Counting Rules, Holographic Wave Functions, Meromorphization and Quark-Hadron Duality*

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We start with study of the meson form factor $F_M(Q^2)$ constructed using recently proposed holographic light-front wave functions. We find that its asymptotic behavior is generated by soft Feynman mechanism rather than by short distance dynamics. This causes a very late onset of the $1/Q^2$ asymptotic behavior at unaccessible momenta $Q^2 \gtrsim 10 \text{ GeV}^2$. We show that this model can be also obtained within the Migdal's "meromorphization" approach if one applies it to 3-point function for scalar currents made of scalar quarks. For spin-1/2 quarks, we demonstrated that resulting form factor $F_M^{\text{spinor}}(Q^2)$ has $1/Q^4$ asymptotic behavior, but owing to late onset of this asymptotic pattern, $F_M^{\text{spinor}}(Q^2)$ imitates the $1/Q^2$ behavior in the few GeV^2 region. In the meromorphization approach, just like in the local quark-hadron duality model for the pion form factor, adding the $\mathcal{O}(\alpha_s)$ correction to the spectral function brings in the hard pQCD contribution that has the dimensional counting $1/Q^2$ behavior at large Q^2 . At accessible Q^2 , the $\mathcal{O}(\alpha_s)$ term is a rather small fraction of the total result. In this scenario, both hard and soft contributions are present, and the "observed" quark counting rules for hadronic form factors is an approximate and transitional phenomenon resulting from long-distance dynamics, Feynman mechanism in its preasymptotic regime.

1. Introduction

Experimental evidence that form factors of hadrons consisting of n quarks have behavior close to $(1/Q^2)^{n-1}$ provokes expectations that there is a fundamental and/or easily visible reason for such a phenomenon, scale invariance being the most natural suspect¹. Hard rescattering in a the-

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ory with dimensionless coupling constant provides a specific dynamical mechanism² that produces a scale invariant behavior. Perturbative QCD predicts the $(\alpha_s/Q^2)^{n-1}$ asymptotic behavior³⁻⁶. In the pion case, the prediction is $F_\pi(Q^2) = (2\alpha_s/\pi)s_0/Q^2$, where $s_0 = 4\pi^2 f_\pi^2 \approx 0.7 \text{ GeV}^2$ is a constant close to $m_\rho^2 \approx 0.6 \text{ GeV}^2$. This indicates that the pQCD asymptotics *is not* the large- Q^2 limit of the phenomenologically successful VMD fit $F_\pi^{\text{VMD}}(Q^2) \sim 1/(1 + Q^2/m_\rho^2)$, but rather looks like $\mathcal{O}(\alpha_s)$ correction to it. Also, the smallness of α_s/π undermines attempts to describe available data solely by pQCD hard mechanism. The growing consensus is that at available Q^2 form factors are dominated by soft contributions described by nonforward parton densities⁷ $\mathcal{F}(x, Q^2)$ (or generalized parton distributions $H(x, \xi; Q^2)$ for zero skewness ξ), and successful fits were obtained⁸⁻¹⁰ in models with $\mathcal{F}(x, Q^2) = f(x) e^{-Q^2 g(x)}$ having exponential behavior for large Q^2 at fixed x . A powerlike asymptotics in this case appears only after integration over x , i.e., it is governed by the Feynman mechanism¹¹, and is determined by the $x \rightarrow 1$ behavior of $f(x)$ in contrast to the hard mechanism for which the subprocess amplitude already has the $(1/Q^2)^{n-1}$ power behavior not affected by subsequent x -integrations.

Another new development is related to applications of AdS/CFT construction to QCD and claims^{12,13} that this framework (“AdS/QCD”) provides a nonperturbative explanation of quark counting rules. Given explicit expressions with the basic scale Λ fixed from fitting the hadron masses, it is straightforward to check the structure of AdS/QCD results for form factors and their potential to describe the features of existing data. This is one of the goals of the present investigation. Another is to study the recently proposed interpretation¹⁴ of AdS/QCD results in terms of light-front wave functions, which opens a possibility to find out whether, in terms of the light-cone momenta $x, \mathbf{k}_\perp$, the AdS/QCD quark counting corresponds to large- $\mathbf{k}_\perp$ hard mechanism or, as we will show, to the $x \rightarrow 1$ soft Feynman/Drell-Yan^{11,15} mechanism. In view of yet another recent observation¹⁶ that some of the results of the AdS/QCD coincide with those of Migdal’s program¹⁷ we apply the extension¹⁸ of this “meromorphization” idea to the 3-current correlators, and establish a connection between this approach and holographic light-front wave functions of Ref.¹⁴.

The meromorphization procedure has features similar to the “local quark-hadron duality” model¹⁹ that successfully describes the pion form factor data and gives a unified description of its soft and hard parts. The soft term $F_\pi^{\text{LD}}(Q^2)$ in this approach dominates at accessible Q^2 , but has the $1/Q^4$ asymptotic behavior. Due to its late onset, the curve $Q^2 F_\pi^{\text{LD}}(Q^2)$

has a wide plateau in a few GeV^2 region, i.e., $F_\pi^{\text{LD}}(Q^2)$ imitates there the $1/Q^2$ behavior. Thus, the desired $1/Q^2$ result for accessible Q^2 is obtained because the nonperturbative term has a faster, $1/Q^4$ asymptotic fall-off.

2. Holographic wave functions and Feynman mechanism

In the hard-wall approximation¹², the expression for the elastic form factor in the holographic duality model is given by

$$F(Q^2) = \int_0^{1/\Lambda} \frac{d\zeta}{\zeta^3} \Phi_{P'}(\zeta) J(Q, \zeta) \Phi_P(\zeta) , \quad (1)$$

where $J(Q, \zeta) = \zeta Q K_1(\zeta Q)$ stays for EM current, and $\Phi_P(\zeta)$, $\Phi_{P'}(\zeta)$ for initial and final states, with $\Phi(\zeta) = C\zeta^2 J_L(\beta_{L,k}\zeta\Lambda)$, $\beta_{L,k}$ being the k th root of the Bessel function $J_L(x)$. On the other hand, in the light-cone (LC) formalism, the meson form factor is given by a 3-dimensional integral. In particular, the 2-body part of a meson form factor is given by

$$F_{(2)}(Q^2) = \int_0^1 dx \int d^2\mathbf{b}_\perp e^{i\bar{x}\mathbf{b}_\perp \cdot \mathbf{q}_\perp} \left| \Psi_2(x, \mathbf{b}_\perp) \right|^2 \equiv \int_0^1 dx \mathcal{F}_{(2)}(x, Q^2) , \quad (2)$$

where $\bar{x} \equiv 1 - x$ and $\mathcal{F}_{(2)}(x, Q^2)$ is the 2-body part of GPD. In Ref.¹⁴ it was noticed that if $\Psi(x, \mathbf{b}_\perp) \sim \sqrt{x\bar{x}} \phi(b\sqrt{x\bar{x}})$, then LC formula converts into the 1-dimensional integral (1) with correct $J(Q, \zeta)$, provided that $\zeta = b\sqrt{x\bar{x}}$. This gives the “holographic” model¹⁴ for light-front wave functions. For the lowest meson, $L = 0, k = 1$ and mass $M = \beta_{0,1}\Lambda$. Then

$$\Psi_M(x, b) = \frac{M\sqrt{x\bar{x}/\pi}}{\beta_{0,1}J_1(\beta_{0,1})} J_0(\sqrt{x\bar{x}}Mb) \theta(\sqrt{x\bar{x}}b \leq \beta_{0,1}/M) . \quad (3)$$

The $\mathbf{k}_\perp$ counterpart of this wave function is given by

$$\tilde{\Psi}_M(x, \mathbf{k}_\perp) = \frac{M}{\sqrt{\pi x\bar{x}}} \frac{J_0(\beta_{0,1}k_\perp/\sqrt{x\bar{x}}M)}{M^2 - k_\perp^2/x\bar{x}} . \quad (4)$$

For small $k_\perp$, this wave function is close to $\exp[-k_\perp^2/2M^2x\bar{x}]/\sqrt{x\bar{x}}$. For large $k_\perp$, it oscillates with the magnitude decreasing as $1/k_\perp^{5/2}$. This result contradicts the statement made in Ref.¹³ that the AdS/QCD construction corresponds to a purely power-law large- $k_\perp$ behavior.

Explicit calculation of the lowest state form factor gives

$$F_M(Q^2) = \frac{4M^2}{Q^2[\beta_{0,1}J_1(\beta_{0,1})]^2} \left[1 - \frac{4M^2}{Q^2} + \frac{9}{8} \left(\frac{4M^2}{Q^2} \right)^2 + \mathcal{O}(M^6/Q^6) \right] . \quad (5)$$

Though this result has a monopole-like structure, the scale $4M^2$ is evidently too large if $M = m_\rho$: the curve for $Q^2 F_M(Q^2)$ is far from being flat in the accessible region $Q^2 \lesssim 10 \text{ GeV}^2$.

The easiest way to see which mechanism is responsible for the $1/Q^2$ asymptotics is to approximate $\tilde{\Psi}_M(x, \mathbf{k}_\perp)$ by a Gaussian function $\tilde{\Psi}_G(x, \mathbf{k}_\perp) \sim \exp[-k_\perp^2/2M^2 x\bar{x}]/\sqrt{x\bar{x}}$. In the impact parameter space, $\Psi_G(x, b) \sim \sqrt{x\bar{x}} \exp[-b^2 M^2 x\bar{x}/2]$. Integrating over *all* (including small) values of $\mathbf{b}_\perp$ in the form factor formula (2) gives $\mathcal{F}_G(x, Q^2) = e^{-\bar{x}Q^2/4xM^2}$, a function that, for any fixed x , vanishes exponentially at large Q^2 . The power-law asymptotics is obtained only after integrating over x , which gives

$$F_G(Q^2) = \sum_{n=1}^{\infty} (-1)^{n-1} n! \left(\frac{4M^2}{Q^2} \right)^n, \quad (6)$$

a result that has the $\mathcal{O}(M^2/Q^2)$ large- Q^2 behavior and structure similar to that of Eq. (5). The crucial role of integration over the $x \sim 1$ region in getting the power-law behaviour is evident: if the integration is restricted to $x < x_0$, the outcome vanishes like $\exp[-\bar{x}_0 Q^2/4x_0 M^2]$ for large Q^2 . Thus, the GPD $\mathcal{F}_G(x, Q^2)$ has the same $\mathcal{F}(x, Q^2) = f(x) e^{-Q^2 g(x)}$ structure as those considered in Refs.^{8–10}. In Ref.²¹ it is shown that the large- Q^2 asymptotics of the meson form factor in the model of Ref.¹⁴ with original, nonapproximated wave function is also governed by the *soft* Feynman mechanism, with the power-law asymptotics determined by the $x \rightarrow 1$ behavior of the prefactor $f(x)$ accompanying a decreasing function of $\bar{x}Q^2/x\Lambda^2$. In the present case, $f(x) = 1$, which gives $F_M(Q^2) \sim 1/Q^2$. Clearly, $f(x) = \mathcal{F}(x, Q^2 = 0)$ is the parton distribution function, i.e., the model of Ref.¹⁴ gives a constant, x -independent parton distribution $f(x) = 1$.

3. Meromorphization and local quark-hadron duality

One may question interpretation of the holographic variable ζ as a particular product $b\sqrt{x\bar{x}}$ of light-cone variables. We are going to show that the picture similar to that of Ref.¹⁴ emerges also within the approach related to Migdal's program¹⁷ of Padé approximating the correlators $\Pi(p^2)$ of hadronic currents calculated in perturbation theory. Recently, it was demonstrated¹⁶ that some of Migdal's results coincide with those of the holographic approach. Migdal's program involves “meromorphization”

$$\Pi(p^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho(s)}{s - p^2} ds \Rightarrow \Pi_{\mathcal{M}}(p^2) = \Pi(p^2) - \frac{1}{\pi Q(p^2)} \int_0^\infty \frac{\rho(s) Q(s)}{s - p^2} ds \quad (7)$$

that substitutes the original correlator $\Pi(p^2)$ by a function $\Pi_{\mathcal{M}}(p^2)$ in which the cut of the original correlator for real positive p^2 is eliminated by the second term in Eq. (7), with zeros of $\mathcal{Q}(p^2)$ at timelike p^2 generating the poles interpreted as hadronic bound states. Explicit Padé construction gives $\mathcal{Q}(p^2) \Rightarrow J_0(\beta_{0,1}\sqrt{p^2}/M)$, with M being the mass of the lowest state. In the lowest order, $\rho(s) = \rho_0 \theta(s)$, with $\rho_0 = 1/16\pi$ for $j = \varphi\varphi$. The coupling constant f_M^2 of the lowest state is given by

$$f_M^2 = \lim_{p^2 \rightarrow M^2} (M^2 - p^2) \Pi_{\mathcal{M}}(p^2) = \frac{4\rho_0 M^2}{\pi[\beta_{0,1} J_1(\beta_{0,1})]^2}. \quad (8)$$

For form factors, it was suggested¹⁸ to “meromorphize” the 3-point function $T(p_1^2, p_2^2, Q^2)$ by adding the term

$$\frac{1}{\pi^2 \mathcal{Q}(p_1^2) \mathcal{Q}(p_2^2)} \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho(s_1, s_2, Q^2) \mathcal{Q}(s_1) \mathcal{Q}(s_2)}{(s_1 - p_1^2)(s_2 - p_2^2)}, \quad (9)$$

that removes its cuts and substitutes them by poles at the same locations as in $\Pi_{\mathcal{M}}(p^2)$. The elastic form factor of the lowest state is extracted using

$$f_M^2 F_M(Q^2) = \lim_{p_1^2 \rightarrow M^2} \lim_{p_2^2 \rightarrow M^2} (p_1^2 - M^2)(p_2^2 - M^2) T_{\mathcal{M}}(p_1^2, p_2^2, Q^2). \quad (10)$$

The spectral densities $\rho(s_1, s_2, Q^2)$ can be calculated²⁰ using the Cutkosky rules and light-cone variables:

$$\rho(s_1, s_2, Q^2) = \rho_0 \int_0^1 dx \frac{n(x)}{x\bar{x}} \int d^2 \mathbf{k}_\perp \delta\left(s_1 - \frac{\mathbf{k}_\perp^2}{x\bar{x}}\right) \delta\left(s_2 - \frac{(k_\perp + \bar{x}\mathbf{q}_\perp)^2}{x\bar{x}}\right), \quad (11)$$

Taking scalar $\varphi\varphi$ currents for hadronized vertices gives $n(x) = 1$ and

$$f_M^2 F_M^{\text{scalar}}(Q^2) = \frac{\rho_0}{\pi^2 [\mathcal{Q}'(M^2)]^2} \int_0^1 \frac{dx}{x\bar{x}} \int d^2 \mathbf{k}_\perp \times \frac{\mathcal{Q}(\mathbf{k}_\perp^2/x\bar{x})}{M^2 - \mathbf{k}_\perp^2/x\bar{x}} \frac{\mathcal{Q}((\mathbf{k}_\perp + \bar{x}\mathbf{q}_\perp)^2/x\bar{x})}{M^2 - (\mathbf{k}_\perp + \bar{x}\mathbf{q}_\perp)^2/x\bar{x}}, \quad (12)$$

i.e., a LC form factor expression. Using $\mathcal{Q}(s) = J_0(\beta_{0,1}\sqrt{s}/M)$ and Eq. (8) for f_M^2 , we obtain that the relevant wave function $\Psi_{\mathcal{M}}^{\text{scalar}}(x, \mathbf{k}_\perp)$ coincides with the holographic wave function (4), thus supporting interpretation in terms of light-cone variables x, b proposed in Ref.¹⁴.

Using spin-1/2 quarks and vector currents j_α, j_β gives tensor amplitude $T_{\alpha\beta}^\mu(p_1, p_2)$. As a simple example, we took the projection $T_{\alpha\beta}^\mu n_\mu n^\alpha n^\beta$, where n is a lightlike vector satisfying $(nq) = 1$. The numerator factor is then $n(x) = 6x\bar{x}$. Explicit calculation gives

$$F_M^{\text{spinor}}(Q^2) = \frac{2M^2}{Q^2[\beta_{0,1} J_1(\beta_{0,1})]^2} \left[0 + 24 \frac{M^2}{Q^2} - 288 \frac{M^4}{Q^4} + \mathcal{O}(M^6/Q^6) \right] \quad (13)$$

Fig. 1. Left: Meson form factors: $Q^2 F_M(Q^2)$ (upper curve), $Q^2 F_M^{\text{spinor}}(Q^2)$ (middle curve); for comparison also shown $Q^2/(1 + Q^2/m_\rho^2)$ (lower curve).

Right: Pion form factor in local quark-hadron duality model: $Q^2 F_\pi^{\text{LD}(\alpha_s)}$ (lower curve), $Q^2 F_\pi^{\text{LD}}$ (middle curve), total contribution (upper curve).

The first term vanishes, and the leading term has $1/Q^4$ behavior, the result that could be traced to an extra $\bar{x}$ factor in GPD $\mathcal{F}(x, Q^2)$. Due to slow convergence, the $1/Q^4$ asymptotics establishes only above 20 GeV^2 . As a result, the $F_M^{\text{spinor}}(Q^2)$ curve in the region of a few GeV^2 imitates the “power counting” $1/Q^2$ behavior much more successfully than Eq. (5) that displays its nominal $1/Q^2$ asymptotics only well outside the few GeV^2 region.

Referring to the “established” $1/Q^2$ behavior of meson form factors, one has in mind data on the pion EM form factor which resemble monopole form $F_\pi(Q^2) \sim 1/(1 + Q^2/m_\rho^2)$. In fact, the data are well described by our local quark-hadron duality model¹⁹, in which the pion form factor is obtained by integrating the perturbative spectral density $\rho(s_1, s_2, Q^2)$ with $\theta(s_i \leq s_0)$ weights, where $s_0 = 4\pi^2 f_\pi^2 \approx 0.7 \text{ GeV}^2$ is the duality interval. This prescription is analogous to that derived in the meromorphization approach, where density was integrated with weights proportional to $J_0(\beta_{0,1}\sqrt{s_i}/M)/(s_i - M^2)$. Explicit expression for the pion form factor in the local duality model is known from Ref.¹⁹:

$$F_\pi^{\text{LD}}(Q^2) = 1 - \frac{1 + 6s_0/Q^2}{(1 + 4s_0/Q^2)^{3/2}} = \frac{6s_0^2}{Q^4} - \frac{40s_0^3}{Q^6} + \frac{210s_0^4}{Q^8} + \mathcal{O}(s_0^4/Q^8). \quad (14)$$

Again, $F_\pi^{\text{LD}}(Q^2)$ behaves like $1/Q^4$ for large Q^2 , but convergence is slow, and for a few Q^2 the $F_\pi^{\text{LD}}(Q^2)$ curve successfully imitates the $1/Q^2$ behavior and goes very close to existing and preliminary experimental data²².

Both in the meromorphization and local duality approaches one should include higher order α_s corrections to spectral densities. Among $\mathcal{O}(\alpha_s)$ contributions, there are gluon-exchange diagrams whose asymptotic large- Q^2 behavior is determined by the hard pQCD mechanism. As a result, the leading $1/Q^2$ term of the spectral density can be written in pQCD-like form.

Substituting it into the local duality relation gives the result coinciding with the pQCD hard gluon exchange contribution $F_{\pi}^{\text{pQCD}}(Q^2) = 8\pi\alpha_s f_{\pi}^2/Q^2 = 2(s_0/Q^2)(\alpha_s/\pi)$ calculated for the asymptotic shape of the pion DA. A very good approximation for the local duality model prediction for $F_{\pi}^{\text{LD}(\alpha_s)}$ at all Q^2 is given by a simple interpolation formula²³ $F_{\pi}^{\text{LD}(\alpha_s)}(Q^2) = (\alpha_s/\pi)/(1 + Q^2/2s_0)$ between the $Q^2 = 0$ value $F_{\pi}^{\text{LD}(\alpha_s)}(0) = \alpha_s/\pi$ (fixed by the Ward identity) and the large- Q^2 asymptotic behavior. With $\alpha_s/\pi \approx 0.1$, the $\mathcal{O}(\alpha_s)$ term is a $\lesssim 30\%$ correction to the $\mathcal{O}(\alpha_s^0)$ term in the $Q^2 \leq 4 \text{ GeV}^2$ region, and their sum is in very good agreement with data²².

In this scenario, both soft and hard contributions are present, and the “observed” quark counting appear because the numerically larger soft term (having faster decrease at asymptotically large Q^2) approximately imitates the $1/Q^2$ behavior in the experimentally accessible region.

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