Amplitude reconstruction from complete experiments and truncated partial-wave expansions

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Abstract

We compare the methods of amplitude reconstruction, for a complete experiment and a truncated partial wave analysis, applied to the photoproduction of pseudo-scalar mesons. The approach is pedagogical, showing in detail how the amplitude reconstruction (observables measured at a single energy and angle) is related to a truncated partial-wave analysis (observables measured at a single energy and a number of angles).

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I. INTRODUCTION AND MOTIVATION

A model-independent determination of amplitudes from experimental data is mathematically possible, ignoring experimental errors, if one measures a sufficient number of observables at a given energy and angle. This has been done in nucleon-nucleon scattering [1] and can be done [2, 3], in principle, using pseudo-scalar meson photoproduction data [4–6].

The complete experiment analysis (CEA) determines helicity or transversity amplitudes only up to an overall phase. This is a problem if one actually wants partial-wave amplitudes, as the undetermined phase may be different at each reconstructed energy and angle. In the analysis of pseudo-scalar photoproduction data, we *do* require multipole amplitudes in order to search for resonance content, and this has led to a renewed interest [7, 8] in the properties of a truncated partial-wave analysis (TPWA), as has been described by Omelaenko [9] and Grushin [10].

The number of required observables is different for the CEA and TPWA. The reason for this is obscured by the fact that very different methods have been used to derive the necessary conditions for a solution. Here, we have used several methods to clarify the connections between the two approaches. The first non-trivial example reveals many of these connections.

II. AMPLITUDES USED IN PSEUDO-SCALAR MESON PHOTOPRODUCTION

Before comparing the CEA and TPWA approaches, we review the notation used to analyze pseudo-scalar photoproduction data. The multipoles and helicity amplitudes are related by [11, 12]

$$H_1 = \frac{1}{\sqrt{2}} \cos\frac{\theta}{2} \sin\theta \sum_{\ell=1}^{\infty} \left[E_{\ell+} - M_{\ell+} - E_{(\ell+1)-} - M_{(\ell+1)-}\right] \left(P_\ell'' - P_{\ell+1}''\right) \ , \tag{1a}$$

$$H_2 = \frac{1}{\sqrt{2}} \cos\frac{\theta}{2} \sum_{\ell=0}^{\infty} \left[(\ell+2)E_{\ell+} + \ell M_{\ell+} + \ell E_{(\ell+1)-} - (\ell+2)M_{(\ell+1)-}\right] \left(P_\ell' - P_{\ell+1}'\right) \ , \tag{1b}$$

$$H_3 = \frac{1}{\sqrt{2}} \sin\frac{\theta}{2} \sin\theta \sum_{\ell=1}^{\infty} \left[(E_{\ell+} - M_{\ell+} + E_{(\ell+1)-} + M_{(\ell+1)-}\right] \left(P_\ell'' + P_{\ell+1}''\right) \ , \tag{1c}$$

$$H_4 = \frac{1}{\sqrt{2}} \sin\frac{\theta}{2} \sum_{\ell=0}^{\infty} \left[(\ell+2)E_{\ell+} + \ell M_{\ell+} - \ell E_{(\ell+1)-} + (\ell+2)M_{(\ell+1)-}\right] \left(P_\ell' + P_{\ell+1}'\right) \ . \tag{1d}$$

From these one can construct the transversity amplitudes [3],

$$b_1 = \frac{1}{2}\left[(H_1 + H_4) \ + \ i \ (H_2 - H_3)\right] \ , \tag{2a}$$

$$b_2 = \frac{1}{2}\left[(H_1 + H_4) \ - \ i \ (H_2 - H_3)\right] \ , \tag{2b}$$

$$b_3 = \frac{1}{2}\left[(H_1 - H_4) \ - \ i \ (H_2 + H_3)\right] \ , \tag{2c}$$

$$b_4 = \frac{1}{2}\left[(H_1 - H_4) \ + \ i \ (H_2 + H_3)\right] \ . \tag{2d}$$

In Table I, expressions for the observables of Type S (cross section and single-polarization), BT (beam-target polarization), BR (beam-recoil polarization), and TR (target-recoil polarization) are given in terms of both helicity and transversity amplitudes.

Transversity amplitudes often simplify the discussion of amplitude reconstruction, as the type-S observables determine their moduli. Another simplification is the property

$$b_2(\theta) = -b_1(-\theta) \quad \text{and} \quad b_4(\theta) = -b_3(-\theta) \ , \tag{3}$$

which allows one to parameterize only two of the four transversity amplitudes. The form introduced by Omelaenko,

$$b_1 = c\, a_{2L} \frac{e^{i\theta/2}}{(1+x^2)^L} \prod_{i=1}^{2L} (x - \alpha_i) \ , \tag{4a}$$

$$b_3 = -c\, a_{2L} \frac{e^{i\theta/2}}{(1+x^2)^L} \prod_{i=1}^{2L} (x - \beta_i) \ , \tag{4b}$$

with $x = \tan(\theta/2)$ and L being the upper limit for ℓ, is convenient for a truncated partial wave analysis, as the ambiguities can be linked to the conjugation of the complex roots of the above relations, with a constraint

$$\prod_{i=1}^{2L} \alpha_i \ = \ \prod_{i=1}^{2L} \beta_i \ . \tag{5}$$

The quantities a_{2L} and c above will be clarified in an explicit example described in Sec. III below.

In a complete experiment analysis (CEA), one attempts to determine the transversity or helicity amplitudes, based on the relations in Table I, at a particular energy and angle. Barker, Donnachie and Storrow [3] (BDS) showed how this could be done with 9 well-chosen observables. For example, the case of I, $\check{P}$, $\check{\Sigma}$, $\check{T}$, $\check{E}$, $\check{F}$, $\check{G}$, $\check{L}_x$, and $\check{L}_z$ was worked

TABLE I. Spin observables expressed in terms of helicity and transversity amplitudes. Helicity amplitudes follow Walker [11] and the SAID convention [12]. The relations in Barker, Donnachie and Storrow [13] are adopted for observables with the replacement $N \to H_2$, $S_1 \to H_1$, $S_2 \to H_4$, $D \to H_3$. The transversity representation for H corrects a typographical error in Ref. [13]. $I = (k/q)\, d\sigma/d\Omega$, k and q being the photon and pion center-of-mass momenta. The checked observables, $\check{O}$, are defined by $\check{O} = I\, O$.

Observable	Helicity representation	Transversity representation	Type
I	$\frac{1}{2}\left(\lvert H_1\rvert^2 + \lvert H_2\rvert^2 + \lvert H_3\rvert^2 + \lvert H_4\rvert^2\right)$	$\frac{1}{2}(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2)$	
$\check{\Sigma}$	$\mathrm{Re}(H_1H_4^* - H_2H_3^*)$	$\frac{1}{2}(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 - \lvert b_4\rvert^2)$	S
$\check{T}$	$\mathrm{Im}(H_1H_2^* + H_3H_4^*)$	$\frac{1}{2}(\lvert b_1\rvert^2 - \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2)$	
$\check{P}$	$-\mathrm{Im}(H_1H_3^* + H_2H_4^*)$	$\frac{1}{2}(\lvert b_1\rvert^2 - \lvert b_2\rvert^2 + \lvert b_3\rvert^2 - \lvert b_4\rvert^2)$	
$\check{G}$	$-\mathrm{Im}(H_1H_4^* + H_2H_3^*)$	$\mathrm{Im}(b_1b_3^* + b_2b_4^*)$	
$\check{H}$	$-\mathrm{Im}(H_1H_3^* - H_2H_4^*)$	$\mathrm{Re}(b_1b_3^* - b_2b_4^*)$	BT
$\check{E}$	$\frac{1}{2}(-\lvert H_1\rvert^2 + \lvert H_2\rvert^2 - \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$-\mathrm{Re}(b_1b_3^* + b_2b_4^*)$	
$\check{F}$	$\mathrm{Re}(H_1H_2^* + H_3H_4^*)$	$\mathrm{Im}(b_1b_3^* - b_2b_4^*)$	
$\check{O}_x$	$-\mathrm{Im}(H_1H_2^* - H_3H_4^*)$	$-\mathrm{Re}(b_1b_4^* - b_2b_3^*)$	
$\check{O}_z$	$\mathrm{Im}(H_1H_4^* - H_2H_3^*)$	$-\mathrm{Im}(b_1b_4^* + b_2b_3^*)$	BR
$\check{C}_x$	$-\mathrm{Re}(H_1H_3^* + H_2H_4^*)$	$\mathrm{Im}(b_1b_4^* - b_2b_3^*)$	
$\check{C}_z$	$\frac{1}{2}(-\lvert H_1\rvert^2 - \lvert H_2\rvert^2 + \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$-\mathrm{Re}(b_1b_4^* + b_2b_3^*)$	
$\check{T}_x$	$\mathrm{Re}(H_1H_4^* + H_2H_3^*)$	$\mathrm{Re}(b_1b_2^* - b_3b_4^*)$	
$\check{T}_z$	$\mathrm{Re}(H_1H_2^* - H_3H_4^*)$	$\mathrm{Im}(b_1b_2^* - b_3b_4^*)$	TR
$\check{L}_x$	$-\mathrm{Re}(H_1H_3^* - H_2H_4^*)$	$\mathrm{Im}(b_1b_2^* + b_3b_4^*)$	
$\check{L}_z$	$\frac{1}{2}(\lvert H_1\rvert^2 - \lvert H_2\rvert^2 - \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$\mathrm{Re}(b_1b_2^* + b_3b_4^*)$	

out explicitly in Ref. [3]. More recently, a counter-example to this scheme was noticed in Ref. [14], which led to the finding, by Chiang and Tabakin [2], that it was possible to perform a CEA with one measurement less. In the case presented by BDS, Chiang and Tabakin demonstrated a solution with only I, $\check{P}$, $\check{\Sigma}$, $\check{T}$, $\check{F}$, $\check{G}$, $\check{T}_x$, and $\check{L}_x$ being required.

In a truncated partial-wave analysis (TPWA), the multipole expansion of helicity or transversity amplitudes is cut off at some upper limit L. Here one finds the amplitudes, for all angles, at a particular energy. Omelaenko showed how this can be done, eliminating the root-conjugation ambiguities associated with the transversity amplitudes in Eqs. (4), using an L-dependent number of angular measurements of five observables, such as I, $\check{P}$, $\check{\Sigma}$, $\check{T}$, and $\check{F}$. As the methods of proof are very different, in the CEA and TPWA problems, it is not obvious how these results can be compared. In the following, we compare the CEA and TPWA results in such a way that the differences can be more easily understood.

III. AMPLITUDE RECONSTRUCTION

A. Trivial case: $L = 0$

It is instructive to compare methods starting with the trivial $\ell = 0$ case of a single E_{0+} multipole and build up to the case studied by Omelaenko [9] including the E_{0+}, M_{1-}, E_{1+}, and M_{1+} multipoles. If only one complex amplitude (E_{0+}) is included, from Eq. (1) we see that there are 2 non-zero helicity amplitudes (H_2 and H_4) which are related by a real factor. Here, we may simply measure the cross section at a single angle. While this gives only one real number, and the amplitudes are complex, the fact that observables involve only bilinear products of amplitudes, i.e. terms of the form A^*B, prevents the measurement of any overall phase associated with the amplitudes. This solves both the CEA and TPWA with the same experimental input.

B. Simplest non-trivial case: $J = 1/2$

The first non-trivial case includes the E_{0+} and M_{1-} multipoles, i.e. partial waves with $J = 1/2$. This combination again produces 2 non-zero helicity amplitudes (H_2 and H_4). In this case, however, the amplitudes are independent. The corresponding transversity amplitudes are given by[1]

$$b_1 = \frac{i}{\sqrt{2}} \left(-e^{i\theta/2} E_{0+} + e^{-i\theta/2} M_{1-} \right) \ , \tag{6}$$

[1] The corresponding expression in Ref. [9] differs by an overall phase ($-i$) and a factor $\sqrt{2}c$ which incorporates the kinematic factor of Table I, here converting $d\sigma/dt$ to I, into the definition of the transversity amplitudes.

TABLE II. Spin observables in terms of helicity amplitudes for a CEA and multipole amplitudes for a TPWA with $J = 1/2$. Here we have used $b_3 = -b_1$ and $b_4 = -b_2$

Observable	CEA (Helicity)	CEA (Transversity)	TPWA	Type
I	$\frac{1}{2}\left(\lvert H_2\rvert^2 + \lvert H_4\rvert^2\right)$	$\lvert b_1\rvert^2 + \lvert b_2\rvert^2$	$\left(\lvert E_{0+}\rvert^2 + \lvert M_{1-}\rvert^2\right) - 2\cos\theta\,\mathrm{Re}\left(E_{0+}M_{1-}^*\right)$	
$\check{\Sigma}$	0	0	0	S
$\check{T}$	0	0	0	
$\check{P}$	$-\mathrm{Im}(H_2H_4^*)$	$\lvert b_1\rvert^2 - \lvert b_2\rvert^2$	$2\sin\theta\,\mathrm{Im}\left(E_{0+}M_{1-}^*\right)$	
$\check{G}$	0	0	0	
$\check{H}$	$\mathrm{Im}(H_2H_4^*)$	$-\lvert b_1\rvert^2 + \lvert b_2\rvert^2$	$-2\sin\theta\,\mathrm{Im}\left(E_{0+}M_{1-}^*\right)$	BT
$\check{E}$	$\frac{1}{2}\left(\lvert H_2\rvert^2 + \lvert H_4\rvert^2\right)$	$\lvert b_1\rvert^2 + \lvert b_2\rvert^2$	$\left(\lvert E_{0+}\rvert^2 + \lvert M_{1-}\rvert^2\right) - 2\cos\theta\,\mathrm{Re}\left(E_{0+}M_{1-}^*\right)$	
$\check{F}$	0	0	0	
$\check{O}_x$	0	0	0	
$\check{O}_z$	0	0	0	BR
$\check{C}_x$	$-\mathrm{Re}(H_2H_4^*)$	$-2\mathrm{Im}b_1b_2^*$	$\sin\theta\,\left(\lvert E_{0+}\rvert^2 - \lvert M_{1-}\rvert^2\right)$	
$\check{C}_z$	$\frac{1}{2}\left(-\lvert H_2\rvert^2 + \lvert H_4\rvert^2\right)$	$2\mathrm{Re}b_1b_2^*$	$2\mathrm{Re}\left(E_{0+}M_{1-}^*\right) - \cos\theta\,\left(\lvert E_{0+}\rvert^2 + \lvert M_{1-}^*\rvert^2\right)$	
$\check{T}_x$	0	0	0	
$\check{T}_z$	0	0	0	TR
$\check{L}_x$	$\mathrm{Re}(H_2H_4^*)$	$2\mathrm{Im}b_1b_2^*$	$-\sin\theta\,\left(\lvert E_{0+}\rvert^2 - \lvert M_{1-}\rvert^2\right)$	
$\check{L}_z$	$\frac{1}{2}\left(-\lvert H_2\rvert^2 + \lvert H_4\rvert^2\right)$	$2\mathrm{Re}b_1b_2^*$	$2\mathrm{Re}\left(E_{0+}M_{1-}^*\right) - \cos\theta\,\left(\lvert E_{0+}\rvert^2 + \lvert M_{1-}^*\rvert^2\right)$	

with $b_3 = -b_1$, and with (b_2, b_4) given by Eq. (3). Below, in Table II, we give the observables both in terms of the helicity/transversity amplitudes (CEA) and the 2 included multipoles (TPWA).

Here the CEA requires 4 measurements at a given energy and angle. For example, I, $\check{P}$, plus either the Beam-Recoil sets ($\check{C}_x$ and $\check{C}_z$) or the Target-Recoil ($\check{L}_x$ and $\check{L}_z$). The TPWA requires one fewer observable, a possible choice being I, $\check{P}$, and $\check{C}_x$ or $\check{L}_x$, compensated by a second angular measurement of the cross section.

For both the CEA and TPWA, closed expressions for the solution of the inverse problem can be obtained in this special case $J = 1/2$. It is instructive to work them out explicitly.

For the quantities I, $\check{P}$ and $\check{C}_x$ in the TPWA, it is possible to parametrize the angular dependence given in Table II as

$$I = \sigma_0 + \sigma_1 \cos\theta \ , \quad \check{P} = P_0 \sin\theta \ , \quad \check{C}_x = C_{x0} \sin\theta \ , \tag{7}$$

where each coefficient carries the energy dependence of the multipoles. It is clear that to extract values for σ_0, σ_1, P_0 and C_{x0}, both spin asymmetries and the cross section are needed at the same angle, with an additional angular measurement required for the cross section.

Having obtained the four coefficients, the zeroth order quantities of I and $\check{C}_x$ can be directly solved for the moduli of the multipoles (cf. Table II),

$$|E_{0+}| = \sqrt{\frac{\sigma_0 + C_{x0}}{2}} \ , \quad |M_{1-}| = \sqrt{\frac{\sigma_0 - C_{x0}}{2}} \ . \tag{8}$$

The relative phase $\phi_{E,M} \equiv \phi_E - \phi_M$ between the multipoles E_{0+} and M_{1-} is obtainable via the remaining two coefficients, both containing information on the real and imaginary parts of the bilinear product $E_{0+}M_{1-}^*$. The additional angular measurement for the cross section fixes the real part,

$$\mathrm{Re}\left(E_{0+}M_{1-}^*\right) = |E_{0+}|\,|M_{1-}|\,\mathrm{Re}\left(e^{i\phi_{E,M}}\right) = -\frac{1}{2}\sigma_1 \ , \tag{9}$$

while the imaginary part can be extracted from the single measurement of $\check{P}$,

$$\mathrm{Im}\left(E_{0+}M_{1-}^*\right) = |E_{0+}|\,|M_{1-}|\,\mathrm{Im}\left(e^{i\phi_{E,M}}\right) = \frac{1}{2}P_0 \ . \tag{10}$$

Together, these define the exponential of the relative phase, provided that none of the moduli vanish,

$$e^{i\phi_{E,M}} = \frac{-\sigma_1 + iP_0}{\sqrt{\sigma_0 + C_{x0}}\sqrt{\sigma_0 - C_{x0}}} \ . \tag{11}$$

This function can be inverted uniquely on the interval $[0, 2\pi)$. Therefore, no quadrant ambiguity remains. The multipoles have been extracted up to an overall phase.

The CEA proceeds in a mathematically exactly analogous way. The observables I, $\check{P}$, $\check{C}_x$ and $\check{C}_z$ yield the moduli and relative phase $\phi_{1,2} \equiv \phi_{b_1} - \phi_{b_2}$ of the transversity amplitudes using exactly the same calculation (cf. Table II)

$$|b_1| = \sqrt{\frac{I + \check{P}}{2}} \ , \quad |b_2| = \sqrt{\frac{I - \check{P}}{2}} \ , \tag{12}$$

$$e^{i\phi_{1,2}} = \frac{\check{C}_z - i\check{C}_x}{\sqrt{I + \check{P}}\sqrt{I - \check{P}}} \ . \tag{13}$$

A crucial difference, however, lies in the kinematical regions over which the CEA and TPWA operate. For a fixed energy, the CEA extracts amplitudes from observables at exactly the same angle and it is completely blind to what may happen at neighbouring angles. The TPWA uses the angular distributions of the observables which, in the present case of $I(\theta)$, is linear in $\cos\theta$. One seemingly obtains a reduction from 4 to 3 observables, but this is bought at the price of having to measure angular distributions which become, for the higher truncation orders, increasingly complicated.

The difference in the nature of these analyses also becomes obvious in considering the end results they yield. The CEA returns transversity amplitudes only at a single angle, up to an energy- and angle-dependent overall phase, cf. Eqs. (12) and (13). However, from the result of the TPWA, the moduli Eq. (8) and relative phase Eq. (11) of multipoles, it is possible to infer transversity amplitudes at all angles, this time up to an energy-dependent phase.

C. Unique features of the $J = 1/2$ case

It is useful to compare the special case of $J = 1/2$ to more general results for the CEA and TPWA in Refs. [2] and [9]. In Ref. [2], a complete set of 8 experiments, explicitly derived and compared to the corresponding BDS case (requiring 9 experiments) is: (I, $\check{\Sigma}$, $\check{P}$, $\check{T}$, $\check{G}$, $\check{F}$, $\check{L}_x$, $\check{T}_x$). Here, with a truncation to $J = 1/2$, this set becomes (I, 0, $\check{P}$, 0, 0, 0, $\check{L}_x$, 0), which does not contain sufficient information, as can be seen directly from Table II. However, the older BDS set, which exchanges $\check{T}_x$ for $\check{E}$ and $\check{L}_z$, *does* constitute a complete experiment. This failure of a set of 8 experiments is due to the number of zero quantities in Table II. The effect can be seen in constraint equation (4.10) employed in the derivation of Ref. [2]. Many Fierz identities listed in Ref. [2] similarly revert to zero-equals-zero relations in this special case.

Similarly, the TPWA conditions for a complete set [9], derived for a case including the E_{0+}, M_{1-}, E_{1+} and M_{1+} multipoles, do not directly reduce to the result given here if the E_{1+} and M_{1+} multipoles are simply set to zero. In Refs. [7, 9], a complete set is given as (I, $\check{P}$, $\check{\Sigma}$, $\check{T}$, $\check{G}$), which again is insufficient in this special case.

To understand how a truncation to $J = 1/2$ changes the result, it is instructive to repeat Omelaenko's analysis [9], which led to the general parametrizations of Eq. (4a) and Eq. (4b), under the constraint in Eq. (5), for all $L \geq 1$.

Expressing $\cos\theta$ and $\sin\theta$ in terms of $x = \tan(\theta/2)$, one can write

$$e^{i\theta} = \frac{(1+ix)^2}{1+x^2} \ . \tag{14}$$

Starting from the expression for b_1 in terms of multipoles given in Eq. (6), we have

$$\begin{aligned} b_1 &= \frac{i}{\sqrt{2}}\left(-e^{i\theta/2}E_{0+} + e^{-i\theta/2}M_{1-}\right) \\ &= \frac{ie^{i\theta/2}}{\sqrt{2}}\left(-E_{0+} + \frac{1-x^2-2ix}{(1+x^2)}M_{1-}\right) \\ &= \frac{-i}{\sqrt{2}}\frac{e^{i\theta/2}}{(1+x^2)}\left(E_{0+}+M_{1-}\right)\left(x^2 + \frac{2iM_{1-}}{E_{0+}+M_{1-}}x + \frac{E_{0+}-M_{1-}}{E_{0+}+M_{1-}}\right) \\ &\equiv \frac{-i}{\sqrt{2}}\frac{e^{i\theta/2}}{(1+x^2)}a_2\left(x^2+\hat{a}_1x+\hat{a}_0\right) \ . \end{aligned} \tag{15}$$

Note that the coefficients a_2, $\hat{a}_1$ and $\hat{a}_0$, defining the amplitude in the last step, are fully equivalent to the multipoles. Decomposing the polynomial into a product of linear factors defined by two complex roots α_1 and α_2, the Omelaenko decomposition of the amplitude b_1 is obtained as

$$b_1 = \frac{-i}{\sqrt{2}}\,a_2 e^{i\theta/2}\frac{\prod_{i=1}^{2}(x-\alpha_i)}{1+x^2}. \tag{16}$$

The expression for the only remaining non-redundant amplitude, b_2, for $J = 1/2$, is obtained by invoking the symmetry in Eq. (3),

$$b_2(\theta) = -b_1(-\theta) = \frac{i}{\sqrt{2}}a_2 e^{-i\theta/2}\frac{\prod_{i=1}^{2}(x+\alpha_i)}{1+x^2}. \tag{17}$$

Therefore, for $J = 1/2$ there are only two α-roots, no β-roots and the constraint Eq. (5) no longer appears.

In view of the already obtained results Eq. (8) and Eq. (11), the observable $\check{C}_x$ will have to be tested for its response to discrete ambiguity transformations. The full Omelaenko decomposition of this observable becomes

$$\check{C}_x = -2\mathrm{Im} b_1 b_2^* = \frac{|a_2|^2}{(1+x^2)^2}\,\mathrm{Im}\left[e^{i\theta}\prod_{i=1}^{2}(x-\alpha_i)(x+\alpha_i^*)\right] \ . \tag{18}$$

The decompositions of amplitudes b_i in terms of roots α_1 and α_2 given by Eqs. (16) and (17) facilitate a study of the discrete ambiguities of the quantities I and $\check{P}$ (as well as $\check{E}$ and $\check{H}$), since they are just linear combinations of the squared moduli $|b_1|^2$ and $|b_2|^2$ (see

Table II). The ambiguities are obtained by the complex conjugation of subsets of roots, as stated below Eqs. (4).

Note that the multipoles E_{0+} and M_{1-}, with an undetermined overall phase that can be arbitrarily fixed, correspond to 3 real numbers. For the variables $(a_2, \alpha_1, \alpha_2)$ of the Omelaenko decomposition, where the phase of a_2 cannot be determined, one counts 5 real degrees of freedom. The general constraint equation (5), which is true for an expansion in ℓ for all $L \geq 1$, is missing here. So, there must be another way in which the effective number of real degrees of freedom is reduced from 5 to 3.

One can learn more by considering the equations which relate the Omelaenko roots (α_1, α_2) to the multipoles (E_{0+}, M_{1-}). Utilizing the notation of Eqs. (15) and (16), we have

$$\hat{a}_1 \equiv -(\alpha_1 + \alpha_2) = \frac{2iM_{1-}}{E_{0+} + M_{1-}} \ , \quad \hat{a}_0 \equiv \alpha_1 \alpha_2 = \frac{E_{0+} - M_{1-}}{E_{0+} + M_{1-}}. \tag{19}$$

These relations lead to a quadratic equation with two solutions given by the roots

$$\alpha_1^{(\mathrm{I})} = i\,\frac{E_{0+} - M_{1-}}{E_{0+} + M_{1-}} \ , \quad \alpha_2^{(\mathrm{I})} = -i \quad \text{and} \quad \alpha_1^{(\mathrm{II})} = -i \ , \quad \alpha_2^{(\mathrm{II})} = i\,\frac{E_{0+} - M_{1-}}{E_{0+} + M_{1-}} \ . \tag{20}$$

Both solutions remove the overcounting mentioned above. Two real degrees of freedom are always removed since one of the roots has a fixed value. Only one of the two roots depends on the multipoles.

Solutions I and II of (20) are not distinct, as both are equivalent by a simple re-labelling of the roots. Taking solution I, for which α_2 is fixed to $-i$, only one discrete ambiguity remains in the Omelaenko formulation for $J = 1/2$, represented by the transformation $\alpha_1 \rightarrow \alpha_1^*$,

Using solution I, the full Omelaenko decomposition of $\check{C}_x$, Eq. (18), simplifies significantly. Again writing the exponential $e^{i\theta}$ in terms of $x = \tan(\theta/2)$, see Eq. (14), we have the identity

$$e^{i\theta}\,(x - \alpha_2)(x + \alpha_2^*) = \frac{(1 + ix)^2}{1 + x^2}\,(x + i)^2 = -\left(1 + x^2\right) \ . \tag{21}$$

The expression for $\check{C}_x$, in terms of the only non-redundant Omelaenko root, α_1, then becomes

$$\check{C}_x = \frac{-\left|a_2\right|^2}{1 + x^2} \mathrm{Im}\left[(x - \alpha_1)(x + \alpha_1^*)\right] \ . \tag{22}$$

For the discrete symmetry, $\alpha_1 \rightarrow \alpha_1^*$, we see that expression (22) changes sign, $\check{C}_x \rightarrow -\check{C}_x$, once the ambiguity transformation is applied. Furthermore, $\check{C}_x$ generally only remains invariant at the angles $\theta = 0$ and $\theta = \pi$, where it vanishes by definition (see Table II).

Another interesting special case is found if one requires the transformation $\alpha_1 \rightarrow \alpha_1^*$ to produce no ambiguity, which can only be fulfilled for a real root. Once this condition is evaluated for the explicit form of α_1 in terms of multipoles, given in Eq. (20), one finds that $\alpha_1^* = \alpha_1$ is equivalent to $|E_{0+}| = |M_{1-}|$.

The Omelaenko decomposition of $\check{C}_x$, as well as the explicit form of this quantity written in terms of multipoles (see Table II), shows that in this particular case $\check{C}_x$ vanishes for all angles. Here, while the sign information associated with $\check{C}_x$ may be missing, it is not required, as the discrete symmetry, which is resolved precisely by this sign, no longer exists. Also, Eqs. (8) and (11) imply that in this special case, i.e. $|E_{0+}| = |M_{1-}|$ or equivalently $C_{x0} = 0$, the moduli of both multipoles, as well as the relative phase $\phi_{E,M}$, are determined by I and $\check{P}$ alone. This case is, however, the only situation where a solution of the inverse problem is possible with just 2 observables.

In summary, both the explicit inversion of the TPWA, Eqs. (8) and (11), and the study of the discrete ambiguities, according to Omelaenko's method, yield consistent results for $J = 1/2$. This has been exemplified by the solvability of the example set I, $\check{P}$ and $\check{C}_x$. The case $J = 1/2$ is special as it allows all three analyses, the CEA, TPWA and ambiguity study, to be performed using simple algebra. For the higher orders $L \geq 1$, Chiang and Tabakin [2] have published a solution for the CEA which holds apart from the special case discussed above.

An algebraic inversion of the TPWA, i.e. the extraction of the bilinear products of multipoles by an effective linearization of the problem, followed by a simple evaluation of moduli and relative phases, does not appear to be possible for $L \geq 1$. The only principle that carries through to the higher orders is the study of discrete ambiguities [7, 9], using the expressions in Eqs. (4a), (4b) and (5).

In this way, complete sets of observables can still be proposed. However, the actual completeness of such sets should, in any case, be checked by a full solution of the inverse problem which, for the higher truncation orders, can only be done numerically.

D. Counting Observables

In examining the $J = 1/2$ case, it was found that a formal solution was possible with only $\check{P}$, $\check{C}_x$, and $\check{C}_z$ (3 rather than 4 quantities), measured at one angle, if one used the overall phase freedom to make one amplitude real and positive. This result could be understood by refining how the counting of observables is done. If a measurement, done with a fixed beam, target and detector setup, produces an 'observable', then the measurement of a polarization asymmetry (spin up versus spin down) is actually two observables. These two measurements can then be combined to form both the asymmetry and the cross section. Once the cross section is known, a second asymmetry can, in principle, be determined from only one of the two possible (such as spins parallel versus anti-parallel) measurements. Thus, the set ($\check{P}$, $\check{C}_x$, $\check{C}_z$) requires 2+1+1=4 measurements, compared to the set (I, $\check{P}$, $\check{C}_x$, $\check{C}_z$), requiring 1+1+1+1=4 measurements.

IV. COMPARING CEA AND TPWA BEYOND $J = 1/2$

In Table III, the examples discussed in detail above are generalized to higher angular-momentum cutoffs. The examples with one, two, and three multipoles show that in the CEA and TPWA approaches, the number of measurements is the same. In cases where a TPWA is possible with all measurements at a single energy and angle, the results are directly related. Note that in the case of 3 multipoles, only 3 of the helicity/transversity amplitudes are independent. This is also true for the standard set of 4 multipoles (E_{0+}, M_{1-}, E_{1+}, M_{1+}) as can be most easily seen if, instead, one writes out the CGLN amplitudes,

$$F_1(\theta) = E_{0+} + 3\,(M_{1+} + E_{1+}) \cos\theta \ , \tag{23a}$$

$$F_2(\theta) = 2\,M_{1+} + M_{1-} \ , \tag{23b}$$

$$F_3(\theta) = 3\,(E_{1+} - M_{1+}) \ , \tag{23c}$$

$$F_4(\theta) = 0 \ . \tag{23d}$$

With $F_4 = 0$, only 3 independent amplitudes can be extracted in a CEA. Consequently, also only 3 linear combinations of multipoles can be obtained in an experiment at a single angle.

Extending the expansion of observables, given in Eq. (8), to higher orders in $\cos\theta$ up to the highest powers for a given L, we have

$$I = \sigma_0 + \sigma_1 \, \cos\theta + \sigma_2 \, \cos^2\theta + \cdots + \sigma_{2L} \, \cos^{2L}\theta \, , \tag{24a}$$

$$\check{\Sigma} = \sin^2\,\theta \left(\Sigma_0 + \cdots + \Sigma_{2L-2} \, \cos^{2L-2}\theta\right) \ , \tag{24b}$$

$$\check{T} = \sin\,\theta \left(T_0 + T_1 \, \cos\theta + \cdots + T_{2L-1} \, \cos^{2L-1}\theta\right) \ , \tag{24c}$$

TABLE III. Examples of measurements at a single energy for CEA and TPWA. The number of different measurements (n), different observables (m) and different angles (k) needed for a complete analysis are given as $n(m)k$. Entries with a † do not allow the comparison CEA $\leftrightarrow$ TPWA. For cases with only one angle, the CEA and TPWA are equivalent. The number of necessary distinct angular measurements is given in brackets.

Set	Included Partial Waves	CEA	TPWA	Complete Sets for TPWA
1	$L = 0$ (E_{0+})	1(1)	1(1)1	$I[1]$
2	$J = 1/2$ (E_{0+}, M_{1-})	4(4)	4(4)1	$I[1]\,, \check{P}[1]\,, \check{C}_x[1]\,, \check{C}_z[1]$
			4(3)2	$I[2]\,, \check{P}[1]\,, \check{C}_x[1]$
3	$L = 0, 1$ (E_{0+}, M_{1-}, E_{1+})	6(6)	6(6)1	$I[1]\,, \check{\Sigma}[1]\,, \check{T}[1]\,, \check{P}[1]\,, \check{F}[1]\,, \check{G}[1]$
			6(4)2	$I[2]\,, \check{\Sigma}[1]\,, \check{T}[2]\,, \check{P}[1]$
			6(3)3	$I[3]\,, \check{\Sigma}[1]\,, \check{T}[2]$
4	$L = 0, 1$ $(E_{0+}, M_{1-}, E_{1+}, M_{1+})$	†		TPWA at 1 angle not possible
	full set of 4 S, P wave multipoles		8(5)2	$I[2]\,, \check{\Sigma}[1]\,, \check{T}[2]\,, \check{P}[2]\,, \check{F}[1]$
			8(4)3	$I[3]\,, \check{\Sigma}[1]\,, \check{F}[2]\,, \check{H}[2]$
5	$L = 0, 1, 2$ $(E_{0+}, M_{1-}, E_{1+}, E_{2-})$	8(8)	8(8)1	$I[1]\,, \check{\Sigma}[1]\,, \check{T}[1]\,, \check{P}[1]\,, \check{F}[1]\,, \check{G}[1]\,, \check{C}_x[1]\,, \check{O}_x[1]$
			8(4)2	$I[2]\,, \check{\Sigma}[2]\,, \check{T}[2]\,, \check{P}[2]$
			8(3)3	$I[3]\,, \check{\Sigma}[2]\,, \check{T}[3]$
6	$J \leq 3/2$ $(E_{0+}, M_{1-}, E_{1+}, M_{1+}, E_{2-}, M_{2-})$	†		TPWA at 1 or 2 angles not possible
			12(5)3	$I[3]\,, \check{\Sigma}[2]\,, \check{T}[3]\,, \check{P}[2]\,, \check{F}[2]$
			12(4)4	$I[4]\,, \check{\Sigma}[2]\,, \check{F}[3]\,, \check{H}[3]$
7	$L = 0, 1, 2$ $(E_{0+}, \ldots, M_{2+})$	†		TPWA at 1 or 2 angles not possible
	full set of 8 S, P, D wave multipoles		16(6)3	$I[3]\,, \check{\Sigma}[3]\,, \check{T}[3]\,, \check{P}[3]\,, \check{F}[3]\,, \check{G}[1]$
			16(5)4	$I[4]\,, \check{\Sigma}[3]\,, \check{T}[3]\,, \check{P}[3]\,, \check{F}[3]$
			16(4)5	$I[5]\,, \check{\Sigma}[3]\,, \check{F}[4]\,, \check{H}[4]$

$$\check{P} = \sin\,\theta \left(P_0 + P_1\,\cos\theta + \cdots + P_{2L-1}\,\cos^{2L-1}\theta\right)\,. \tag{24d}$$

The remaining double-polarization observables, ($\check{E}$, $\check{C}_x$, $\check{O}_x$, $\check{T}_z$, $\check{L}_x$) behave like I, ($\check{F}$, $\check{H}$, $\check{O}_z$, $\check{T}_x$) like $\check{T}$, $\check{G}$ like $\check{\Sigma}$, while $\check{C}_z$ and $\check{L}_z$ exhibit the highest powers up to $\cos^{2L+1}\theta$.

Table III gives examples of measurement sets involving from one to a generalized number of $4L$ multipoles using one or more angles in the TPWA. Set 1 is the trivial case and set 2 with $J = 1/2$ has already been discussed in detail. Besides the $J = 1/2$ TPWA set with the minimal number of 3 observables, but more than one angle, a solution also exists at one angle with 4 observables, which is fully equivalent to the CEA. In a set with 3 S, P wave multipoles, E_{0+}, E_{1+}, M_{1-}, 3 amplitudes are linearly independent, e.g. F_1, F_2, F_3, and both CEA and TPWA are again equivalent. Also, with TPWA at more than one angle, the number of observables can be reduced. Taking the full angular distribution, a minimal set of 3 polarization observables is already complete.

The next logical set is the full set 4 of S and P wave multipoles, E_{0+}, M_{1-}, E_{1+}, and M_{1+}, but this yields a surprising result. As already discussed, from Eqs. (23) only 3 amplitudes are linearly independent, leading to a CEA, which is not sufficient to resolve all 4 multipoles. This is only possible by using the angular distribution of the observables, in the minimal case by measurements at a second angle. At this point, it is very interesting to note that solutions with only 4 observables are also possible [15]. Here we give the set of observables, (I, $\check{\Sigma}$, $\check{F}$, $\check{H}$), providing a solution with no recoil measurements required. This is a very surprising result, as it goes beyond the studies of Omelaenko [7, 9], where unique solutions were found only with 5 or more observables.

A simple set with 4 multipoles and 4 independent amplitudes is set 5 of Table III with E_{0+}, E_{1+}, M_{1-}, and E_{2-}. In this case also $F_4 = -3E_{2-}$ is finite. For this set an equivalent set of 8 observables yields unique solutions for a CEA, with 4 transversity amplitudes, and a TPWA, at a single angle, with 4 multipoles. However, taking into account the angular distribution, the number of necessary observables can be reduced to only 3, I, $\check{\Sigma}$ and $\check{T}$, where no recoil measurement would be needed.

Truncating the multipole series in the total spin J (instead of angular momentum L) leads to set 6 with limit $J = 3/2$. This set contains 6 multipoles, and a CEA at one angle is certainly no longer sufficient to determine all of them. The last set 7 of Table III is the full set of 8 multipoles for $L = 2$ and can be generalized for any higher L. Here also the CEA is no longer related to the TPWA.

The Omelaenko method [7, 9] can be applied to any given L. This method proves, in general, a unique solution is possible with 5 observables measured over the full angular range, i.e. at enough angles to determine the $\cos\theta$ or alternatively the Legendre coefficients.

These are 4 observables from group S, the unpolarized cross section and the 3 single-spin polarizations, plus one more double polarization observable from any other group, except $\check{E}$ and $\check{H}$. The 5th observable is needed to resolve, first of all, the double ambiguity. The new solution with only 4 observables [15], which was found to provide a solution for set 4, has been found to solve set 7 as well, and can most likely be generalized for any higher L.

However, as discussed in Refs. [7, 9], an increasing number of 4^{2L} accidental ambiguities can occur, which leads to enormous numerical problems for $L > 2$. This problem can partly be solved by extending the set of observables. However, the accidental ambiguities depend on the dynamics of the underlying models and the physics involved, and unique solutions cannot be guaranteed in many cases, so elaborate numerical methods need to be applied. Since experimental data contain sizable statistical errors, and in most cases also systematic errors, a unique solution by this method will become increasingly difficult for larger L. Therefore, in practice, higher partial waves have to be fixed by models or if possible by theoretical constraints such as unitarity, analyticity and fixed-t dispersion relations.

Instead of doing model applications, the results of Table III have been obtained in a more general numerical simulation procedure. The underlying multipoles numbering from two to eight were randomly chosen as complex numbers with integer values for their real and imaginary parts. From these multipoles, all observables and their coefficients were calculated and the inverse solution was searched with numerical minimization techniques using random search with the help of Mathematica. Sets 1 to 6 were quickly obtained but set 7, for $L = 2$, required a significant increase in computation time. Nevertheless, the uniqueness of the solution in terms of the squared numerical deviation is found to be of order 10^{-20}.

V. CONCLUSIONS

We have explored the CEA and TPWA, applying a number of approaches, in order to compare the information required for a complete solution. The connection is seen most easily in the first non-trivial case, $J = 1/2$, involving the interference of two multipoles or helicity/transversity amplitudes. The reduced number of observable types for a TPWA is compensated by additional angular measurements. From a physical standpoint, the appearance of θ-dependent factors in Eq. (6) is due to rotational symmetry, as contained in the rotation matrices used to construct the helicity amplitudes [11, 16].

This matching of information required to determine either the multipoles or helicity/transversity amplitudes holds only when the number of independent helicity/transversity amplitudes, for a CEA, is the same as the number of multipoles used in their construction. The number of angular measurements for a TPWA grows with increasing angular momentum cutoff, as described in Refs. [7, 9]. With greater than four multipole amplitudes included, the TPWA and CEA problems are fundamentally different and the information required for a solution is not comparable.

Our pedagogical study of the simple $J = 1/2$ case, generalized to higher angular-momentum cutoffs, has revealed further solutions of the TPWA problem addressed by Omelaenko [9], which require only 4 well selected polarization observables. These will be examined in detail in a future publication [15].

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