GDH sum rule in QED and ChPT

Vladimir Pascalutsa^{ab} *

^aPhysics Department, The College of William & Mary, Williamsburg, VA 23187, USA

^bTheory Group, Jefferson Lab, 12000 Jefferson Ave, Newport News, VA 23606, USA

The celebrated Gerasimov-Drell-Hearn (GDH) sum rule (SR) relates the anomalous magnetic moment (a.m.m.) of a spin-1/2 particle to a dispersion integral of its polarized total photoabsorption cross-section:

$$\frac{e^2 \kappa^2}{2M^2} = \frac{1}{\pi} \int_0^\infty d\omega \frac{\Delta\sigma(\omega)}{\omega}, \quad (1)$$

where $\Delta\sigma = \sigma_{3/2} - \sigma_{1/2}$ (subscript 3/2 or 1/2 refers to whether the photon beam polarization is in parallel or antiparallel to the polarization of the initial nucleon), for more details see, e.g., [1].

Let us consider the *lhs* and *rhs* of the GDH SR for the electron in QED. To lowest order in the fine-structure constant, $\alpha = e^2/4\pi$, the photoabsorption cross-section is given by the tree-level Compton scattering cross-section [2]:

$$\Delta\sigma(\omega) = \frac{2\pi\alpha^2}{M\omega} \left[2 + \frac{2\omega^2}{(M+2\omega)^2} - \left(1 + \frac{M}{\omega} \right) \ln \left(1 + \frac{2\omega}{M} \right) \right] + O(\alpha^3). \quad (2)$$

On the other hand, the one-loop contribution to the electron a.m.m. is of order α and therefore the *lhs* has no contribution of α^2 . Fortunately, the GDH integral over the tree-level cross-section Eq. (2) *vanishes*, and thus, at this order, everything works out.

At order α^3 the *lhs* receives the contribution in the form of Schwinger's correction: $\kappa = \alpha/2\pi$. The calculation of cross sections at this order is quite a formidable task as it requires the knowledge of Compton scattering amplitude to one-loop, inclusion of the pair-production channel, and so on, cf. [3]. Instead, we follow a much simpler way to do calculations at this level [4].

Let us introduce a 'classical' (or 'trial') value of the a.m.m., κ_0 . At the level of the Lagrangian this amounts to introducing a Pauli term for our spin-1/2 field:

$$\mathcal{L}_{\text{Pauli}} = (i\kappa_0/4M) \bar{\psi} \sigma_{\mu\nu} \psi F^{\mu\nu}, \quad (3)$$

where F is the electromagnetic field tensor and $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$. In the end we can put κ_0 equal to zero, but for now the total value of the a.m.m. is $\kappa = \kappa_0 + \delta\kappa$, with $\delta\kappa$ being the quantum effects. Note that $\delta\kappa$ and the total cross-sections become explicitly

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dependent on κ_0 . To get something new out of this we need to start taking derivatives of the GDH SR with respect to κ_0 :

$$(4\pi^2\alpha/M^2) \kappa \kappa' = \int_0^\infty \frac{d\omega}{\omega} \Delta\sigma'(\omega), \quad (4)$$

$$(4\pi^2\alpha/M^2) (\kappa'^2 + \kappa \kappa'') = \int_0^\infty \frac{d\omega}{\omega} \Delta\sigma''(\omega), \quad (5)$$

and so on. Now observe that to lowest order in α these relations simply read as

$$(4\pi^2\alpha/M^2) n \kappa^{(n-1)} = \int_0^\infty \frac{d\omega}{\omega} \Delta\sigma^{(n)}(\omega), \quad (6)$$

where n is the order of the derivative with respect to κ_0 . This allows us in principle to compute κ to order α^k by using k th, $(k-1)$ th, etc., derivatives of the cross-section computed, respectively, to order α^{k+1} , α^k , etc. In this way, to lowest order we have the following sum rule:

$$(4\pi^2\alpha/M^2) \kappa = \int_0^\infty \frac{d\omega}{\omega} \Delta\sigma'(\omega)|_{\kappa_0=0}. \quad (7)$$

The striking feature of this sum rule is the *linear* relation between the a.m.m. and the photoabsorption cross section, in contrast to the GDH SR where the relation is quadratic.

Although the cross-section quantity $\Delta\sigma'$ is not an observable, it is very clear how to determine it within a given theory. The first derivative of the tree-level cross-section with respect to κ_0 , at $\kappa_0 = 0$, in QED takes the form [4]:

$$\Delta\sigma'(\omega)|_{\kappa_0=0} = \frac{2\pi\alpha^2}{M\omega} \left[6 - \frac{2M\omega}{(M+2\omega)^2} - \left(2 + \frac{3M}{\omega} \right) \ln \left(1 + \frac{2\omega}{M} \right) \right]. \quad (8)$$

It is then not difficult to find that

$$\frac{1}{\pi} \int_0^\infty \frac{d\omega}{\omega} \Delta\sigma'(\omega)|_{\kappa_0=0} = \frac{2\alpha^2}{M^2}. \quad (9)$$

Substituting this result in the linearized SR, Eq. (7), we obtain $\kappa = \alpha/2\pi$. Thus, the Schwinger's one-loop result is reproduced here by computing only a (derivative of the) tree-level Compton scattering cross-section and then performing a GDH integral.

Consider now the theory of nucleons interacting with pions via pseudovector coupling:

$$\mathcal{L}_{\pi NN} = \frac{g}{2M} \bar{\psi} \gamma^\mu \gamma^5 \tau^a \psi \partial_\mu \pi^a, \quad (10)$$

where g is the pion-nucleon coupling constant, τ^a are isospin Pauli matrices, π^a is the isovector pion field. For our purposes this Lagrangian is sufficient to obtain the leading order results of chiral perturbation theory (ChPT).

To lowest order in g the photoabsorption cross section in this theory is dominated by the single pion photoproduction graphs as displayed in Fig. 1. We find for the corresponding

Figure 1. Tree-level pion photoproduction graphs. The circled vertex corresponds to the Pauli coupling.

GDH cross sections:

$$\Delta\sigma^{(\pi^0 p)} = \frac{\pi C}{M^2 x^2} \left[(2\alpha\bar{s} + 1 - x) \ln \frac{\alpha + \lambda}{\alpha - \lambda} - 2\lambda[x(\alpha - 2) + \bar{s}(\alpha + 2)] \right], \quad (11)$$

$$\Delta\sigma^{(\pi^+ n)} = \frac{2\pi C}{M^2 x^2} \left[-\mu^2 \ln \frac{\beta + \lambda}{\beta - \lambda} + 2\lambda(\bar{s}\beta - x\alpha) \right], \quad (12)$$

$$\Delta\sigma^{(\pi^0 n)} = 0, \quad (13)$$

$$\Delta\sigma^{(\pi^- p)} = \frac{2\pi C}{M^2 x^2} \left[-\mu^2 \ln \frac{\beta + \lambda}{\beta - \lambda} + (2\alpha\bar{s} - 1 - x) \ln \frac{\alpha + \lambda}{\alpha - \lambda} - 2\bar{s}\lambda \right], \quad (14)$$

where $C = (eg/4\pi)^2$, $\mu = m_\pi/M$, m_π is the pion mass, and

$$s = M^2 + 2M\omega, \quad \bar{s} = s/M^2, \quad (15)$$

$$\alpha = (s + M^2 - m_\pi^2)/2s, \quad (16)$$

$$\beta = (s - M^2 + m_\pi^2)/2s = 1 - \alpha, \quad (17)$$

$$\lambda = (1/2s) \sqrt{s - (M + m_\pi)^2} \sqrt{s - (M - m_\pi)^2}. \quad (18)$$

As in the case of QED, the anomalous magnetic moment corrections start at $\mathcal{O}(g^2)$, implying that the lhs of the GDH SR begins at $\mathcal{O}(g^4)$. Since the tree-level cross sections are $\mathcal{O}(g^2)$, we must require that

$$\int_{\omega_{\text{th}}}^{\infty} \frac{d\omega}{\omega} \Delta\sigma^{(I)}(\omega) = 0, \quad \text{for } I = \pi^0 p, \pi^+ n, \pi^0 n, \pi^- p, \quad (19)$$

where $\omega_{\text{th}} = m_\pi(1 + m_\pi/2M)$ is the threshold of the pion photoproduction reaction. This requirement is indeed verified for the expressions given in Eq. (1) —the consistency of GDH SR is maintained in this theory for each of the pion production channels.

We now turn our attention to the linearized GDH sum rule. In this case we first introduce Pauli moments κ_{0p} and κ_{0n} for the proton and the neutron, respectively. The dependence of the cross-sections on these quantities can generally be presented as:

$$\begin{aligned} \Delta\sigma(\omega; \kappa_{0p}, \kappa_{0n}) &= \Delta\sigma(\omega) + \kappa_{0p} \Delta\sigma_{1p}(\omega) + \kappa_{0n} \Delta\sigma_{1n}(\omega) \\ &+ \kappa_{0p}^2 \Delta\sigma_{2p}(\omega) + \kappa_{0n}^2 \Delta\sigma_{2n}(\omega) + \kappa_{0p} \kappa_{0n} \Delta\sigma_{1p1n}(\omega) + \dots \end{aligned} \quad (20)$$

Furthermore, we introduce proton and neutron photoproduction cross sections $\Delta\sigma^{(p)}$ and $\Delta\sigma^{(n)}$ and express the corresponding GDH SRs and their first derivatives. The first derivatives of the cross-sections that enter in the linearized GDH SR, to leading order in g , arise through the interference of Born graphs Fig. 1(a) with the graphs in Fig. 1(b) and we find:

$$\Delta\sigma_{1p}^{(p)} \equiv \Delta\sigma_{1p}^{(\pi^0 p)} + \Delta\sigma_{1p}^{(\pi^+ n)} = \frac{\pi C}{M^2 x^2} \{2x\lambda[4 + (1 - 2\alpha)(2 + \bar{s} + 2x)] - 2\bar{s}\lambda(\alpha + 2) - \mu^2 x \ln \frac{\beta + \lambda}{\beta - \lambda} + (2\alpha\bar{s} + 1 - x) \ln \frac{\alpha + \lambda}{\alpha - \lambda}\}, \quad (21)$$

$$\Delta\sigma_{1n}^{(n)} \equiv \Delta\sigma_{1n}^{(\pi^0 n)} + \Delta\sigma_{1n}^{(\pi^- p)} = \frac{\pi C}{M^2 x} \left\{ 2\lambda(2 + 2x - \bar{s}) + \mu^2 \ln \frac{\beta + \lambda}{\beta - \lambda} - \ln \frac{\alpha + \lambda}{\alpha - \lambda} \right\}.$$

Employing the linearized SRs, we obtain:

$$\begin{aligned} \kappa_p &= \frac{M^2}{\pi e^2} \int_{\omega_{\text{th}}}^{\infty} \frac{d\omega}{\omega} \Delta\sigma_{1p}^{(p)} \\ &= \frac{g^2}{(4\pi)^2} \left\{ 1 - \frac{\mu(4 - 11\mu^2 + 3\mu^4)}{\sqrt{1 - \frac{1}{4}\mu^2}} \arccos \frac{\mu}{2} - 6\mu^2 + 2\mu^2(-5 + 3\mu^2) \ln \mu \right\}, \\ \kappa_n &= \frac{M^2}{\pi e^2} \int_{\omega_{\text{th}}}^{\infty} \frac{d\omega}{\omega} \Delta\sigma_{1n}^{(n)} = \frac{-2g^2}{(4\pi)^2} \left\{ 2 - \frac{\mu(2 - \mu^2)}{\sqrt{1 - \frac{1}{4}\mu^2}} \arccos \frac{\mu}{2} - 2\mu^2 \ln \mu \right\}. \end{aligned} \quad (22)$$

On the other hand, when using the standard Feynman-parameter technique to evaluate the one-loop contributions to a.m.m., we have

$$\begin{aligned} \kappa_p &= 2\kappa^{(\pi)} - \kappa^{(N)}, \\ \kappa_n &= -2[\kappa^{(\pi)} + \kappa^{(N)}], \end{aligned} \quad (23)$$

with

$$\begin{aligned} \kappa^{(\pi)} &= \frac{g^2}{(4\pi)^2} \int_0^1 dx \frac{2x(1-x)^2}{x\mu^2 + (1-x)^2}, \\ \kappa^{(N)} &= \frac{g^2}{(4\pi)^2} \int_0^1 dx \frac{2x^3}{(1-x)\mu^2 + x^2}, \end{aligned} \quad (24)$$

where the superscript denotes whether the photon couples to the pion or the nucleon in the one-loop graph. We have thus verified that, for this theory as well, the sum-rule and the Feynman-parameter calculation give equivalent results.

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