THE GENERALIZED GDH SUM FOR $^3$HE

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The Burkhardt–Cottingham, Bjorken and generalized GDH sum rules are all consequences of the $Q^2$-dependent dispersion relations for the virtual photon Compton amplitudes. These integrals are investigated for a $^3$He target at low $Q^2$.

1. The E94010 Experiment

We have investigated the spin structure of $^3$He in Hall A of the Thomas Jefferson National Accelerator Facility (JLab) by performing an inclusive polarized cross section measurement at low momentum transfer (0.1 < $Q^2$ < 0.9 GeV$^2$) in the quasielastic (QE) and resonance regions. Longitudinally polarized (~71%) electrons of incident energy from 0.9 to 5.0 GeV were scattered from a high pressure polarized $^3$He target. The target achieved an average polarization of 35% with an extremely high luminosity ($\mathcal{L} \approx 10^{38}$ (cm$^2$·s)$^{-1}$). Longitudinal and transverse target polarizations were maintained, allowing a precision determination of both $g_1(x, Q^2)$ and $g_2(x, Q^2)$. Further experimental details can be found in refs. [1, 2].

2. The Gerasimov-Drell-Hearn (GDH) Sum Rule

The GDH sum rule for real photon scattering relies on only a few general assumptions, as discussed for example in ref. [3]. Drell and Hearn [4] modestly describe their role in its derivation as "very simply ... joining the dispersion relation and the low-energy theorem with the no-subtraction assumption". Gerasimov [5], along with Hosoda and Yamamoto [6], inde-

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pendently derived this same relation which we present for a spin-1/2 target:

$$\int_{\nu_{th}}^{\infty} \frac{\sigma_{3/2}(\nu) - \sigma_{1/2}(\nu)}{\nu} d\nu = -2\pi^2 \alpha \left( \frac{\kappa}{M} \right)^2 \quad (1)$$

Here $\sigma_{3/2}$ ($\sigma_{1/2}$) is the cross section for photoabsorption with the photon helicity anti-parallel (parallel) to the target spin. $\nu_{th}$ represents the inelastic threshold, which is quasi-free pion production (photodisintegration) for a nucleonic (nuclear) target. Eq. 1 reveals that a non-zero anomalous magnetic moment $\kappa$ is intimately linked to the presence of an internal target structure with a complicated excitation spectrum. It is remarkable, in that it relates static properties of the ground state to dynamic properties of all the excited states. Due to the $\nu$-weighting we expect that the low-lying mass resonances will have the most significant contribution to the sum rule.

The sum rule prediction for the neutron and $^3$He are $-234 \ \mu$b and $-469 \ \mu$b respectively. To get an idea of the relative size of the “nuclear” contribution to eq. 1, we divide the $^3$He integral into two regions. Region I extends from two-body breakup ($W_{2bb}$) to the nucleonic pion production threshold, and region II extends from the pion production threshold to $\infty$. Since $^3$He roughly approximates a free neutron due to the spin pairing of the protons, we can expect the contribution from region II to be approximately $-234 \ \mu$b. Therefore the contribution in region I is necessarily quite large in order to satisfy the sum rule prediction for $^3$He. We note that the only absorption channel available to real photons in region I is photodisintegration. Indeed, Arenhövel [7] points out that the disintegration channel must become significant at low $Q^2$ in order to satisfy the sum rule prediction for the deuteron. We anticipate similar behaviour from the $^3$He nucleus.

3. Dispersive Sum Rules at Arbitrary $Q^2$

The GDH sum rule provides a natural reference at the real photon point which can be extended to a more general treatment for electron scattering at arbitrary $Q^2$. Ji and Osborne [8] suggest a generalization that takes advantage of the relationship between the forward virtual Compton amplitudes and the spin dependent structure functions. They point out that since eq. 1 is derived from the dispersion relation [9] for the invariant Compton amplitude $S_1(\nu, Q^2 = 0)$ at the real photon point, a generalized sum rule can also be constructed from the same dispersion relation at nonzero $Q^2$. This approach leads to a set of $Q^2$-dependent dispersion relations [9] for the spin structure functions:
\[ S_1(\nu, Q^2) = 4 \int_0^\infty \frac{d\nu'}{\nu'^2 - \nu^2} G_1(\nu', Q^2) \] 
\[ S_2(\nu, Q^2) = 4 \int_0^\infty \frac{d\nu}{\nu^2 - \nu'^2} G_2(\nu', Q^2) \]

where \( G_1, G_2 \) are related to the corresponding dimensionless quantities via
\[ g_1(x, Q^2) = M^2 G_1(x, Q^2) \] and \( g_2(x, Q^2) = \nu^2 G_2(x, Q^2) \).

3.1. The Generalized GDH Sum

We now consider \( \nu = 0 \) in eq. 2 to obtain the \( Q^2 \) dependent relation:
\[ \overline{S}_1(0, Q^2) = \frac{8}{Q^2} \int_0^{1-x} g_1(x, Q^2) dx \]

where the overbar signifies exclusion of the elastic contribution. This serves as a natural extension to virtual-photon scattering, and represents a \( Q^2 \)-dependent sum rule provided predictions for the \( S_1 \) amplitude can be extended beyond the low-energy theorem results at \( Q^2 = 0 \). An alternate extension of the GDH sum to finite \( Q^2 \) is often presented in the literature:
\[ I_A(Q^2) = \frac{16\pi^2\alpha}{Q^2} \int_0^{x_{th}} \left[ g_1(x, Q^2) - \frac{4M^2x^2}{Q^2} g_2(x, Q^2) \right] dx \]

where \( x_{th} = Q^2/(2M\mu_h) \). It is interesting to note that the standard \( I_A \) generalization of the GDH sum is also directly related to the Compton amplitudes [10]: \( I_A(Q^2) = 2\pi^2\alpha \left[ S_1(\nu, Q^2) - \overline{S}_1(\nu, Q^2) \right] \).

3.2. The Bjorken Sum Rule

As \( Q^2 \to \infty \), the following relation holds for the difference of the proton and neutron Compton amplitudes [8]: \( S_1^n(0, Q^2) - S_1^p(0, Q^2) = 4/3Q^2g_A \).

Eq. 4 transforms this into the Bjorken sum rule [11], which was originally derived using Current Algebra but which also follows from QCD. When evolved to finite \( Q^2 \), non-singlet QCD radiative corrections [12, 13] arise and (at leading twist) the sum rule takes the form:
\[ \Gamma^n(0, Q^2) - \Gamma^p(0, Q^2) = \frac{g_A}{6} \cdot C_{\text{MS}}(\alpha_s(\mu)) \]

This relation has been confirmed experimentally to 10% [14, 15]. Because of isospin symmetry, it is also possible to write a “Bjorken” sum rule for any pair of mirror nuclei, \(^3\text{He} \) and \(^3\text{H} \) for example [16, 17]:
\[ \Gamma_1{}^{3\text{He}}(Q^2) - \Gamma_1{}^{3\text{H}}(Q^2) = \frac{g_{A1}}{6} \cdot C_{\text{MS}}(\alpha_s(\mu)) \]
where \( g_A^{tri} \) is the axial vector coupling constant of the triton, measured in tritium decay. The advantage of eq. 7 lies in its freedom from nuclear corrections. Also, if we take the ratio of the \( A = 1 \) to \( A = 3 \) Bjorken sum rules the QCD radiative corrections cancel exactly. Both \( g_A \) and \( g_A^{tri} \) have been precisely measured and their ratio is \( 0.956 \pm 0.004 \). Comparison of the experimental value of the ratio with the predictions using exact \(^3\)He wavefunctions have been used to evaluate the size of off-shell corrections in \(^3\)He and the effect of non-nucleonic degrees of freedom \([16, 18]\).

3.3. The Burkhardt–Cottingham Sum Rule

Equation 3 for the \( S_2 \) Compton amplitude leads directly\(^a\) to the following “super-convergence” relation valid for all \( Q^2 \):

\[
0 = \int_0^1 g_2(x, Q^2) dx
\]

which is the Burkhardt–Cottingham \([19]\) (BC) sum rule. Note that the integration includes the elastic target contribution. There is a healthy skepticism of the BC sum rule in the literature\(^b\). However, it has been shown to be satisfied at first order in \( \alpha_s \) for a quark target in pQCD \([21]\), and to lowest order in \( \alpha_{em} \) in QED \([22]\). The sum rule was also tested with a complete one loop calculation for a nucleon target in ref. \([23]\).

4. Preliminary Results

Fig. 1 displays the polarized cross sections in the QE region compared to the PWIA calculation of ref. \([24, 25]\) and the full Faddeev calculation of ref. \([26, 27]\). The full calculation reproduces our data extremely well for our lowest incident energy, where at the QE peak \( Q^2 \approx 0.04 \text{ GeV}^2 \). The PWIA calculation predicts the general features of the data (the cross-over for example) but differs in absolute magnitude by several sigma. In the right panel, \( Q^2 \approx 0.19 \) at the QE peak, and we are reaching the limits of applicability of the full calculation. The most important contribution missing from the PWIA curve is expected to be the FSI, and we presumably see their importance at low \( \nu \) compared to the full calculation.

Fig. 2 (left) compares the E94010 \( g_1 \) results to the DIS results of earlier SLAC measurements \([28]\). We note that the resonance curves approximately

\(^a\)See for example ref. \([9]\).

\(^b\)See for example ref. \([20]\).
Figure 1. Polarized cross section differences in the quasielastic region. **Left:** $E_0 = 0.9$ GeV. **Right:** $E_0 = 1.7$ GeV. Dot-Dashed curve: PWIA. Solid curve: full Faddeev calculation. Inner (outer) error bars represent statistical (total) uncertainty.

average to the DIS scaling curve. This is similar to behaviour observed in the unpolarized structure functions [29], but represents the first indication of Bloom-Gilman duality in polarized $^3$He structure functions. JLab experiment E01012 examined this phenomenon in closer detail. See ref. [30].

Fig. 2 (right) displays $\Gamma_1$, the first moment of $g_1^{N\pi}$. The points labeled E94010 represent our measured data for $W_{266} < W < 2$ GeV. Statistical uncertainties are shown on the data points, while the systematic error of the measured (total) integral is represented by the light (dark) grey band. Previous SLAC [28] results are compatible with our data but have considerably larger errors. The total integral includes an estimate of the contribution from the unmeasured DIS region [31] up to $W = \sqrt{1000}$ GeV. At $Q^2 = 0$ we have plotted the slope predicted by the GDH sum rule for $^3$He. Our data is trending away from zero at low $Q^2$ and indicate that there must be a dramatic transition occurring below 0.10 GeV$^2$. To shed light on this intriguing behaviour we eagerly await the results of recently completed JLab experiment E97-110 [32], an extension of E94010 to the range $0.02 < Q^2 < 0.3$ GeV$^2$. The phenomenological model MAID [33] integrated to a maximum $W$ of 2 GeV agrees very well with our data at all $Q^2$. The model of Burkert and Ioffe [34] reproduces the full integral within errors, while the model of Soffer and Teryaev [35] tracks the data but exhibits oscillations at large $Q^2$ not observed in the measured integral. For all three models we have estimated the contribution below the pion threshold by assuming that the quasielastic reaction can be described as an incoherent sum of nucleon elastic scattering. We note that at large momentum transfer, $\Gamma_1(Q^2)$ appears
to be nearly independent of $Q^2$. This is consistent with an OPE interpretation in which the higher twist effects become negligible at large $Q^2$ and the evolution is driven by logarithmic pQCD effects alone. It is however somewhat surprising to observe this behaviour below 1.0 GeV$^2$ in a region where the higher twist effects are conventionally expected to be quite significant. Finally, we point out that our $\Gamma_1(Q^2)$ data is a direct test of the sum rule of eq. 4, and should provide a valuable constraint on future \chi PT and lattice gauge calculations of the $S_1$ Compton amplitude.

Experimental measurements of $g_2$ are scarce and only recently has the BC sum rule been evaluated for the first time. The SLAC E1xx collaborations [36] measured $\Gamma_2(Q^2)$ at $Q^2 = 5$ GeV$^2$. They found the BC sum rule to be satisfied for the deuteron, while a violation of almost $3\sigma$ was found for the proton. In fig. 3 (left) we plot $\Gamma_2$ using the same symbols as for $\Gamma_1$. In this case, the unmeasured DIS contribution is estimated via the leading twist Wandzura-Wilczek relation. The total integral includes the elastic contribution which has been evaluated using the form factors of ref. [37]. MAID follows the trend of our measured points but differs in absolute value. We note that all of our data points are consistent with the Burkhardt–Cottingham sum rule. Data from this same experiment has also been used to test the BC sum rule for the neutron [38]. It is interesting to notice that in the case of the neutron the sum rule is satisfied primarily due to the cancellation of the resonance and elastic contributions, while for $^3$He the elastic contribution is mostly unimportant. Instead, we see a balance struck between the large positive contribution from the integral above the
pion threshold and a large negative contribution from the QE region.

Fig. 3 (right) displays our measured values of the extended GDH sum as defined in eq. 5. Including the DIS contribution [31] has only a minor effect due to the $1/\mu$-weighting of the integrand. The negative sum rule prediction at $Q^2 = 0$ stands in stark contrast to the large positive value of our lowest point. On the other hand, the MAID model tracks the data well. The $^3$He GDH integral is dominated by a positive QE contribution which largely outweighs the negative contribution of the resonances. If we assume the continuity of $\sigma_{\text{PT}}^I$ as $Q^2 \to 0$, as in the nucleonic case [39], our results indicate the necessity of a dramatic turnover in $I_A$ at very low $Q^2$.

5. Conclusion

We have presented $^3$He results which are consistent with the Burkhardt-Cottingham sum rule. The GDH integrals display intriguing behaviour at low $Q^2$ which is also predicted by several models. In the future, the E94010 data set will provide valuable constraints on $^3$He $\chi$PT and lattice calculations. Measurement of the $g_1$ structure function of tritium in the same kinematic range as E94010 would allow a high precision test of the $A = 3$ Bjorken sum rule, free from nuclear effects. Also this isospin difference could extend the applicability of $\chi$PT to higher $Q^2$ as discussed in ref. [40].

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