow is the same for each stage, although greater than $\dot{m}_L$. Also, as mentioned before, the Carnot step (i.e., temperature ratio) is the same for each stage. Note that these equations are not valid for 100 percent refrigeration, since $f = 0$, which makes β_r indeterminate.

The reason for the Carnot step being the same for the two previous cases [i.e., (a) perfect HX, but non-isentropic equal-efficiency expanders and (b) non-perfect HX with either isentropic or non-isentropic equal-efficiency expanders] is that the expanders inherently establish the temperature ratio by expanding from the high to low pressure. However, as stated above in the former case (with perfect HX's), the expander mass flows are the same (for each stage), but in the latter (with non-perfect HX's) they are not the same. The reason is that the non-perfect HX's introduce an irreversibility that must be

propagated up through (i.e., 'dealt with by') each stage (i.e., from the cold end or load up to the ambient temperature) and eventually dissipated to the 'zero-grade' temperature (i.e., 300K) by way of the compressors. Since this irreversibility is the same *relative to the temperature of each stage*, the expander mass flow increases by a constant (multiplier) factor at each stage.

APPLICATION

The practical use for these equations can be found by examining the magnitude of β_r . For a total temperature ratio $\Theta = 80/4.2$ (i.e., LN₂ pre-cooling is used), four Carnot steps (or expansion stages, $N = 4$), expander temperature ratio of $\theta_x = 1.5$, $\tau = 0.02$, $\alpha = 1$ and $\varepsilon_K = 0$; $\theta = 2.120$ and $\beta_r = 1.047$. So, the expander mass flow increases by approximately 5 percent every step (or stage), from the coldest to warmest step. As an approximation, since the warmest step expander flow is only 15 percent greater than the coldest step, all expander flows (for each step) could be considered equal. In fact, as discussed elsewhere [1], the real gas effects (non-idealities) at the cold end counter the effect of the increasing expander flow (due to step irreversibilities). What this allows is a linear superposition of loads. For example, consider a shield load in addition to a liquefaction and refrigeration load. The expander flow in the stage whose expander inlet and outlet temperatures meet the shield load requirements would be increased by the amount necessary for the shield load (the left hand side of FIGURE 5 is helpful in visualizing this linear superposition). The same reasoning would apply for a cold compressor load. The expander stages below the cold compressor return would see a liquefaction load equal to the cold compressor flow; and the expanders above the cold compressor return temperature would see the additional high and low side flow as a refrigeration load.

CONCLUSION

The importance and applications of the Carnot step have been discussed and apply for new system designs and for assessing the existing system operations.

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