TOPICS IN THEORY OF GENERALIZED PARTON DISTRIBUTIONS

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Abstract

Several topics in the theory of generalized parton distributions (GPDs) are reviewed. First, we give a brief overview of the basics of the theory of generalized parton distributions and their relationship with simpler phenomenological functions, *viz.* form factors, parton densities and distribution amplitudes. Then, we discuss recent developments in building models for GPDs that are based on the formalism of double distributions (DDs). A special attention is given to a careful analysis of the singularity structure of DDs. The DD formalism is applied to construction of a model GPDs with a singular Regge behavior. Within the developed DD-based approach, we discuss the structure of GPD sum rules. It is shown that separation of DDs into the so-called "plus" part and the D-term part may be treated as a renormalization procedure for the GPD sum rules. This approach is compared with an alternative prescription based on analytic regularization.

1 Introductory remarks

The basic role played by the generalized parton distributions (GPDs) [1, 2, 3, 4, 5, 6, 7] is to access the fundamental physics related to the structure of hadrons. This is a rather general statement, and one may wish to confront it with a more specific one. A classic example of such a specific case is the celebrated search for the Higgs boson (HB) performed currently at the

Large Hadron Collider (LHC). The motivation for the search is that HB is supposed to be responsible for generation of fermion masses, in particular, quark masses.

Now, with the announced discovery [8, 9] of the Higgs particle, can we say that the problem of generation of visible mass is completely solved? Unfortunately, no! In fact, by far the largest part of the visible mass is due to the nucleons, and out of 940 MeV of the nucleon mass, the origin of less than 30 MeV (current quark masses) may be related to the Higgs boson. The remaining more than 97% of the nucleon mass is due to gluons – which are represented as massless fields in the QCD Lagrangian !
This is a characteristic illustration of the situation in hadron physics:
i) All the relevant particles are already established: no "higgses" to find.
ii) The QCD Lagrangian is known.
iii) However, we still need to understand how QCD works,
i.e., to understand hadronic structure in terms of quark and gluon fields.

The evident thing to do is to project quark and gluon fields $q(z_1)$, $q(z_2)$, ... onto hadronic states $|p, s\rangle$. This gives matrix elements:

$$\langle\, 0\,|\,\bar{q}_\alpha(z_1)\,q_\beta(z_2)\,|\,M(p), s\,\rangle \quad , \quad \langle\, 0\,|\,q_\alpha(z_1)\;q_\beta(z_2)\,q_\gamma(z_3)|\,B(p), s\,\rangle \qquad (1)$$

that can be interpreted as hadronic wave functions. In particular, in the light-cone (LC) formalism [10], a hadron is described by its Fock components in the infinite-momentum frame. For the nucleon, one can schematically write:

$$|P\,\rangle = \Psi_{qqq}|q(x_1P, k_{1\perp})q(x_2P, k_{2\perp})q(x_3P, k_{3\perp})\rangle + \Psi_{qqqG}|qqqG\rangle + \Psi_{qqq\bar{q}q}|qqq\bar{q}q\rangle + \dots\,, \qquad (2)$$

where x_i are momentum fractions satisfying $\sum_i x_i = 1$; $k_{i\perp}$ are transverse momenta, $\sum_i k_{i\perp} = 0$. In principle, solving the bound-state equation $H|P\,\rangle = E|P\,\rangle$ one should get the wave function $|P\,\rangle$ that contains complete information about the hadron structure. In practice, however, the equation (involving an infinite number of Fock components) has not been solved yet in the realistic 4-dimensional case. Moreover, the LC wave functions are not directly accessible experimentally.

The way out in this situation is the description of hadron structure in terms of phenomenological functions. Among the "old" functions used for a long time we can list form factors, usual parton densities, and distribution amplitudes. The "new" functions, generalized parton distributions (for reviews, see Refs.[11, 12, 13, 14, 15]), are hybrids of form factors, parton densities and distribution amplitudes. Furthermore, the "old" functions are limiting cases of the "new" ones.

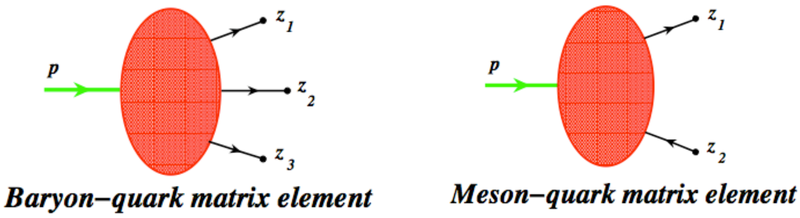

Figure 1: Hadron-to-quarks matrix elements.

The relation of GPDs to more simple "old" functions is an essential element in constructing realistic models of GPDs. One of the most restrictive constraints is imposed by the formula [2] relating GPDs to the usual parton densities, which may be treated as a "forward" limit of GPDs. A nontrivial observation here is that GPDs contain contributions which are "invisible" in the forward limit, such as the D-term [16]. In addition to the requirements of reproducing "old" functions in specified limits ("reduction relations"), such models should satisfy other constraints, such as polynomiality [11], and correspondence with Regge behavior of usual parton densities in the region of small parton momenta.

The polynomiality constraint is highly nontrivial, but it is automatically satisfied if GPDs are built from so-called "double distributions" [1, 4, 6]. However, imposing on DDs the constraints dictated by correspondence with the Regge behavior one faces rather singular functions, and this raises a lot of questions related to the singularity structure of GPDs in general.

The goal of the present paper is, first, to give a brief overview of the basics of the theory of generalized parton distributions and their relationship with previously used phenomenological functions, and, second, to describe a recent development [17] in modeling GPDs based on their formulation in terms of double distributions, with emphasis on careful disentangling their singularity structure. To this end, in Section 2 we start with an overview, of "old" phenomenological functions. Their relation with generalized parton distributions is discussed in Section 3 . The formalism of double distributions is outlined in Section 4. Before switching to the discussion of more technical issues related to modeling GPDs within DD formalism, a brief summary of the content of Sections 2–4 is given in Section 5. A specific problem of building model GPDs with a Regge behavior is addressed in Section 6. The model described there provides a particular example of singularities that one may encounter in GPD construction. It also gives a nontrivial example of a situation when the part of the D-term (that is formally invisible in the forward limit), comes from a term generated by the correspondence with the usual ("forward") parton densities. As shown in recent papers [18, 19, 20, 21, 22], the D term also appears as a subtraction constant in dispersion sum rules for GPDs. In Section 7, we study these sum rules within the DD formalism used in previous sections, in particular, we show that separation of DDs into the so-called "plus" part and the D-term part may be treated as a renormalization procedure for the GPD sum rules. In Section 8, we compare it to the alternative prescription based on analytic regularization used in Refs. [20, 23, 24]. Our conclusions are formulated in Section 9.

2 "Old" phenomenological functions

Form factors. The form factors are defined through matrix elements of electromagnetic (EM) and weak currents between hadronic states. In particular, the nucleon electromagnetic form factors are given by

$$\langle\, p',\, s' \,|\, J^\mu(0) \,|\, p,\, s \,\rangle = \bar{u}(p',s') \left[\gamma^\mu F_1(t) + \frac{r^\nu \sigma^{\mu\nu}}{2m_N} F_2(t) \right] u(p,s) \ , \tag{3}$$

where $r = p - p'$ is the momentum transfer and $t = r^2$. The electromagnetic current is given by the sum of its flavor components:

$$J^\mu(z) = \sum_f e_f \bar{\psi}_f(z) \gamma^\mu \psi_f(z) \, . \tag{4}$$

The nucleon helicity non-flip form factor $F_1(t)$ can also be written as a sum $\sum_f e_f F_{1f}(t)$. A similar decomposition holds for the helicity flip form factor $F_2(t) = \sum_f e_f F_{2f}(t)$. At $t = 0$, these functions have well known limiting values. In particular, $F_1(t=0) = e_N = \sum_f N_f e_f$ gives total electric charge of the nucleon (N_f is the number of valence quarks of flavor f) and $F_2(t=0) = \kappa_N$ gives its anomalous magnetic moment. The form factors are measurable through elastic eN scattering.

Usual parton densities. The parton densities are defined through forward matrix elements of quark/gluon fields separated by light-like distances. In particular, in the unpolarized case we have

$$\langle\, p \,|\, \bar{\psi}_a(-z/2) \gamma^\mu \psi_a(z/2) \,|\, p \,\rangle\big|_{z^2=0} = 2p^\mu \int_0^1 \left[e^{-ix(pz)} f_a(x) - e^{ix(pz)} f_{\bar{a}}(x) \right] dx \ . \tag{5}$$

In the local limit $z = 0$, the operators in this definition coincide with the operators contributing into the non-flip form factor F_1. Since $t = 0$ for the forward matrix element, we obtain the sum rule for the numbers of valence quarks:

$$\int_0^1 [f_a(x) - f_{\bar{a}}(x)]\, dx = N_a\,. \tag{6}$$

The definition of parton densities has the form of the plane wave decomposition. This observation allows one to give the momentum space interpretation: $f_{a(\bar{a})}(x)$ is the probability to find $a\,(\bar{a})$-quark with momentum xp inside a nucleon with momentum p. The classic process to access the usual parton densities is deep inelastic scattering (DIS) $\gamma^* N \to X$.

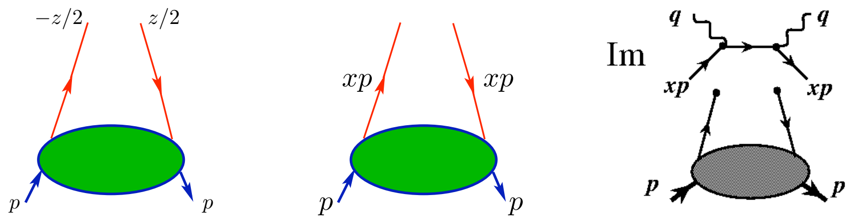

Figure 2: Matrix element defining parton densities, their momentum-space interpretation and lowest order pQCD factorization for DIS.

Using the optical theorem, the $\gamma^* N \to X$ cross section is given by the imaginary part of the forward virtual Compton scattering amplitude. The momentum transfer q is spacelike $q^2 \equiv -Q^2$, and when it is sufficiently large, perturbative QCD factorization works. At the leading order, one deals with the so-called handbag diagram, see Fig. 2. Through simple algebra,

$$\frac{1}{\pi}\mathrm{Im}\; 1/(q+xp)^2 \approx \frac{\delta(x - x_B)}{2(pq)}\ ,$$

one finds that DIS measures parton densities at the point $x = x_B$, where the parton momentum fraction equals the Bjorken variable $x_B = Q^2/2(pq)$. Comparing parton densities to form factors, we note that the latter have a point vertex instead of a light-like separation and $p \neq p'$.

Distribution amplitudes Another example of nonperturbative functions describing the hadron structure are the distribution amplitudes (DAs). They can be interpreted as light cone wave functions integrated over transverse momentum, or as $\langle 0|\ldots|p\rangle$ matrix elements of light cone operators. In the pion case, we have

$$\langle\, 0\,|\,\bar{\psi}_d(-z/2)\gamma_5\gamma^\mu\psi_u(z/2)\,|\,\pi^+(p)\,\rangle\big|_{z^2=0} = ip^\mu f_\pi \int_{-1}^{1} e^{-i\alpha(pz)/2}\varphi_\pi(\alpha)\, d\alpha\,, \tag{7}$$

with $x_1 = (1+\alpha)/2$, $x_2 = (1-\alpha)/2$ being the fractions of the pion momentum carried by the quarks. The distribution amplitudes describe the hadrons in situations when the pQCD hard scattering approach is applicable to exclusive processes. The classic example is the $\gamma^*\gamma \to \pi^0$ transition; its amplitude is proportional to the $1/(1-\alpha^2)$ moment of $\varphi_\pi(\alpha)$, see Fig. 3, right.

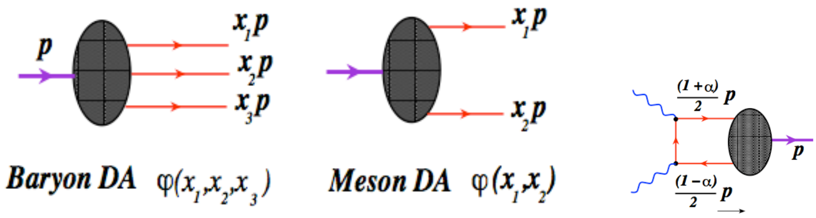

Figure 3: Left: baryon and meson distribution amplitudes. Right: lowest-order pQCD factorization for $\gamma^*\gamma \to \pi^0$ transition form factor.

3 Generalized parton distributions

The classic process that requires the description of the hadron structure in terms of the generalized parton distributions is the deeply virtual Compton scattering (DVCS) $\gamma^*N \to \gamma N$. It is convenient to visualize DVCS in the γ^*N center-of-mass frame, with the initial hadron and the virtual photon moving in opposite directions along the z-axis. When the momentum transfer t is small, the hadron and the real photon in the final state also move close to the z-axis. This means that the virtual photon momentum $q = q' - x_B p$ has the component $-x_B p$ canceled by the momentum transfer r. In other words, the momentum transfer r has the longitudinal component $r^+ = x_B p^+$, where $x_B = Q^2/2(pq)$ is the DIS Bjorken variable. One can say that DVCS has a skewed kinematics in which the final hadron has the "plus" momentum $(1-\zeta)p^+$ that is smaller than that of the initial hadron. In the particular case of DVCS, we have $\zeta = x_B$.

Nonforward parton distributions. The parton picture for DVCS has some similarity to that of DIS, with the main difference that the plus-momenta of the incoming and outgoing quarks in DVCS are not equal; they are Xp^+ and $(X-\zeta)p^+$, see Fig. 4. Another difference is that the invariant momentum transfer t in DVCS is nonzero: the matrix element of partonic fields is essentially nonforward.

Thus, the nonforward parton distributions (NFPDs) $\mathcal{F}_\zeta(X,t)$ describing the hadronic structure in DVCS depend on X (the fraction of p^+ carried by

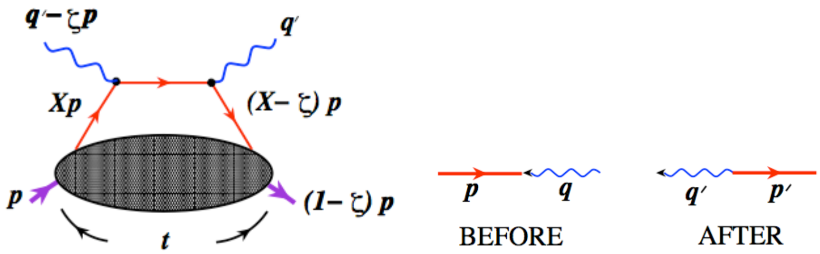

Figure 4: Lowest-order DVCS in terms of nonforward parton distributions.

the outgoing quark), ζ (the skewness parameter characterizing the difference between initial and final hadron momenta), and t (the invariant momentum transfer). In the forward $r = 0$ limit, we have a reduction formula

$$\mathcal{F}^a_{\zeta=0}(X, t = 0) = f_a(X) \tag{8}$$

relating NFPDs with the usual parton densities. The nontriviality of this relation is that $\mathcal{F}_\zeta(X, t)$ appear in the amplitude of the exclusive DVCS process, while the usual parton densities are measured from the cross section of the inclusive DIS reaction. The local limit relates NFPDs to form factors:

$$\int_0^1 \mathcal{F}^a_\zeta(X, t)\, dX = F^a_1(t)\,(1 - \zeta/2)\,. \tag{9}$$

Off-forward parton distributions. The description in terms of NFPDs has the advantage of using the variables most close to those of the usual parton densities. However, the initial and final hadron momenta are not treated symmetrically in this scheme. X. Ji [2] proposed to use symmetric variables in which the plus-momenta of the hadrons are $(1 + \xi)P^+$ and $(1 - \xi)P^+$, and those of the active partons are $(x + \xi)P^+$ and $(x - \xi)P^+$, P being the average momentum $P = (p + p')/2$, see Fig. 5. The relevant functions were called originally "off-forward parton distributions" (OFPDs). At present, most researchers use OFPDs, referring to them simply as GPDs. (When the NFPDs conventions are used, the functions are still called GPDs, but specifying that the variables X, ζ correspond to definitions of Ref. [6].) In the simplified case of scalar fields, the GPD parametrization of the nonforward matrix element is

$$\langle P + r/2|\psi(-z/2)\psi(z/2)|P - r/2\rangle = \int_{-1}^1 e^{-ix(Pz)} H(x, \xi)\, dx + \mathcal{O}(z^2)\,. \tag{10}$$

To take into account the spin properties of hadrons and quarks, one needs four generalized parton distributions $H, E, \tilde{H}, \widetilde{E}$, each of which is a function of x, ξ, and t. The skewness parameter $\xi \equiv r^+/2P^+$ can be expressed in terms of the Bjorken variable, $\xi = x_B/(2 - x_B)$, but it does not coincide with the latter.

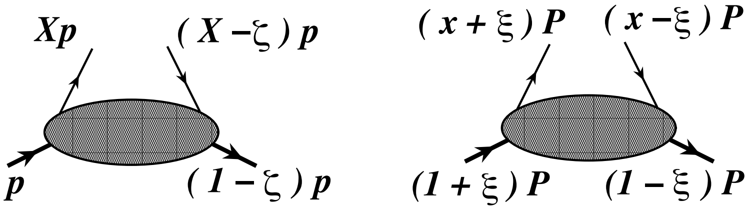

Figure 5: Comparison of NFPDs and OFPDs.

Depending on the value of x, each GPD has 3 distinct regions. When $\xi < x < 1$, GPDs are analogous to usual quark distributions; when $-1 < x < -\xi$, they are similar to antiquark distributions. In the region $-\xi < x < \xi$, the "returning" quark has a negative momentum and should be treated as an outgoing antiquark with momentum $(\xi - x)P$. The total $q\bar{q}$ pair momentum $r = 2\xi P$ is shared by the quarks in fractions $r(1 + x/\xi)/2$ and $r(1 - x/\xi)/2$. Hence, a GPD in the region $-\xi < x < \xi$ is similar to a distribution amplitude $\Phi(\alpha)$ with $\alpha = x/\xi$.

4 Double distributions

Double distributions as hybrids of parton densities and distribution amplitudes. The main idea behind the double distributions [1, 4, 5, 25, 26] is a "superposition" of P^+ and r^+ momentum flows, i.e., the representation of the parton momentum $k^+ = \beta P^+ + (1 + \alpha)r^+/2$ as the sum of a component βP^+ due to the average hadron momentum P (flowing in the s-channel) and a component $(1 + \alpha)r^+/2$ due to the t-channel momentum r, see Fig. 6. In the simplified case of scalar fields, the DD parametrization reads

$$\langle P - r/2|\psi(-z/2)\psi(z/2)|P + r/2\rangle = \int_\Omega F(\beta, \alpha)\, e^{-i\beta(Pz) - i\alpha(rz)/2}\, d\beta\, d\alpha + \mathcal{O}(z^2) \; . \tag{11}$$

Thus, the double distribution $f(\beta, \alpha)$ (we consider here for simplicity the $t = 0$ limit) looks like a usual parton density with respect to β and like a distribution amplitude with respect to α. The support region Ω is specified

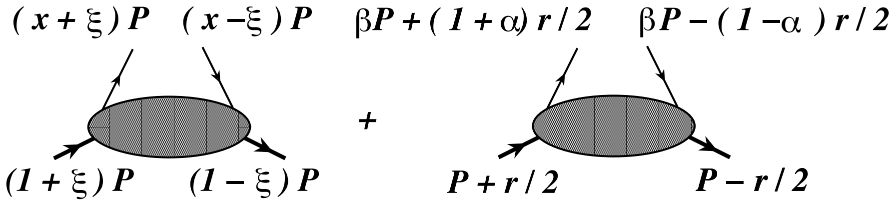

Figure 6: Comparison of GPD and DD descriptions.

by $|\beta| + |\alpha| \leq 1$. The connection between the DD variables β, α and the GPD variables x, ξ is obtained from $r^+ = 2\xi P^+$, which results in the basic relation $x = \beta + \xi\alpha$. The formal connection between DDs and GPDs is

$$H(x, \xi) = \int_\Omega F(\beta, \alpha)\, \delta(x - \beta - \xi\alpha)\, d\beta\, d\alpha \ . \tag{12}$$

Local operators and DDs. The definition of DDs may also be given through parameterization of symmetric-traceless part $\psi(0)\{\overset{\leftrightarrow}{\partial}_{\mu_1} \ldots \overset{\leftrightarrow}{\partial}_{\mu_n}\}\psi(0)$ (denoted by { }) of the composite local operators resulting from the Taylor expansion of the bilocal operator used in the definition given above. For a scalar target, one may write

$$\begin{aligned}&\langle P + r/2|\psi(0)\{\overset{\leftrightarrow}{\partial}_{\mu_1} \ldots \overset{\leftrightarrow}{\partial}_{\mu_n}\}\psi(0)|P - r/2\rangle \\ &= \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n-1} A_{nl}\{P_{\mu_1} \ldots P_{\mu_{n-l}} r_{\mu_{n-l+1}} \ldots r_{\mu_n}\} + A_{nn}\{r_{\mu_1} \ldots r_{\mu_n}\}\right] .\end{aligned} \tag{13}$$

In the momentum representation, the derivative $\overset{\leftrightarrow}{\partial}_\mu$ converts into the average $\bar{k}_\mu = (k_\mu + k'_\mu)/2$ of the initial k and final k' quark momenta. After integration over k, $(\bar{k})^n$ should produce the P and r factors in the r.h.s. of the equation above. In this sense, one may treat $(\bar{k})^n$ as $(\beta P + \alpha r/2)^n$ and define DDs through

$$\frac{n!}{(n-l)!\, l!\, 2^l} \int_\Omega F(\beta, \alpha) \beta^{n-l} \alpha^l\, d\beta\, d\alpha = A_{nl} \tag{14}$$

as a function whose $\beta^{n-l}\alpha^l$ moments are proportional to the coefficients A_{nl}.

D-term, scalar quarks. Parameterizing the matrix element (13), one may wish to separate the A_{nn} terms that are accompanied by tensors built from the momentum transfer vector r only (and, thus, invisible in the forward $r = 0$ limit), and introduce the *D-term* [16]

$$\int_{-1}^{1} D(\alpha)\,(\alpha/2)^n\,d\alpha = A_{nn} \tag{15}$$

as a function whose $(\alpha/2)^n$ moments give A_{nn}. Within the DD-parameterization, the separation of the D-term can be made by simply using

$$e^{-i\beta(Pz)} = [e^{-i\beta(Pz)} - 1] + 1 \; .$$

The D-term is then given by

$$D(\alpha) = \int_{-1+|\alpha|}^{1-|\alpha|} F(\beta,\alpha)\,d\beta \; , \tag{16}$$

and the DD-parameterization converts into a "DD plus D" parameterization

$$\begin{aligned}\langle P - r/2|\psi(-z/2)\psi(z/2)|P + r/2\rangle &= \int_\Omega [F(\beta,\alpha)]_+\, e^{-i\beta(Pz)-i\alpha(rz)/2}\,d\beta\,d\alpha \\ &+ \int_{-1}^{1} D(\alpha)\,e^{-i\alpha(rz)/2}\,d\alpha + \mathcal{O}(z^2) \; , \end{aligned} \tag{17}$$

where

$$[F(\beta,\alpha)]_+ = F(\beta,\alpha) - \delta(\beta)\int_{-1+|\alpha|}^{1-|\alpha|} F(\gamma,\alpha)\,d\gamma \tag{18}$$

is the DD with subtracted D-term. Mathematically, $[F(\beta,\alpha)]_+$ is a "plus distribution" with respect to β. It satisfies the condition

$$\int_{-1+|\alpha|}^{1-|\alpha|} [F(\beta,\alpha)]_+\,d\beta = 0 \; , \tag{19}$$

guaranteeing that no D-term can be constructed from $[F(\beta,\alpha)]_+$.

Spin-1/2 quarks: two-DD representation. In the simple model with scalar quarks discussed above, one may just use the original DD $F(\beta,\alpha)$ without splitting it into the "plus" part and the D-term. In models with spin-1/2 quarks, it is more difficult to avoid an explicit introduction of extra functions producing a D-term. The basic reason [16] is that the matrix element of the bilocal operator, even in the case of spin-0 hadrons, should have two parts

$$\begin{aligned}&\langle P - r/2|\bar\psi(-z/2)\gamma_\mu\psi(z/2)|P + r/2\rangle|_{\text{twist}-2} \\ &= 2P_\mu f\big((Pz),(rz),z^2\big) + r_\mu g\big((Pz),(rz),z^2\big) \; . \end{aligned} \tag{20}$$

This suggests to introduce a parametrization with two DDs corresponding to f and g functions [16]. For the matrix element (20) multiplied by z^μ – which is exactly what one obtains doing the leading-twist factorization for the Compton amplitude [27] – this gives

$$
\begin{aligned}
& z^\mu \langle P - r/2 | \bar{\psi}(-z/2)\gamma_\mu \psi(z/2) | P + r/2 \rangle \\
& = \int_\Omega e^{-i\beta(Pz) - i\alpha(rz)/2} \left[2(Pz)F(\beta,\alpha) + (rz)G(\beta,\alpha) \right] d\beta \, d\alpha \; + \mathcal{O}(z^2). \qquad (21)
\end{aligned}
$$

The separation into F- and G-parts in this case is not unique: expanding the exponential in powers of (Pz) and (rz), one may obtain the same $(Pz)^m(rz)^l$ term both from the F-type and G-type parts. This leads to possibility of "gauge transformations": one can change [28]

$$
F(\beta,\alpha) \to F(\beta,\alpha) + \partial\chi(\beta,\alpha)/\partial\alpha \; , \qquad (22)
$$

$$
G(\beta,\alpha) \to G(\beta,\alpha) - \partial\chi(\beta,\alpha)/\partial\beta \; , \qquad (23)
$$

using a gauge function $\chi(\beta,\alpha)$ that is odd in α. Still, the terms $(Pz)^0(rz)^l$ cannot be produced from the F-type contribution. The maximum of what can be done is to absorb all $m \neq 0$ contributions into the F-type term. As a result, Eq. (21) is converted into a "DD plus D" parameterization [16] in which the term in the square brackets is substituted by the

$$
2(Pz)F_D(\beta,\alpha) + (rz)\delta(\beta)D(\alpha) \qquad (24)
$$

combination, with $D(\alpha)$ given by the β-integral of $G(\beta,\alpha)$ and $F_D(\beta,\alpha)$ related to the original DDs through the gauge transformation (cf. Refs.[28, **?**]).

Spin-1/2 quarks: single-DD representation. In fact, since the Dirac index μ is symmetrized in the local twist-two operators $\bar{\psi}\{\gamma_\mu \overset{\leftrightarrow}{\partial}_{\mu_1} \ldots \overset{\leftrightarrow}{\partial}_{\mu_n}\}\psi$ with the μ_i indices related to the derivatives, one may expect that it also produces the factor $\beta P_\mu + \alpha r_\mu/2$. As shown by the authors of Ref.[29], this is precisely what happens. In their construction, not only the exponential produces the z-dependence in the combination $\beta(Pz) + \alpha(rz)/2$, but also the pre-exponential terms come in the $\beta(Pz) + \alpha(rz)/2$ combination, i.e., the result is a representation in which

$$
2(Pz)F(\beta,\alpha) + (rz)G(\beta,\alpha) = [2\beta(Pz) + \alpha(rz)]f(\beta,\alpha) \; , \qquad (25)
$$

that corresponds to $F(\beta,\alpha) = \beta f(\beta,\alpha)$ and $G(\beta,\alpha) = \alpha f(\beta,\alpha)$. Thus, formally, one deals with just one DD $f(\beta,\alpha)$. In principle, though, this single function may be a sum of several components, e.g., $\delta(\alpha)f(\beta)/\beta+\delta(\beta)D(\alpha)/\alpha$ (the result of the pioneering D-term paper [16] for the pion DD in an effective chiral model corresponds to $f^{I=0}(\beta,\alpha)=\delta(\alpha)/|\beta|-\delta(\beta)/|\alpha|$).

In the two-DD approach, GPDs are introduced through

$$H(x,\xi)=\int_\Omega \big[F(\beta,\alpha)+\xi G(\beta,\alpha)\big]\,\delta(x-\beta-\xi\alpha)\,d\beta\,d\alpha\ , \tag{26}$$

which converts into

$$H(x,\xi)=x\int_\Omega f(\beta,\alpha)\,\delta(x-\beta-\xi\alpha)\,d\beta\,d\alpha \tag{27}$$

in the "single-DD" formulation. The D-term in the single-DD case is given by

$$D(\alpha)=\alpha\int_{-1+|\alpha|}^{1-|\alpha|} f(\beta,\alpha)\,d\beta\ , \tag{28}$$

and one may write $f(\beta,\alpha)$ as a sum

$$f(\beta,\alpha)=[f(\beta,\alpha)]_+ +\delta(\beta)D(\alpha)/\alpha \tag{29}$$

of its "plus" part $[f(\beta,\alpha)]_+$ (cf. Eq.(18)) and D-term part $\delta(\beta)D(\alpha)/\alpha$.

Getting GPDs from DDs. The relation between DDs and GPDs can be illustrated on the DD support rhombus $|\beta|+|\alpha|\leq 1$ (see Fig. 7).

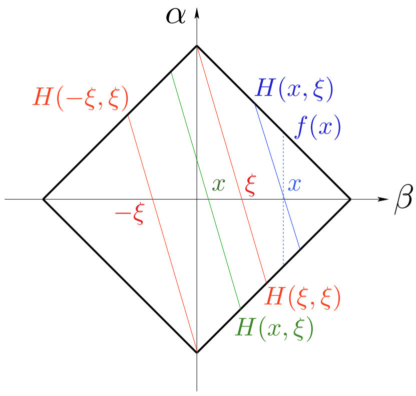

Figure 7: Support region for double distributions and lines producing $f(x), H(x,\xi)$ (for $x>\xi$ and $x<\xi$), $H(\xi,\xi)$ and $H(-\xi,\xi)$.

The delta-function in Eq. (27) specifies the line of integration in the $\{\beta,\alpha\}$ plane. To get $H(x,\xi;t)$, one should integrate $f(\beta,\alpha)$ over α along a straight line $\beta = x - \xi\alpha$. Fixing some value of ξ, one deals with a set of parallel lines intersecting the β-axis at $\beta = x$. The upper limit of the α-integration is determined by intersection of this line either with the line $\beta + \alpha = 1$ (this happens if $x > \xi$) or with the line $-\beta + \alpha = 1$ (if $x < \xi$). Similarly, the lower limit of the α-integration is set by the line $\beta - \alpha = 1$ for $x > -\xi$ or by the line $\beta + \alpha = -1$ for $x < -\xi$. The lines corresponding to $x = \pm\xi$ separate the rhombus into three parts generating the three components of $H(x, \xi; t)$:

$$
\begin{aligned}
H_a(x, \xi; t) = \theta(\xi \leq x \leq 1) \int_{-\frac{1-x}{1+\xi}}^{\frac{1-x}{1-\xi}} f_a(x - \xi\alpha, \alpha)\, d\alpha \\
+\theta(-\xi \leq x \leq \xi) \int_{-\frac{1-x}{1+\xi}}^{\frac{1+x}{1+\xi}} f_a(x - \xi\alpha, \alpha)\, d\alpha \\
+\theta(-1 \leq x \leq -\xi) \int_{-\frac{1+x}{1-\xi}}^{\frac{1+x}{1+\xi}} f_a(x - \xi\alpha, \alpha)\, d\alpha.
\end{aligned}
\tag{30}
$$

For $x > \xi > 0$, the integration lines lie completely inside the right half of the rhombus. The line producing GPD at the "border" point $x = \xi$ starts at its upper corner, while the lines corresponding to $|x| < \xi$ cross the line $\beta = 0$. Thus, one deals with the "outer" regions $x > \xi$ and $x < -\xi$ (in this case, the whole line is in the left half of the rhombus) and the central region $-\xi < x < \xi$, when the integration lines in the (β, α) plane lie in both halves of the rhombus and intersect the $\beta = 0$ line.

The forward limit $r = 0$ corresponds to $\xi = 0$, and GPD $H(x, \xi)$ converts into the usual parton distribution $f(x)$. Using DDs, we may write

$$
f(x) = \int_{-1+|x|}^{1-|x|} F(x, \alpha)\, d\alpha = x \int_{-1+|x|}^{1-|x|} f(x, \alpha)\, d\alpha\ . \tag{31}
$$

Thus, the forward distributions $f(x)$ are obtained by integrating DDs over vertical lines $\beta = x$ in the (β, α) plane. For nonzero ξ, GPDs are obtained from DDs through integrating them along the lines $\beta = x - \xi\alpha$ having $1/\xi$ slope, i.e. the family of $H(x, \xi)$ functions for different values of ξ is obtained by "scanning" the same DD at different angles.

In GPD variables (x, ξ), the momentum fraction $x - \xi$ carried by the final quark is positive for the right outer region, and negative for the central region, i.e., in the latter case it should be interpreted as an outgoing antiquark rather than incoming quark [4], i.e. GPD in the central region describes emission of a quark-antiquark pair with total plus momentum r^+ shared in fractions $(1 + x/\xi)/2$ and $(1 - x/\xi)/2$, like in a meson distribution amplitude.

From this physical interpretation, one may expect that the behavior of a GPD $H(x, \xi)$ in the central region is unrelated to that in the outer region. But, since the GPD in both regions is obtained from the same DD, one may expect, to the contrary, that the set of GPDs for all "outer" x's and all ξ's contains the same information as the set of GPDs for all central x's and all ξ's. This "holographic" picture (cf. Refs.[20, 23]) may be violated by terms contributing to GPDs in the central region and not contributing to GPDs in the outer regions: by the terms with support on the $\beta = 0$ line, i.e., those proportional to $\delta(\beta)$ (and, in principle, its derivatives), in particular, by the D-term. For this reason, the usual approach is to build separate models for the D-term and for the remaining part of DD.

Recall that integrating the DD $f(x, \alpha; t = 0)$ over a vertical line gives the usual parton density $f(x)$. To get the $t = 0$ GPDs one should scan the same DD along the lines having a ξ-dependent slope. Thus, one can try to build models for SPDs using information about usual parton densities. Note, however, that the usual parton densities are insensitive to the meson-exchange type contributions $H_M(x, \xi; t)$ coming from the singular $x = 0$ parts of DDs. Thus, information contained in GPDs originates from two physically different sources: meson-exchange type contributions $H_M(x, \xi; t)$ and the functions $H_M(x, \xi; t)$ obtained by scanning the $x \neq 0$ parts of DDs $f(x, \alpha; t)$. The support of exchange contributions is restricted to $|x| \leq \xi$. Up to rescaling, the function $H_M(x, \xi; t)$ has the same shape for all ξ, e.g., $\varphi_M(x/\xi; t)/|\xi|$. For any nonvanishing ξ, these exchange terms become invisible in the forward limit $\xi \to 0$. On the other hand, interplay between x and ξ dependences of the component resulting from integrating the $x \neq 0$ part of DDs is quite nontrivial. Its support in general covers the whole $-1 \leq x \leq 1$ region for all ξ including the forward limit ξ in which they convert into the usual (forward) densities $f^a(x)$, $f^{\bar{a}}(x)$. The latter are rather well known from inclusive measurements. at small t.

Factorized DD Ansatz. The reduction formula (31) suggests a model

$$f(\beta, \alpha) = h(\beta, \alpha)\, f(\beta)/\beta \ , \tag{32}$$

where $f(\beta)$ is the forward distribution, while $h(\beta, \alpha)$ determines DD profile in the α direction and satisfies the normalization condition

$$\int_{-1+|\beta|}^{1-|\beta|} h(\beta, \alpha)\, d\alpha = 1 \ . \tag{33}$$

Since the plus component of the momentum transfer r is shared between the quarks in fractions $(1 + \alpha)/2$ and $(1 - \alpha)/2$, like in a meson distribution amplitude, it was proposed [25, 26] to model the shape of the profile function by

$$h_N(\beta,\alpha) \sim \frac{[(1-|\beta|)^2-\alpha^2]^N}{(1-|\beta|)^{2N+1}} \ , \tag{34}$$

that vanishes at the sides of the support rhombus $|\alpha|+|\beta| \leq 1$, with N being a parameter governing the width of the profile.

Such a factorized DD Ansatz (FDDA) was originally applied [25, 26] to an analog of the $F(\beta,\alpha)$ function of the two-DD formalism, which corresponds to a model $F(\beta,\alpha) = f(\beta)h(\beta,\alpha)$ and $G(\beta,\alpha) = 0$. Later, it was corrected by addition of the D-term [16], which formally corresponds to the "gauge" (23) in which $G(\beta,\alpha) \to G_D(\beta,\alpha) = \delta(\beta)D(\alpha)$, and $F(\beta,\alpha) \to F_D(\beta,\alpha)$. Note that if $F = \beta f$ and $G = \alpha f$, the model $F_D(\beta,\alpha) = f(\beta)\,h(\beta,\alpha)$ does not coincide with the model $f(\beta,\alpha) = f(\beta)\,h(\beta,\alpha)/\beta$, since the gauge function $\chi_D(\beta,\alpha)$ (see Eq. (22)) is nontrivial.

Thus, there is a question whether the FDDA should be applied to $F_D(\beta,\alpha)$ (as it was done so far) or to the DD $f(\beta,\alpha)$ of the single-DD formulation. It should be confessed that no enthusiasm has been observed to use FDDA in the form of the single-DD formula (32). This observation has a simple explanation: the function $f(\beta)/\beta$ is not integrable for $\beta = 0$, even if $f(\beta)$ is finite for $\beta = 0$. The reason is that the DVCS amplitude contains singlet GPDs, which are odd functions of β. Hence, $f(\beta)/\beta$ should be an even function, and the principal value prescription does not work. Moreover, for small β one would expect that the forward distribution $f(\beta)$ has a singular $f(\beta) \sim 1/\beta^a$ Regge behavior, which makes the problem even worse. We will address these questions in the second part of our review. Before proceeding to it, we give below a brief summary of the first part.

5 GPDs and phenomenological functions

Hadronic structure is a complicated subject, and it requires a study from many sides and in many different types of experiments. The description of specific aspects of hadronic structure is provided by several different functions: form factors, usual parton densities, distribution amplitudes. Generalized parton distributions provide a unified description: all these functions can be treated as particular or limiting cases of GPDs $H(x,\xi,t)$.

Usual parton densities $f(x)$ correspond to the case $\xi = 0, t = 0$. They describe a hadron in terms of probabilities $\sim |\Psi|^2$. However, QCD is a quantum theory: GPDs with $\xi \neq 0$ describe correlations $\sim \Psi_1^*\Psi_2$. Taking only the point $t = 0$ corresponds to integration over impact parameters $b_\perp$ — information about the transverse structure is lost.

Form factors $F(t)$ contain information about the distribution of partons in the transverse plane, but $F(t)$ involve integration over momentum fraction x — information about longitudinal structure is lost.

A simple "hybridization" of usual densities and form factors in terms of NPDs $\mathcal{F}(x,t)$ (GPDs with $\xi = 0$) shows that the behavior of $F(t)$ is governed both by transverse and longitudinal distributions. GPDs provide adequate description of nonperturbative soft mechanism. They also allow to study transition from soft to hard mechanism.

Distribution amplitudes $\varphi(x)$ provide quantum-level information about the longitudinal structure of hadrons. In principle, they are accessible in exclusive processes at large momentum transfer, when hard scattering mechanism dominates. GPDs have DA-type structure in the central region $|x| < \xi$.

Generalized parton distributions $H(x,\xi,t)$ provide a 3-dimensional picture of hadrons. GPDs also provide some novel possibilities, such as "magnetic distributions" related to the spin-flip GPD $E(x,\xi,t)$. In particular, the structure of nonforward density $E(x,\xi = 0,t)$ determines the t-dependence of $F_2(t)$. Recent JLab data give $F_2(t)/F_1(t) \sim 1/\sqrt{-t}$ rather than $1/t$ expected in hard pQCD and many models — a puzzle waiting to be resolved. The forward reductions $\kappa^a(x)$ of $E(x,\xi,t)$ look as fundamental as $f^a(x)$ and $\Delta f^a(x)$: Ji's sum rule involves $\kappa^a(x)$ on equal footing with $f(x)$. Magnetic properties of hadrons are strongly sensitive to dynamics providing a testing ground for models. Another novel possibility is the study of flavor-nondiagonal distributions, e.g., proton-to-neutron GPDs accessible through processes like exclusive charged pion electroproduction, proton-to-Λ GPDs (they appear in kaon electroproduction), and proton-to-Δ GPDs — these can be related to form factors of proton-to-Δ transition. The GPDs for $N \to N + \text{soft } \pi$ processes can be used for testing the soft pion theorems and physics of chiral symmetry breaking.

An interesting problem is the separation and flavor decomposition of GPDs. The DVCS amplitude involves all four types of GPDs, $H, E, \widetilde{H}, \widetilde{E}$, so we need to study other processes involving different combinations of GPDs. An important observation is that, in hard electroproduction of mesons, the spin nature of produced meson dictates the type of GPDs involved, e.g., for pion electroproduction, only $\widetilde{H}, \widetilde{E}$ appear, with $\widetilde{E}$ dominated by the pion pole at small t. This gives an access to (generalization of) polarized parton densities without polarizing the target.

Summarizing above discussion, we want to emphasize that the structure of hadrons is the fundamental physics to be accessed via GPDs. GPDs describe hadronic structure on the quark-gluon level and provide a three-dimensional picture ("tomography") of the hadronic structure. GPDs adequately reflect the quantum-field nature of QCD (correlations, interference). They also provide new insights into spin structure of hadrons (spin-flip distributions, orbital angular momentum). GPDs are sensitive to chiral symmetry breaking effects, a fundamental property of QCD. Furthermore, GPDs unify existing ways of describing hadronic structure. The GPD formalism provides nontrivial relations between different exclusive reactions and also between exclusive and inclusive processes.

6 Modeling GPDs

Preliminaries. The general idea of extracting GPDs from experiments is to build some models for GPDs, and fix the parameters of such models by comparing their predictions with experimental data.

There are two approaches used to model GPDs. One is based on a direct calculation of parton distributions in specific dynamical models, such as bag model [30], chiral soliton model [31], light-cone formalism [32], etc. Another approach [25, 33, 34] is a phenomenological construction based on reduction formulas relating GPDs to usual parton densities $f(x), \Delta f(x)$ and form factors $F_1(t), F_2(t), G_A(t), G_P(t)$. The most convenient way to construct such models is to start with double distributions $f(\beta, \alpha; t)$.

Let us concentrate on the limiting case $t = 0$. As we discussed earlier, the interpretation of the β-variable as the fraction of the P momentum and the reduction formula (31) stating that the integral of $f_a(\beta, \alpha)$ over α gives the usual parton density $f_a(\beta)$ suggests the *factorized DD Ansatz* (32) in which $f(\beta, \alpha) = h(\beta, \alpha)\, f(\beta)$, where the function $h(\beta, \alpha)$ describes the α-profile normalized to 1 according to Eq.(33) The profile function should be symmetric with respect to $\alpha \to -\alpha$ because DDs $f(\beta, \alpha)$ are even in α [33, 26]. For a fixed β, the function $h(\beta, \alpha)$ describes how the longitudinal momentum transfer r^+ is shared between the two partons. Hence, the shape of $h(\beta, \alpha)$ should look like a symmetric meson distribution amplitude $\varphi(\alpha)$. Recalling that DDs have the support restricted by $|\alpha| \le 1 - |\beta|$, to get a more complete analogy with DAs, it makes sense to rescale α as $\alpha = (1 - |\beta|)\gamma$ introducing the variable γ with β-independent limits: $-1 \le \gamma \le 1$. The simplest model is to assume that the γ–profile is a universal function $g(\gamma)$ for all β. Possible simple choices for $g(\gamma)$ may be $\delta(\gamma)$ (no spread in γ-direction), $\frac{3}{4}(1 - \gamma^2)$ (characteristic shape for asymptotic limit of nonsinglet quark distribution amplitudes), $\frac{15}{16}(1 - \gamma^2)^2$ (asymptotic shape of gluon distribution amplitudes), etc. In the variables β, α, this gives

$$h^{(\infty)}(\beta,\alpha) = \delta(\alpha) \ , \ h^{(1)}(\beta,\alpha) = \frac{3}{4}\frac{(1-|\beta|)^2-\alpha^2}{(1-|\beta|)^3} \ ,$$
$$h^{(2)}(\beta,\alpha) = \frac{15}{16}\frac{[(1-|\beta|)^2-\alpha^2]^2}{(1-|\beta|)^5} \ . \tag{35}$$

These models can be treated as specific cases of the general profile function

$$h^{(N)}(\beta,\alpha) = \frac{\Gamma(2N+2)}{2^{2N+1}\Gamma^2(N+1)}\frac{[(1-|\beta|)^2-\alpha^2]^N}{(1-|\beta|)^{2N+1}} \ , \tag{36}$$

whose width is governed by the parameter N.

Simple models. Let us analyze the structure of GPDs obtained from these simple models. In particular, taking $f^{(\infty)}(\beta,\alpha) = \delta(\alpha)f(\beta)$ gives the simplest model $H^{(\infty)}(x,\xi;t=0) = f(x)$ in which OFPDs at $t=0$ have no ξ-dependence.

In case of the $b=1$ and $b=2$ models, simple analytic results can be obtained only for some explicit forms of $f(x)$. For the "valence quark"-oriented ansatz $h^{(1)}(\beta,\alpha)$, the following choice of a normalized usual density

$$f^{(1)}(\beta) = \frac{\Gamma(5-a)}{6\,\Gamma(1-a)}\,\beta^{-a}(1-\beta)^3 \tag{37}$$

is both close to phenomenological quark distributions and produces a simple expression for the double distribution since the denominator $(1-\beta)^3$ factor in Eq. (35) is canceled. As a result, the integral in Eq. (30) is easily performed and we get [34]

$$\begin{aligned} H_V^1(x,\xi)|_{|x|\geq\xi} &= \frac{1}{\xi^3}\left(1-\frac{a}{4}\right)\left\{\left[(2-a)\xi(1-x)(x_+^{2-a}+x_-^{2-a})\right.\right. \\ &+ \left.\left.(\xi^2-x)(x_+^{2-a}-x_-^{2-a})\right]\,\theta(x\geq\xi)+(x\to -x)\right\} \end{aligned} \tag{38}$$

for $x| \geq \xi$ and

$$H_V^1(x,\xi)|_{|x|\leq\xi} = \frac{1}{\xi^3}\left(1-\frac{a}{4}\right)\left\{x_+^{2-a}[(2-a)\xi(1-x)+(\xi^2-x)]+(x\to -x)\right\} \tag{39}$$

in the middle $-\xi \leq x \leq \xi$ region. We use here the notation

$$x_+ = (x+\xi)/(1+\xi) \quad , \quad x_- = (x-\xi)/(1-\xi) \ . \tag{40}$$

To extend these expressions onto negative values of ξ, one should substitute ξ by $|\xi|$. One can check, however, that no odd powers of $|\xi|$ would appear in the x^N moments of $H_V^1(x,\xi)$. Furthermore, these expressions are explicitly non-analytic for $x = \pm\xi$. This is true even if a is integer. Discontinuity at $x = \pm\xi$, however, appears only in the second derivative of $H_V^1(x,\xi)$ i.e., the model curves for $H_V^1(x,\xi)$ look very smooth (see Fig. 8).

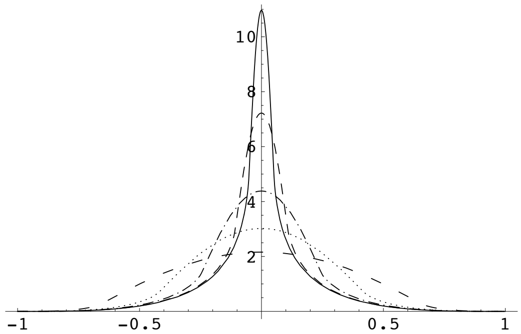

Figure 8: Valence quark distributions $H_V^1(x,\xi)$ with $a = 0.5$ for several values of $\xi = x_{Bj}/(2 - x_{Bj})$ corresponding to values $x_{Bj} = 0.1,\ 0.2,\ \ 0.4,\ 0.6,\ \ 0.8.$ Lower curves correspond to larger values of x_{Bj}.

For $a = 0$, the $x > \xi$ part of GPD has the same x-dependence as its forward limit, differing from it by an overall ξ-dependent factor only,

$$H_V^1(x,\xi)|_{a=0} = 4\,\frac{(1-|x|)^3}{(1-\xi^2)^2}\,\theta(|x| \geq \xi)\, + 2\,\frac{\xi + 2 - 3x^2/\xi}{(1+\xi)^2}\,\theta(|x| \leq \xi)\,. \quad (41)$$

The $(1 - |x|)^3$ behavior can be trivially continued into the $|x| < \xi$ region. However, the actual behavior of $H_V^1(x,\xi)|_{a=0}$ in this region is given by a different function. In other words, $H_V^1(x,\xi)|_{a=0}$ can be represented as a sum of a function analytic at border points and a contribution whose support is restricted by $|x| < \xi$. It should be emphasized that despite its DA-like appearance, this contribution should not be treated as an exchange-type term. It is generated by the regular $\beta \neq 0$ part of the DD, and, unlike the $\varphi(x/\xi)/\xi$ functions its shape changes with ξ, the function becoming very small for small ξ.

For the singlet quark distribution, the DDs $f^S(\beta,\alpha)$ should be odd functions of β. Still, we can use the model like (37) for the $\beta > 0$ part, but take $f^S(\beta,\alpha)|_{\beta\neq 0} = A\,f^{(1)}(|\beta|,\alpha)\,\mathrm{sign}(\beta)$. Note, that the integral (30) producing $H^S(x,\xi)$ in the $|x| \leq \xi$ region would diverge for $\alpha \to x/\xi$ if $a \geq 1$, which is the usual case for standard parametrizations of singlet quark distributions for sufficiently large Q^2. However, due to the antisymmetry of $f^S(\beta,\alpha)$ with

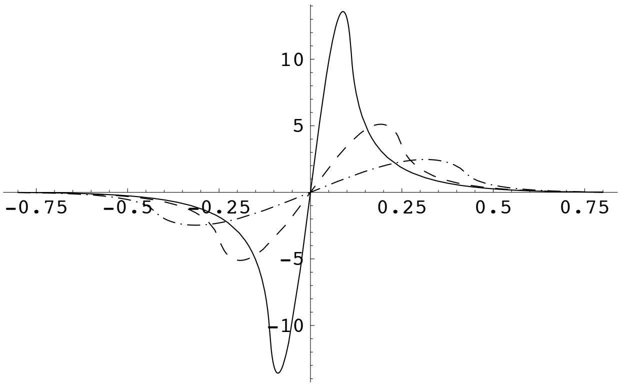

Figure 9: Model for singlet quark distribution $H_S^1(x,\xi)$ for values of ξ corresponding to x_{Bj} equal to 0.2, 0.4, 0.6. Lower curves correspond to larger values of x_{Bj}.

respect to $\beta \to -\beta$ and its symmetry with respect to $\alpha \to -\alpha$, the singularity at $\alpha = x/\xi$ can be integrated using the principal value prescription which in this case produces the $x \to -x$ antisymmetric version of Eqs. (38) and (39). For $a = 0$, its middle part reduces to

$$H_S^1(x,\xi)|_{|x|\leq\xi, a=0} = 2x\,\frac{3\xi^2 - 2x^2\xi - x^2}{\xi^3(1+\xi)^2}\,. \tag{42}$$

The shape of singlet GPDs in this model is shown in Fig. 9.

It should be noted that explicit calculations of generalized parton distributions performed within the chiral soliton model [31] show that the middle region behavior of SPDs strongly resembles that of distribution amplitudes.

GPD model with implanted Regge behavior. The assumptions used in the factorized DD Ansatz are based on the experience with calculating DDs for triangle diagrams [6] and form factors in the light-front formalism models with power-law dependence of the wave function on transverse momentum [35] (see also Ref.[36]).

The simplest triangle diagram (see Fig.10, left) in the scalar model corresponding to Eq. (13) may be used as an example of a model for GPD

$$H(x,\xi) \sim \int \frac{d^4k\,\delta(x-(kn)/(Pn))}{(m_1^2-k_1^2)(m_2^2-k_2^2)(m_3^2-(P-k)^2)}\,. \tag{43}$$

Though the ξ-dependence is not immediately visible here, it appears after integration over k through the $(rn)/2(Pn)$ ratio. The DD $F(\beta,\alpha)$ generated by this diagram is just a constant, see Ref.[6], which corresponds to a flat $N = 0$ profile $h^{(0)}(\beta, \alpha) \sim 1/(1-\beta)$ and $f(\beta) \sim 1-\beta$ forward distribution.

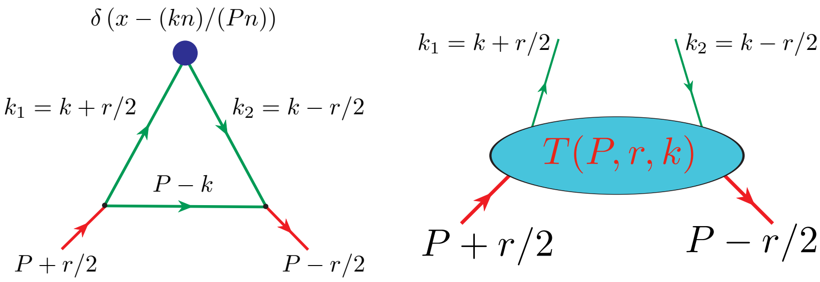

Figure 10: Left: Triangle diagram model for GPD. Right: Hadron-quark scattering amplitude.

The calculation [35] of overlap integrals for light-front wave functions with a power-law behavior $\psi(x, k_\perp) \sim 1/(k_\perp^2)^{1+\kappa}$ resulted in expressions equivalent to using DDs with $N = \kappa$ profile in Eq.(34) and forward distributions behaving like $(1-\beta)^{2\kappa+1}$. The same profile arises [35] if one differentiates a scalar triangle diagram κ times with respect to masses (squared) of each active quark, i.e. substitutes

$$\frac{1}{(m_1^2 - k_1^2)(m_2^2 - k_2^2)} \to \frac{1}{(m_1^2 - k_1^2)^{1+\kappa}(m_2^2 - k_2^2)^{1+\kappa}} . \tag{44}$$

It should be emphasized that $\kappa \neq 0$ models the softer-than-perturbative behavior expected for the transition amplitude relating a bound state with its constituents.

The triangle diagrams, however, do not generate the Regge $f(\beta) \sim 1/\beta^a$ behavior for small β. The latter may be obtained, in particular, by infinite summation of higher-order t-channel ladder diagrams (see, e.g., Ref.[37]). A simpler way was proposed in Ref.[38], where the spectator propagator was substituted by a parton-hadron scattering amplitude $T(P, r, k)$ (see Fig.10, right)) written in the dispersion relation representation. To avoid divergencies generated by the Regge behavior, the subtracted dispersion relation

$$T(P, r, k) \to T((P-k)^2) = T_0 + \int_0^\infty d\sigma \rho(\sigma) \left\{ \frac{1}{\sigma - (P-k)^2} - \frac{1}{\sigma} \right\} \tag{45}$$

was used. The spectral function $\rho(\sigma)$ here should be adjusted to produce a desired Regge-type behavior with respect to $s = (P-k)^2$.

In the light-front formalism, the starting contribution corresponds to a triangle diagram in which the hadron-quark vertices are substituted by the light-front wave functions $\psi(x, k_\perp)$ that bring in an extra fall-off of the integrand at large transverse momenta $k_\perp$. The authors of Ref.[38] intended to reflect this physics in their covariant model. To introduce form factors bringing in a faster fall-off of the k-integrand with respect to quark virtualities k_1^2 and k_2^2, it was proposed to use higher powers of $1/(m_i^2 - k_i^2)$ instead of perturbative propagators, which may be achieved by differentiating the triangle diagram with respect to m_i^2.

The model of Ref.[38] assumes spin-1/2 quarks. It was argued that the Dirac structure of the hadron-parton scattering amplitude in this case should be given by $\not k$, which provides EM gauge invariance of the DVCS amplitude. Thus, the model scattering amplitude has the structure

$$\frac{\not k \, T((P-k)^2}{(m_1^2 - k_1^2)^{N_1+1}(m_2^2 - k_2^2)^{N_2+1}} \, . \tag{46}$$

To treat the two quarks on equal footing, we take $m_1 = m_2$, and the model GPD analyzed below is given by

$$H(x,\xi) = \frac{1}{\pi^2}\frac{N_1!N_2!}{(N_1+N_2)!}\int \frac{(kn)}{(Pn)} \frac{d^4k \, \delta(x-(kn)/(Pn))}{[m^2-(k+r)^2]^{N_1+1}[m^2-(k-r)^2]^{N_2+1}} \left[T_0 + \int_0^\infty d\sigma\rho(\sigma)\left\{\frac{1}{\sigma-(P-k)^2} - \frac{1}{\sigma}\right\}\right] \, . \tag{47}$$

The overall factors were introduced here for future convenience. The T_0 subtraction term gives the D-term-type contribution

$$D_0(x/\xi) = \frac{T_0}{2^{N_1+N_2}(N_1+N_2)}\left(\frac{x}{|\xi|}\right)\left(1-\frac{x}{\xi}\right)^{N_1}\left(1+\frac{x}{\xi}\right)^{N_2}\theta\left(\left|\frac{x}{\xi}\right|<1\right) \tag{48}$$

that vanishes outside the central region and, hence, is invisible in the forward limit. In what follows, we will concentrate on the terms generated by the dispersion integral, but one should remember that the D_0 term can always be added to GPD $H(x,\xi)$, i.e., in all formulas below one should be ready to change $H(x,\xi) \to H(x,\xi) + D_0(x/\xi)$.

The model and DD representation. For equal $N_1 = N_2 = N$, we obtain

$$H(x,\xi) = \frac{x}{2^{2N+1}}\int_0^\infty d\sigma\,\rho(\sigma)\int_0^1 d\beta\int_{-1+\beta}^{1-\beta} d\alpha\,\frac{[(1-\beta)^2-\alpha^2]^N}{(\beta\sigma+(1-\beta)m^2)^{2N+1}} \left\{\delta(x-\beta-\alpha\xi) - \frac{\delta(x-\alpha\xi)}{(1-\beta)^2}\right\} \, . \tag{49}$$

Taking $\xi = 0$, one obtains the usual (forward) parton distributions:

$$H(x, \xi = 0) = \frac{x}{2^{2N+1}} \int_0^\infty d\sigma\, \rho(\sigma) \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} \frac{[(1-\beta)^2 - \alpha^2]^N\, d\alpha}{(\beta\sigma + (1-\beta)m^2)^{2N+1}}$$
$$\times \left\{ \delta\,(x - \beta) - \frac{\delta\,(x)}{(1-\beta)^2} \right\} \,. \tag{50}$$

Treating $x\delta(x)$ as zero, we obtain the representation

$$f(x) = \frac{(N!)^2}{(2N+1)!}\, x\,(1-x)^{2N+1} \int_0^\infty \frac{d\sigma\, \rho(\sigma)}{(x\sigma + (1-x)m^2)^{2N+1}} \,. \tag{51}$$

Now, using Eq.(51), we substitute the σ-integral through forward distribution to get

$$H(x, \xi) = \frac{x}{2^{2N+1}} \frac{(2N+1)!}{(N!)^2} \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha\, \frac{[(1-\beta)^2 - \alpha^2]^N}{(1-\beta)^{2N+1}} \frac{f(\beta)}{\beta}$$
$$\times \left\{ \delta\,(x - \beta - \alpha\xi) - \frac{\delta\,(x - \alpha\xi)}{(1-\beta)^2} \right\} \,. \tag{52}$$

This trick allows one to avoid choosing a specific form of the spectral density $\rho(\sigma)$. It is easy to notice that the factor

$$h_N(\beta, \alpha) \equiv \frac{1}{2^{2N+1}} \frac{(2N+1)!}{(N!)^2} \frac{[(1-\beta)^2 - \alpha^2]^N}{(1-\beta)^{2N+1}} \tag{53}$$

is a normalized profile satisfying Eq.(33). Thus, we can rewrite Eq.(52) as

$$\frac{H(x, \xi)}{x} = \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha\, \frac{f(\beta)}{\beta}\, h_N(\beta, \alpha)$$
$$\times \left\{ \delta\,(x - \beta - \alpha\xi) - \frac{\delta\,(x - \alpha\xi)}{(1-\beta)^2} \right\} \,. \tag{54}$$

The first term here coincides with the factorized DD Ansatz for $H(x, \xi)/x$ in which it is reconstructed from its forward limit $f(x)/x$. The relevant double distribution is given by $f(\beta, \alpha) = h_N(\beta, \alpha) f(\beta)/\beta$. The total contribution is then given by

$$\frac{H(x, \xi)}{x} = \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} d\alpha\, \delta\,(x - \beta - \alpha\xi)$$
$$\times \left\{ f(\beta, \alpha) - \delta(\beta) \int_0^{1-|\alpha|} d\gamma\, \frac{f(\gamma, \alpha)}{(1-\gamma)^2} \right\} \,. \tag{55}$$

Thus, the model of Ref.[38], first, corresponds to the single-DD representation (27), and, second, it has the structure of the factorized DD Ansatz (32).

Results for GPDs. For the model forward distribution

$$f_a(\beta) = (1-\beta)^3/\beta^a \tag{56}$$

and the profile function

$$h_1(\beta,\alpha) = \frac{3}{4}\,\frac{(1-\beta)^2-\alpha^2}{(1-\beta)^3}\ , \tag{57}$$

we obtain, for $x > \xi$:

$$H(x,\xi)|_{x>\xi} = \frac{3}{4}\,\frac{x}{\xi}\int_{\beta_1}^{\beta_2}\frac{d\beta}{\beta^{a+1}}\left\{(1-\beta)^2-\left(\frac{x-\beta}{\xi}\right)^2\right\}\ . \tag{58}$$

Calculating $H(\xi,\xi)$, i.e., the GPD at the border point $x=\xi$, one gets here the $[(1-\beta)^2-(1-\beta/\xi)^2]\sim\beta$ factor from the profile function, and this factor changes the strength of singularity for $\beta=0$. As a result, the integral over β converges as far as $a<1$. This outcome is a consequence of using a profile function that linearly vanishes at the sides of the support rhombus. In its turn, the $N=1$ profile is generated by the assumed $1/(k_1^2k_2^2)^2$ dependence of the k-integrand for large parton virtualities. If one takes the $N=0$ profile, the factor in the curly brackets should be substituted by $1/(1-\beta)$), and the integral producing $H(\xi,\xi)$ diverges. For small, but nonzero $x-\xi$, one obtains the behavior proportional to $1/\beta_1^a \sim (x-\xi)^{-a}$. Turning now to the $|x|<\xi$ region we get, for the $N=1$ profile:

$$\begin{aligned}H(x,\xi)|_{|x|<\xi} =& \frac{3}{4}\,\frac{x}{\xi}\left[\frac{1}{\xi^2}\int_0^{\beta_2}\frac{d\beta}{\beta^a}(2x-\beta)+\int_0^{\beta_2}\frac{d\beta}{\beta^a}\left\{1-\frac{x^2}{\xi^2(1-\beta)^2}\right\}(\beta-2)\right.\\ &\left. -\int_{\beta_2}^{1-|x|/\xi}\frac{d\beta}{\beta^{a+1}}\left\{1-\frac{x^2}{\xi^2(1-\beta)^2}\right\}\right]\ . \end{aligned}\tag{59}$$

Note that as far as $|x|$ is strictly less than ξ, the profile function does not vanish at the singularity point $\beta=0$. The mechanism of softening singularity to $1/\beta^a$ strength is now provided by the $1/\sigma$ subtraction term of the original dispersion relation. To get a model for singlet GPDs, one should take the antisymmetric combination

$$H_S(x,\xi) = H(x,\xi) - H(-x,\xi)\ . \tag{60}$$

The resulting GPDs are shown in Fig. 11, left. For positive x, they are peaking at $x=\xi$. The functions $H_S(x,\xi)$ in this model are continuous at $x=\pm\xi$, but the derivative $dH_S(x,\xi)/dx$ is discontinuous at these points. In a similar way, one can calculate model GPDs for the $N=2$ profile. The resulting GPDs are shown in Fig. 11, right. For positive x, they are peaking at points close to $x=\xi$. In the model with $N=2$ profile, both the functions $H_S(x,\xi)$ and their derivatives $dH_S(x,\xi)/dx$ are continuous at $x=\pm\xi$.

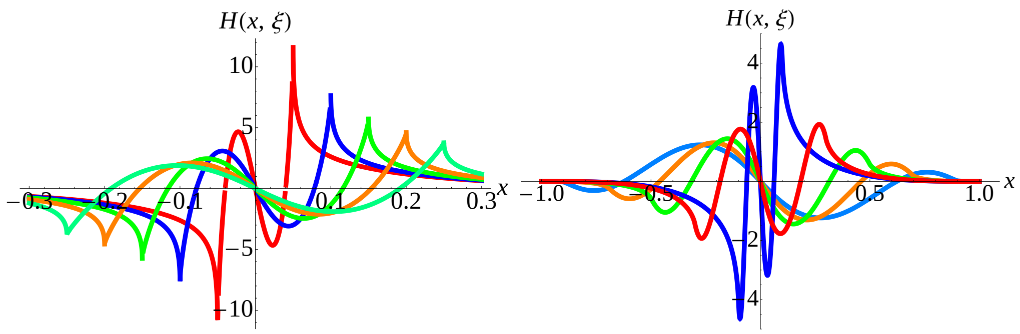

Figure 11: Model singlet GPD $H_S(x,\xi)$ with $N=1$ (left) and $N=2$ (right) DD profile for $a=0.5$ and $\xi=0.05, 0.1, 0.15, 0.2, 0.25$.

Results for D-term. In Eq.(55), we deal with the regularized double distribution

$$f^{\rm reg}(\beta,\alpha) = f(\beta,\alpha) - \delta(\beta)\int_0^{1-|\alpha|} d\gamma \, \frac{f(\gamma,\alpha)}{(1-\gamma)^2} \ . \tag{61}$$

However, due to the $1/(1-\gamma)^2$ factor in the subtraction term, $f^{\rm reg}(\beta,\alpha)$ does not coincide with $f_+(\beta,\alpha)$. Their difference induces the D-term

$$D(\alpha) = \alpha \int_0^{1-|\alpha|} d\beta \, \frac{f(\beta)}{\beta} \, h(\beta,\alpha) \left\{ 1 - \frac{1}{(1-\beta)^2} \right\} \ .$$

Taking the same model forward distribution $f(\beta)=(1-\beta)^3/\beta^a$ and $N=1$ profile function gives

$$D^{\{N=1\}}(\alpha) = \frac{3}{2}\,\alpha \int_0^{1-|\alpha|} \frac{d\beta}{\beta^a} \left[1 - \frac{\alpha^2}{(1-\beta)^2} \right] (\beta-2) \ . \tag{62}$$

A similar expression for the D-term is obtained in the $N=2$ profile model:

$$D^{\{N=2\}}(\alpha) = \frac{15}{8}\,\alpha \int_0^{1-|\alpha|} \frac{d\beta}{\beta^a} \left[1 - \frac{\alpha^2}{(1-\beta)^2} \right]^2 (\beta-2) \ . \tag{63}$$

As one can see in Fig. 12, the two curves are rather close to each other.

The comparison of the total GPD $H(x,\xi)$ and its D-term part is shown in Fig. 13, left. The difference between GPD $H(x,\xi)$ and D-term $D(x/\xi)$ corresponds to the term $H_+(x,\xi)$ obtained from the "plus" part $[f(\beta,\alpha)]_+$ of DD. The shape of the difference for $\xi=0.5$ is shown in Fig. 13, right. Note that, despite the fact that the forward distribution in this model is positive,

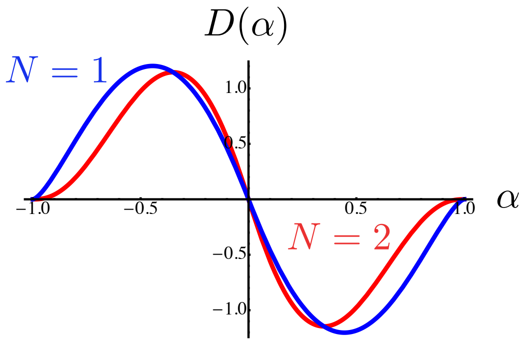

Figure 12: The D-terms in $N = 1$ and $N = 2$ profile models for $a = 0.5$.

there is a region, where the contribution to $H(x, \xi)$ coming from $[f(\beta, \alpha)]_+$ is negative. This is due to the $\delta(\beta)$ subtraction term contained in $[f(\beta, \alpha)]_+$. Also shown is the ratio $H_+(x, \xi)/x$. Looking at the figure, one may suspect that the x-integral of $H_+(x, \xi)/x$ vanishes. In the next section, we show that this, indeed, is the case.

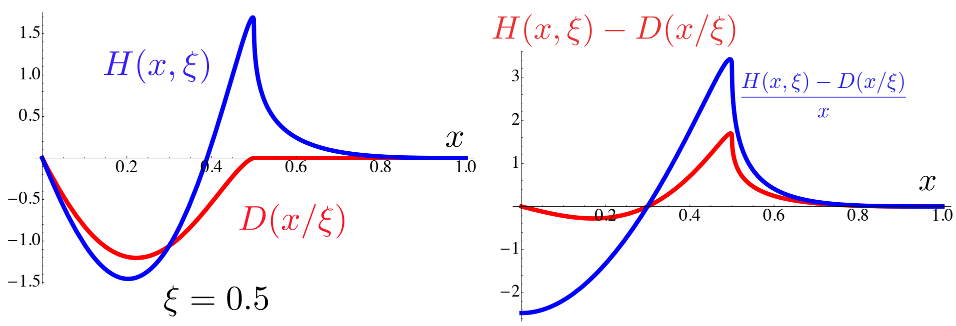

Figure 13: Left: GPD $H(x, \xi)$ and D-term $D(x/\xi)$ for $\xi = 0.5$ and positive x. Right: Difference between GPD $H(x, \xi)$ and D-term $D(x/\xi)$ in the case of the $N = 1$ profile for $\xi = 0.5$ and positive x. The same function divided by x is also shown.

7 GPD sum rules.

Sum rules. The D-term determines the subtraction constant in the dispersion relation for the DVCS amplitude [18, 19, 20, 21, 22]. In particular, it was shown [19] that the original expression for the real part of the DVCS amplitude involving $H(x,\xi)$, and the dispersion integral involving $H(x,x)$ differ by a constant Δ given by the integral of the D-term function $D(\alpha)$:

$$P\int_{-1}^{1} \frac{H(x,x)-H(x,\xi)}{x-\xi}\,dx = \Delta \equiv \int_{-1}^{1} \frac{D(\alpha)}{1-\alpha}\,d\alpha\ . \tag{64}$$

Here, P denotes the principal value prescription. In Ref.[19], this relation was derived using polynomiality properties of GPDs. It was also pointed out there that it can be obtained by incorporating representation of GPDs in the two-DD formalism (which is basically again the use of the polynomiality).

Taking $\xi = 0$, one formally arrives at the sum rule

$$\int_{-1}^{1} \frac{H(x,x)-H(x,0)}{x}\,dx = \int_{-1}^{1} \frac{D(\alpha)}{1-\alpha}\,d\alpha\ . \tag{65}$$

Since both $H(x,0)/x$ and $H(x,x)/x$ are even functions of x, their singularities for $x=0$ cannot be regularized by the principle value prescription. Moreover, there are no indications that singularities of these two functions may cancel each other. On the contrary, as emphasized in Ref.[39], there are arguments that the ratio $H(x,x)/H(x,0)$ does not tend to 1 for small x.

"Plus + D" decomposition. To begin with, we remind the basic formulas:

$$H(x,\xi)/x = \int_{\Omega} f(\beta,\alpha)\,\delta(x-\beta-\xi\alpha)\,d\beta\,d\alpha\ , \tag{66}$$

the expression producing GPDs from DDs, and the decomposition of DD

$$f(\beta,\alpha) = [f(\beta,\alpha)]_{+} + \delta(\beta)D(\alpha)/\alpha \tag{67}$$

into the "plus" part $[f(\beta,\alpha)]_{+}$ and the D-term part $\delta(\beta)D(\alpha)/\alpha$.

Correspondingly, we split GPD into the part coming from the "plus" part of DD

$$\frac{H_{+}(x,\xi)}{x} \equiv \int_{\Omega} f(\beta,\alpha)\left[\delta(x-\beta-\xi\alpha)-\delta(x-\xi\alpha)\right]d\beta\,d\alpha \tag{68}$$

and that generated by the D-term

$$\frac{H_{D}(x,\xi)}{x} \equiv \int_{-1}^{1} \frac{D(\alpha)}{\alpha}\delta(x-\xi\alpha)\,d\alpha\ . \tag{69}$$

Another important relation

$$\frac{H_D(x,0)}{x} = \delta(x) \int_{-1}^{1} \frac{D(\alpha)}{\alpha}\, d\alpha \tag{70}$$

is obtained by taking $\xi = 0$. Now, Eq. (69) gives

$$\frac{H_D(x,x)}{x} = \delta(x) \int_{-1}^{1} \frac{D(\alpha)}{\alpha(1-\alpha)}\, d\alpha \ . \tag{71}$$

Note that both $H_D(x,0)/x$ and $H_D(x,x)/x$ are proportional to $\delta(x)$, with the coefficients given by integrals of $D(\alpha)$. This means that, unlike the functions $H(x,0)$ and $H(x,x)$, which, for $x \neq 0$, are insensitive to changes of $D(\alpha)$ in the $\delta(\beta)D(\alpha)/\alpha$ term, the (mathematical) distributions $H(x,0)/x$ and $H(x,x)/x$ contain information about such a D-term.

Our next step is to study contributions from different parts of the GPDs involved in the sum rule (65).

"Secondary" sum rule. One can easily see from Eq. (68) that

$$\int_{-1}^{1} \frac{H_+(x,\xi)}{x}\, dx = 0 \tag{72}$$

for any ξ, including $\xi = 0$. Since the integrand is an even function of x, the vanishing of this integral means that we also have

$$\int_{0}^{1} \frac{H_+(x,\xi)}{x}\, dx = 0 \ . \tag{73}$$

Thus, $H_+(x,\xi)$ should be negative in some part of the central region, and this negative contribution should exactly compensate the contribution from the regions, where $H_+(x,\xi)$ is positive. In other words, on the $(0,1)$ interval, $H_+(x,\xi)/x$ has the same property as a "plus distribution" with respect to x. Note, that this does not mean that $H_+(x,\xi)/x$ necessarily contains singular functions like $\delta(x)$. For finite ξ, the function $H_+(x,\xi)/x$ is pretty regular for all x values (see Fig.14). The negative $\delta(x)$ function appears only in the ξ=0 limit, i.e.

$$\frac{H_+(x,0)}{x} = \frac{f(x)}{x} - \delta(x) \int_{-1}^{1} \frac{f(y)}{y}\, dy \equiv \left[\frac{f(x)}{x}\right]_+ \ . \tag{74}$$

For the integral involving the border function, we get

$$\int_{-1}^{1} \frac{H_+(x,x)}{x}\, dx = \int_{-1}^{1} dx \int_\Omega d\beta\, d\alpha\ f(\beta,\alpha) \left\{ \delta\big[x(1-\alpha)-\beta\big] - \frac{\delta(x)}{1-\alpha} \right\} \ . \tag{75}$$

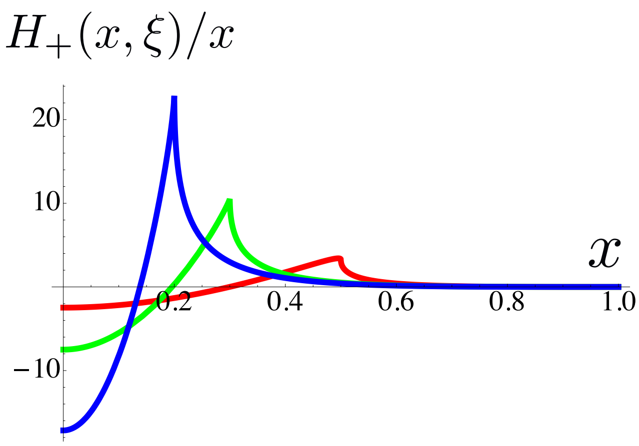

Figure 14: Function $H_+(x,\xi)/x$ in the $N = 1$ profile model for $\xi = 0.2, 0.3, 0.5$ and positive x.

The integrals coming from the two delta-functions cancel each other, and we have

$$\int_{-1}^{1} \frac{H_+(x,x)}{x}\, dx = 0 \ , \tag{76}$$

just like for $H_+(x,\xi)/x$. Unlike $H_+(x,\xi)$, however, the combination $H_+(x,x)/x$ explicitly contains the $\delta(x)$ subtraction term, i.e. it is a genuine "plus distribution" with respect to x, namely, $H_+(x,x)/x = [H(x,x)/x]_+$.

Summarizing, the "plus" parts of both functions entering into the sum rule (65) separately produce vanishing contributions into the x-integral. Furthermore, these zero contributions are due to the fact that $H_+(x,0)/x$ and $H_+(x,x)/x$ are "plus distributions", which results in zero integrals irrespectively of the form of the forward distribution $f(x)$ and the border function $H(x,x)$.

Let us now turn to the D-parts. First, we have

$$\int_{-1}^{1} \frac{H_D(x,\xi)}{x}\, dx = \int_{-1}^{1} \frac{D(\alpha)}{\alpha}\, d\alpha \tag{77}$$

for any fixed ξ, including $\xi = 0$. This result may be obtained by integrating over x the $\delta(x - \xi\alpha)$ factor in the integral representation (69).

For the integral involving the border function, we use Eq. (71), which gives

$$\int_{-1}^{1} \frac{H_D(x,x)}{x}\, dx = \int_{-1}^{1} \frac{D(\alpha)}{\alpha(1-\alpha)}\, d\alpha \ . \tag{78}$$

As a result,

$$\int_{-1}^{1} \frac{H_D(x,x)}{x}\,dx - \int_{-1}^{1} \frac{H_D(x,0)}{x}\,dx = \int_{-1}^{1} \frac{D(\alpha)}{1-\alpha}\,d\alpha\ . \tag{79}$$

Combining this outcome with zero contributions from the "plus" parts, one obtains the sum rule (65).

Thus, our construction confirms the sum rule. Our derivation shows also that the "plus" parts of both terms simply do not contribute to the sum rule whatever the shapes of $f(x)$ and $H(x,x)$ are. Only the D-parts contribute, so there is no surprise that the net result can be expressed in terms of $D(\alpha)$.

An essential point is that both $H_D(x,0)/x$ and $H_D(x,x)/x$ are proportional to the $\delta(x)$-function, with the coefficients given by integrals of the D-term function $D(\alpha)$. In this sense, $H(x,0)/x$ and $H(x,x)/x$ "know" about the D-term.

A simple consequence is that all x^j moments of $H_D(x,0)$ and $H_D(x,x)$ with $j \geq 0$ vanish, and one cannot get the D-part of the sum rule (65) by an analytic continuation of the x^j moments of $H_D(x,0)$ and $H_D(x,x)$ to $j=-1$, i.e., using the procedure of Refs.[20, 23, 24]. In fact, x^j moments of $H_D(x,0)$ and $H_D(x,x)$ are proportional to the Kronecker delta function $\delta_{j,-1}$, a non-analytic function of j.

Need for renormalization. Since $H(x,0)/x$ is given by integrating the DD $f(\beta,\alpha)$ over α along vertical lines $\beta = x$, a subsequent integration over all x gives DD $f(\beta,\alpha)$ integrated over the whole rhombus:

$$\begin{aligned}\int_{-1}^{1} \frac{H(x,0)}{x}\,dx &= \int_{-1}^{1} dx \int_\Omega d\beta\, d\alpha f(\beta,\alpha)\delta(x-\beta) \\ &= \int_\Omega f(\beta,\alpha)\; d\beta\, d\alpha = \int_{-1}^{1} \frac{D(\alpha)}{\alpha}\,d\alpha\ .\end{aligned} \tag{80}$$

On the last step, we used that the β-integral of $f(\beta,\alpha)$ formally gives $D(\alpha)/\alpha$. However, if $f(\beta,\alpha) \sim 1/\beta^{1+a}$, being even in β, one needs a regularization for the β-integral. The "DD+D" separation (66), as we have seen, provides such a regularization. It works like a renormalization: the divergent integral formally giving the D-term is subtracted from the "bare" DD, and substituted by a finite "observable" function $D(\alpha)/\alpha$.

In a similar way, we can treat the second integral:

$$\begin{aligned}\int_{-1}^{1} \frac{H(x,x)}{x}\,dx &= \int_{-1}^{1} dx \int_\Omega d\beta\, d\alpha f(\beta,\alpha)\delta(x-\beta-x\alpha) \\ &= \int_\Omega \frac{f(\beta,\alpha)}{1-\alpha}\; d\beta\, d\alpha = \int_{-1}^{1} \frac{D(\alpha)}{\alpha(1-\alpha)}\,d\alpha\ .\end{aligned} \tag{81}$$

Again, the last step requires a subtraction of the infinite part of the β-integral.

The advantage of using the "DD+D" separation as a renormalization prescription is that it is applied directly to the DD. Hence, it is universal, and may be used for other integrals involving $f(\beta, \alpha)$.

Generic sum rule. Finally, let us apply the "DD+D" separation to the generic relation (64). For the "plus" part, representing $1/(x-\xi) = 1/x + (\xi/x)/(x-\xi)$ and using Eqs. (73), (76), we have

$$\begin{aligned} P\int_{-1}^{1} \frac{H_+(x,x)}{x-\xi}\,dx =& P\int_{-1}^{1} \xi\,\frac{dx}{x-\xi}\int_\Omega f(\beta,\alpha)\;d\beta\,d\alpha \\ &\times\Big[\delta(x(1-\alpha)-\beta) - \delta(x(1-\alpha))\Big] \\ =& \; P\int_\Omega f(\beta,\alpha)\;d\beta\,d\alpha\left[\frac{\xi}{\beta-\xi(1-\alpha)} + \frac{1}{(1-\alpha)}\right] , \end{aligned} \tag{82}$$

and

$$\begin{aligned} P\int_{-1}^{1} \frac{H_+(x,\xi)}{x-\xi}\,dx =& P\int_{-1}^{1} \xi\,\frac{dx}{x-\xi}\int_\Omega f(\beta,\alpha)\;d\beta\,d\alpha \\ &\times\Big[\delta(x-\beta-\xi\alpha) - \delta(x-\xi\alpha)\Big] \\ =& \; P\int_\Omega f(\beta,\alpha)\;d\beta\,d\alpha\left[\frac{\xi}{\beta-\xi(1-\alpha)} + \frac{1}{(1-\alpha)}\right] . \end{aligned} \tag{83}$$

Thus, seemingly different delta-functions have converted $1/(x-\xi)$ into identical expressions (cf. Ref.[21], where a similar result was obtained for the F_D part of the two-DD representation). As a result,

$$P\int_{-1}^{1} \frac{H_+(x,x)}{x-\xi}\,dx - P\int_{-1}^{1} \frac{H_+(x,\xi)}{x-\xi}\,dx = 0\;. \tag{84}$$

In this case, we deal with the situation when the difference of two integrals vanishes, but each integral does not necessarily vanish.

In case of the "D" part, we have, for the integral involving the border function,

$$\begin{aligned} P\int_{-1}^{1} \frac{H_D(x,x)}{x-\xi}\,dx &= P\int_{-1}^{1} \frac{H_D(x,x)}{x}\,\frac{x}{x-\xi}\,dx \\ &= P\int_{-1}^{1} dx\,\frac{x}{x-\xi}\,\delta(x)\int_{-1}^{1} \frac{D(\alpha)}{\alpha(1-\alpha)}\,d\alpha = 0\;. \end{aligned} \tag{85}$$

In simple words, the starting integrand in (85) vanishes for $x \neq 0$ since then $H_D(x,x) = 0$, while for $x = 0$ it is given by the $x\delta(x)$ distribution which produces zero after integration with a function that is finite for $x = 0$, which is the case if $\xi \neq 0$. The second piece is given by

$$
\begin{aligned}
& P\int_{-1}^{1} \frac{H_D(x,\xi)}{x-\xi}\,dx = P\int_{-1}^{1} \frac{H_D(x,\xi)}{x}\,\frac{x}{x-\xi}\,dx \\
& = P\int_{-1}^{1} \frac{x\,dx}{x-\xi}\int_{-1}^{1} \frac{D(\alpha)}{\alpha}\,\delta(x-\xi\alpha)\,d\alpha \\
& = \int_{-1}^{1} \frac{\xi\alpha}{\xi\alpha-\xi}\,\frac{D(\alpha)}{\alpha}\,d\alpha = -\int_{-1}^{1} \frac{D(\alpha)}{1-\alpha}\,d\alpha\,.
\end{aligned}
\tag{86}
$$

Again, the result above may be obtained by simply using

$$
H_D(x,\xi) = \mathrm{sign}(\xi)\,\theta(|x| < |\xi|)D(x/\xi)
$$

and rescaling $x = \alpha\xi$. Also, though the final result of Eq. (86) does not depend on ξ, it does not coincide with the result of the counterpart relation (77). However, for the difference of the two integrals we obtain

$$
P\int_{-1}^{1} \frac{H_D(x,x)}{x-\xi}\,dx - P\int_{-1}^{1} \frac{H_D(x,\xi)}{x-\xi}\,dx = \int_{-1}^{1} \frac{D(\alpha)}{1-\alpha}\,d\alpha\ , \tag{87}
$$

the same result as in Eq. (79). Combining the results for the "plus" and D-parts gives Eq. (64).

8 Analytic regularization

Mellin moments. Another possibility to renormalize the β-integral in Eq.(80) for a singular DD is to use the analytic regularization as proposed in Refs. [20, 23, 24]. Namely, it is assumed that the positive Mellin moments (or conformal moments, see, e.g., Ref.[40])

$$
\Phi(j) \equiv \int_{-1}^{1} x^j\,[H(x,x) - H(x,0)]\,dx \tag{88}
$$

can be analytically continued to the point $j = -1$. The result of such a procedure is equivalent to analytic regularization of the x-integral. However, the assumed analyticity properties of $\Phi(j)$ may be violated by singular or "invisible" terms (cf. Ref.[20]) in the integrand of Eq.(88). For example, a $x\delta(x)$ term gives a non-analytic $\delta_{j,-1}$ contribution into $\Phi(j)$. In the model with implanted Regge behavior, singular terms explicitly emerge as a result of subtractions in the dispersion relation, so one may wish to develop a less restrictive approach to the renormalization problem. In this connection, we would like to stress that the derivation of the sum rule (65) given above was based merely on separation (67) of the DDs into the "plus" part and the D-term. No assumptions about smoothness were made. The essential moment of the derivation was that one should not hurry up to treat $x\delta(x)$ terms in $H(x,x)$ as zero, since they convert into non-negligible $\delta(x)$ contributions in $H(x,x)/x$. The same applies to $H(x,0)/x$.

Comparison of the "plus" prescription and analytic regularization. Analytic regularization works as follows. If we need to integrate a function like $\lambda(x)/x^{a+1}$ with $\lambda(x)$ being finite and nonzero for $x=0$, we subtract from $\lambda(x)$ as many terms of its Taylor expansion as needed to remove the divergence

$$\int_{(0)}^{y} \frac{\lambda(x)}{x^{a+1}}\, dx = \int_0^y dx \frac{\lambda(x) - \lambda(0) - x\lambda'(0) - \dots}{x^{a+1}}$$
$$+ \lambda(0) \int_{(0)}^{y} \frac{dx}{x^{a+1}} + \lambda'(0) \int_{(0)}^{y} \frac{dx}{x^{a}} + \dots \ , \tag{89}$$

and then treat the compensating integrals of x^n/x^{a+1} as convergent, substituting them by $y^{n-a}/(n-a)$. So, let us consider again a DD which is nonzero for positive β only and has the form

$$f(\beta,\alpha) = \frac{\lambda(\beta,\alpha)}{\beta^{a+1}}\, \theta(\beta + |\alpha| \le 1)\, \theta(\beta \ge 0)$$

with $a < 1$. Then the analytic regularization of its integral with some reference function $\Phi(\beta)$ is defined by

$$\int_{(0)}^{1-|\alpha|} \frac{\Phi(\beta)\lambda(\beta,\alpha)}{\beta^{a+1}}\, d\beta = \int_0^{1-|\alpha|} \frac{\Phi(\beta)\lambda(\beta,\alpha) - \Phi(0)\lambda(0,\alpha)}{\beta^{a+1}}\, d\beta$$
$$- \frac{\Phi(0)\lambda(0,\alpha)}{a(1-|\alpha|)^a} \ , \tag{90}$$

which may be rewritten as

$$\int_{(0)}^{1-|\alpha|} \Phi(\beta) \frac{\lambda(\beta,\alpha)}{\beta^{a+1}}\, d\beta = \int_0^{1-|\alpha|} [\Phi(\beta) - \Phi(0)]\, \frac{\lambda(\beta,\alpha)}{\beta^{a+1}}\, d\beta \tag{91}$$
$$+ \Phi(0) \left[\int_0^{1-|\alpha|} \frac{\lambda(\beta,\alpha) - \lambda(0,\alpha)}{\beta^{a+1}}\, d\beta - \frac{\lambda(0,\alpha)}{a(1-|\alpha|)^a} \right] \ .$$

Now, the first contribution on the r.h.s. is generated by the "plus" part of the DD, while the second one comes from a D-term. After adding the $\beta < 0$ part of the DD, the D-term $\mathcal{D}(\alpha)/\alpha$ corresponding to the analytic regularization is given by

$$\frac{\mathcal{D}(\alpha)}{\alpha} = 2\left[\int_0^{1-|\alpha|} \frac{\lambda(\beta,\alpha) - \lambda(0,\alpha)}{\beta^{a+1}}\, d\beta - \frac{\lambda(0,\alpha)}{a(1-|\alpha|)^a}\right] . \tag{92}$$

Thus, the analytic regularization prescription unambiguously fixes the D term, and in this sense it may be called the "analytic renormalization".

In the model with implanted Regge behavior, we also obtained a concrete result for the D-term. But the specific D-term contribution we obtained there came only from the σ-integral part of the dispersion relation for the hadron-parton scattering amplutude subtracted at $(P-k)^2 = 0$. As we pointed out, one should be always ready to add to it the D_0 term coming from the T_0 constant in the dispersion relation (45). In principle, we had no reasons to require that $T_0 = 0$. In this sense, the D-term in that model is not fixed.

On the other hand, the statement, that x^j moments of $H(x,\xi)$ are analytic functions of j, does not explicitly mention fixing any subtraction constants: it sounds like a general principle, and may create an impression that there are no ambiguities in the subtraction of the $\beta = 0$ singularity. However, the analyticity assumption was not shown so far to be a consequence of general principles of quantum field theory. Moreover, as mentioned in Ref.[41], it is not satisfied in the nonlocal chiral soliton model. Still, one may hope that it is valid in QCD.

To see if the $T_0 = 0$ model of the previous section agrees with the analyticity assumption, we should just check whether its D-term is different from that obtained *via* analytic renormalization. In particular, for the $N = 1$ model, we have

$$\lambda(\beta,\alpha) = \frac{3}{4}\left[(1-\beta)^2 - \alpha^2\right] , \tag{93}$$

and, hence,

$$\frac{\mathcal{D}(\alpha)}{\alpha} = \frac{3}{2}\left[\frac{(1-|\alpha|)^{2-a}}{2-a} - 2\frac{(1-|\alpha|)^{1-a}}{1-a} - \frac{1-\alpha^2}{a(1-|\alpha|)^a}\right] . \tag{94}$$

In Fig.15, we compare this result (for $a = 0.5$) with the result obtained by single subtraction in the dispersion relation (45) with $T_0 = 0$.

Our main point is that representing $H(x,\xi)$ as the sum $H_+(x,\xi) + H_D(x,\xi)$ one can derive the GPD sum rule (65) without using the analyticity assumption. But since our derivation, so to say, works for any D-term, it also works for the D-term following from the analyticity assumption.

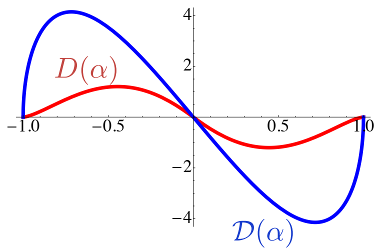

Figure 15: The D-terms in the model with $N = 1$ profile and $a = 0.5$: $\mathcal{D}(\alpha)$ was obtained using analytic regularization, and $D(\alpha)$ was obtained for $T_0 = 0$ in the model of the previous section.

Summary on sum rules. Thus, the calculation described above confirms the generic GPD sum rule (64) derived in Refs. [19, 21]. It also supports the $\xi = 0$ sum rule (65) suggested in Ref. [19]. It should be emphasized that the integrals present in the generic sum rule have a singularity for $x = \xi$, which is inside the region of integration, so the integrals may be taken using the principal value prescription. Since $H(x, 0)/x$ and $H(x, x)/x$ are even functions of x, the $\xi = 0$ sum rule may be written through an integral from 0 to 1, and its $1/x$ singularity is at the end-point of the integration region, which means that the P-prescription cannot regulate it. Just because of this fact alone, the sum rule (65) cannot be a straightforward consequence of the generic sum rule (64).

In the presented derivation, the finite expressions were obtained for each term involved. In particular, we established that though $H_D(x, x)$ and $H_D(x, \xi)$ contributions to the generic sum rule (64) are ξ-independent, they do not coincide with their counterparts from the secondary sum rule (65), i.e., the latter cannot be obtained by formally continuing to $\xi = 0$ the ξ-independent results for each term of the generic GPD sum rule.

In our derivation, we did not make an assumption about analyticity of the Mellin moments of GPDs. We have obtained GPD sum rules as a consequence of the polynomiality of GPDs that follows from Lorentz invariance and is encoded in the DD representation. The analyticity is a much stronger restriction. One may try to find out whether it can be tested experimentally and it is also worth trying to prove it in QCD.

9 Conclusions

In Sections 6–8, we discussed some basic aspects of building models for GPDs using the factorized DD Ansatz (FDDA) within the "single-DD" formulation. The main difficulty in the implementation of such a construction is the necessity to deal with projection onto a more singular function $f(\beta)/\beta$ (rather than just onto $f(\beta)$) in the forward limit. This leads to two problems. First, one encounters non-integrable singularities for $\beta = 0$ in the integrals producing GPDs in the central region $|x| < |\xi|$. The difficulty is exaggerated by necessity to consider forward distributions $f(\beta)$ that have a singular β^{-a} Regge behavior at small β. Second, if there are no factors suppressing the $\beta \sim 0$ region for the integration line corresponding to $x = \xi$, the combined $1/\beta^{1+a}$ singularity leads to a singular $(x - \xi)^{-a}$ behavior for GPDs in the outer region $x > \xi$ near the border point $x = \xi$. Such a behavior was found in the model of Ref.[38].

In our analysis, we found that this model gives the single-DD-type representation for the model GPD, and thus above reasoning is applicable to it. But we argued, that a proper softening of the hadron-quark vertices produces a profile function $h_N(\beta, \alpha)$ that results, for $x = \xi$, in the $\mathcal{O}(\beta^N)$ suppression factor securing a finite value of the GPD $H(x, \xi)$ at the border point.

However, the profile factor has no impact on the combined $1/\beta^{1+a}$ singularity on the $\beta = 0$ line inside the support rhombus, which one faces when calculating GPDs in the $|x| < |\xi|$ region. The advantage of the model of Ref. [38] is that it implants the Regge behavior through a subtracted dispersion relation for the hadron-quark scattering amplitude. We found that the subtraction provides the regularization necessary for the calculation of GPDs in the central region, and illustrated the behavior of resulting GPDs in models with $N = 1$ and $N = 2$ profiles.

We also observed that this model produces a D-term contribution, despite the fact that it uses only the forward distribution as an input. This D-term contribution appears because the subtraction generated by the dispersion relation differs from the subtraction that converts the original DD into a "plus" distribution $[f(\beta, \alpha)]_+$. The latter, by definition, cannot generate a D-term. We have shown that the GPD $H_+(x, \xi)$ generated by the $[f(\beta, \alpha)]_+$ part of the original DD (i.e., GPD $H(x, \xi)$ with the D-term contribution $D(x/\xi)$ subtracted) has a remarkable property that the integral of $H_+(x, \xi)/x$ over positive values $0 \le x \le 1$ vanishes. As a result, $H_+(x, \xi)$ must be negative in some part of the central region, a feature that is absent in previous FDDA models based on two-DD formulation.

Within the single-DD formalism, it is very natural to separate the relevant DD $f(\beta, \alpha)$ into the "plus" part $[f(\beta, \alpha)]_+$ and the D-term. We demonstrated that this separation can be used to rederive the GPD sum rule related to the dispersion relation for the real part of the DVCS amplitude, and we also gave a derivation of another sum rule proposed as the $\xi \to 0$ limit of that generic sum rule. Our derivation shows that this "secondary" sum rule is not a straightforward consequence of the generic one. In particular, the principal value prescription used in the generic sum rule needs to be substituted by another prescription, like the "plus" prescription. The "plus" prescription, in fact, is automatically generated by the separation of DDs into the "plus" part and the D-term. We also demonstrated that the contributions into the two sum rules generated by the same functions are not in a one-to-one correspondence.

Summarizing, using (intentionally) simplified models, we developed the basic tools that can be used in building realistic GPD models based on the factorized DD Ansatz within the single-DD formalism. Future developments in this direction should include the extension of the presented methods onto the cases with $a > 1$ Regge behavior, which would require an extra subtraction in the dispersion relation, and building models for nucleons and other targets with a non-zero spin.

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