Quasi-PDFs, momentum distributions and pseudo-PDFs

A. V. Radyushkin
Physics Department, Old Dominion University, Norfolk, VA 23529, USA and
Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA

We show that quasi-PDFs may be treated as hybrids of PDFs and primordial rest-frame momentum distributions of partons. This results in a complicated convolution nature of quasi-PDFs that necessitates using large $p_3 \gtrsim 3$ GeV momenta to get reasonably close to the PDF limit. As an alternative approach, we propose to use pseudo-PDFs $\mathcal{P}(x, z_3^2)$ that generalize the light-front PDFs onto spacelike intervals and are related to Ioffe-time distributions $\mathcal{M}(\nu, z_3^2)$, the functions of the Ioffe time $\nu = p_3 z_3$ and the distance parameter z_3^2 with respect to which it displays perturbative evolution for small z_3. In this form, one may divide out the z_3^2 dependence coming from the primordial rest-frame distribution and from the problematic factor due to lattice renormalization of the gauge link. The ν-dependence remains intact and determines the shape of PDFs.

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Introduction. The parton distribution functions (PDFs) $f(x)$ [1] are related to matrix elements of bilocal operators on the light cone $z^2 = 0$, which prevents a straightforward calculation of these functions in the lattice gauge theory formulated in Euclidean space. The usual way out is to calculate their moments. However, recently, X. Ji [2] suggested a method allowing to calculate PDFs as functions of x. To this end, he proposes to use purely space-like separations $z = (0, 0, 0, z_3)$. Then one deals with quasi-PDFs $Q(y, p_3)$ describing sharing of the p_3 hadron momentum component, and tending to PDFs $f(y)$ in the $p_3 \to \infty$ limit. The same method can be applied to distribution amplitudes (DAs). The results of lattice calculations of quasi-PDFs were reported in Refs. [3–5] and of the pion quasi-DA in Ref. [6].

In our recent papers [7, 8], we have studied nonperturbative evolution of quasi-PDFs and quasi-DAs using the formalism of virtuality distribution functions [9, 10]. We found that quasi-PDFs can be obtained from the transverse momentum dependent distributions (TMDs) $\mathcal{F}(x, k_\perp^2)$. We built models for the nonperturbative evolution of quasi-PDFs using simple models for TMDs. Our results are in qualitative agreement with the p_3-evolution patterns obtained in lattice calculations.

In the present paper, our first goal is to develop a picture for quasi-PDFs as hybrids of PDFs and primordial momentum distributions of partons in a hadron at rest. As an intermediate step, we demonstrate that the connection between TMDs and quasi-PDFs [7] is a mere consequence of Lorentz invariance. Then we show that, when the hadron is moving, the parton k_3 momentum comes from two sources. The motion of the hadron as a whole gives the xp_3 part, governed by the dependence of the TMD $\mathcal{F}(x, \kappa^2)$ on its x argument. The remaining part $k_3 - xP$ is governed by the dependence of the TMD on its second argument, κ^2, governing the primordial rest-frame momentum distribution. The convolution nature of quasi-PDFs results in a rather complicated pattern of their p_3 evolution, necessitating rather large values $p_3 \sim 3$ GeV for getting close to the PDF limit.

Thus, our second goal is to propose an alternative approach for lattice PDF extraction. To this end, we introduce *pseudo-PDFs* $\mathcal{P}(x, z_3^2)$ that generalize the light-cone PDFs $f(x)$ onto spacelike intervals like $z = (0, 0, 0, z_3)$. The pseudo-PDFs are Fourier transforms of the *Ioffe-time* [11] *distributions* [12] $\mathcal{M}(\nu, z_3^2)$ that are basically given by generic matrix elements like $\langle p|\phi(0)\phi(z)|p\rangle$ written as functions of $\nu = p_3 z_3$ and z_3^2. Unlike quasi-PDFs, the pseudo-PDFs have the "canonical" $-1 \le x \le 1$ support for all z_3^2. They tend to PDFs when $z_3 \to 0$, showing in this limit a usual perturbative evolution with $1/z_3$ serving as an evolution parameter. Finally, we discuss how these properties of pseudo-PDFs may be used for extraction of PDFs on the lattice.

Generic matrix element and Lorentz invariance. Historically [1], PDFs were introduced to describe spin-1/2 quarks. Since complications related to spin do not affect the very concept of parton distributions, we start with a simple example of a scalar theory. In that case, information about the target is accumulated in the generic matrix element $\langle p|\phi(0)\phi(z)|p\rangle$. By Lorentz invariance, it is a function of two invariants, (pz) and z^2 (or $-z^2$ if we want a positive value for spacelike z):

$$\langle p|\phi(0)\phi(z)|p\rangle = \mathcal{M}((pz), -z^2)\,. \qquad (1)$$

It can be shown [7, 13] that, for all contributing Feynman diagrams, its Fourier transform $\mathcal{P}(x, -z^2)$ with respect to (pz) has the $-1 \le x \le 1$ support, i.e.,

$$\mathcal{M}((pz), -z^2) = \int_{-1}^{1} dx\, e^{-ix(pz)}\, \mathcal{P}(x, -z^2)\ . \qquad (2)$$

Note that Eq. (2) gives a covariant definition of x. There is no need to assume that $p^2 = 0$ or $z^2 = 0$ to define x.

Collinear PDFs. Choosing some special cases of p and z, one can get expressions for various parton distributions, all in terms of the same function $\mathcal{M}((pz), -z^2)$. In particular, taking a light-lke z, e.g., that having the light-front minus component z_- only, we parameterize the matrix element by the twist-2 parton distribution $f(x)$

$$\mathcal{M}(p_+z_-,0) = \int_{-1}^{1} dx\, f(x)\, e^{-ixp_+z_-} \ , \tag{3}$$

with $f(x)$ having the usual interpretation of probability that the parton carries the fraction x of the target momentum component p_+. The inverse relation is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{ix\nu}\, \mathcal{M}(\nu,0) = \mathcal{P}(x,0) \ . \tag{4}$$

Since $f(x) = \mathcal{P}(x,0)$, the function $\mathcal{P}(x,-z^2)$ generalizes PDFs onto non-lightlike intervals z^2, and we will call it *pseudo-PDF*. The variable ν is called the *Ioffe time* [11], and $\mathcal{M}(\nu,-z^2)$ is the *Ioffe-time distribution* [12].

Note that the definition of $\mathcal{P}(x,-z^2)$ is simpler than that of $f(x)$ because it does not require taking a subtle $z^2 \to 0$ limit. In renormalizable theories, the function $\mathcal{M}(\nu,z^2)$ has $\sim \ln z^2$ singularities generating perturbative evolution of parton densities. Within the operator product expansion (OPE) approach, the $\ln z^2$ singularities are subtracted using some prescription, say, dimensional renormalization, and the resulting PDFs depend on the renormalization scale μ, i.e., $f(x) \to f(x,\mu^2)$.

Transverse momentum dependent distributions. Treating the target momentum p as longitudinal, $p = (E, \mathbf{0}_\perp, P)$, one can introduce transverse degrees of freedom. Taking z that has z_- and $z_\perp = \{z_1, z_2\}$ components only, one defines the *TMD* $\mathcal{F}(x,k_\perp^2)$

$$\mathcal{M}(\nu, z_1^2+z_2^2) = \int_{-1}^{1} dx\ e^{-ix\nu} \int_{-\infty}^{\infty} dk_1 e^{-ik_1z_1} \times \int_{-\infty}^{\infty} dk_2\, e^{-ik_2z_2} \mathcal{F}(x, k_1^2+k_2^2) \ . \tag{5}$$

The $\sim \ln z_\perp^2$ terms in $\mathcal{M}(\nu,z_\perp^2)$ are produced by the $\sim 1/k_\perp^2$ hard tail of $\mathcal{F}(x,k_\perp^2)$. Thus, it makes sense to visualize $\mathcal{M}(\nu,z_\perp^2)$ as a sum of a soft part $\mathcal{M}^{\rm soft}(\nu,z_\perp^2)$, that has a finite $z_\perp^2 \to 0$ limit and a hard part reflecting the evolution. For TMDs, soft part decreases faster than $1/k_\perp^2$, say, like a Gaussian $e^{-k_\perp^2/\Lambda^2}$. In the $z_\perp$ space, the distributions are then concentrated in $z_\perp \sim 1/\Lambda$ region.

Quasi-Distributions. Since one cannot have light-like separations on the lattice, it was proposed [2] to consider spacelike separations $z = (0,0,0,z_3)$ [or, for brevity, $z = z_3$]. Then, in the $p = (E, 0_\perp, P)$ frame, one introduces quasi-PDF $Q(y,P)$ through a parametrization

$$\langle p|\phi(0)\phi(z_3)|p\rangle = \int_{-\infty}^{\infty} dy\, Q(y,P)\, e^{iyPz_3} \ . \tag{6}$$

The inverse Fourier transformation

$$Q(y,P) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\ e^{iy\nu}\, \mathcal{M}(\nu,\nu^2/P^2) \tag{7}$$

indicates that $Q(y,P)$ tends to $f(y)$ in the $P \to \infty$ limit, as far as $\mathcal{M}(\nu,\nu^2/P^2) \to \mathcal{M}(\nu,0)$. The deviation of quasi-PDF $Q(y,P)$ from the PDF $f(y)$ may be described in terms of TMDs. To this end, we substitute Eq. (5) with $z_1 = 0$ and $z_2 = \nu/P$ into Eq. (7) to convert it into the expression for quasi-PDFs in terms of TMDs

$$Q(y,P)/P = \int_{-\infty}^{\infty} dk_1 \int_{-1}^{1} dx\, \mathcal{F}(x, k_1^2 + (y-x)^2P^2) \ . \tag{8}$$

Originally, this relation was derived in Ref. [7] using a Nakanishi-type representation of Refs. [9, 10]. Now, we see that it is a mere consequence of Lorentz invariance.

Quantum chromodynamics (QCD) case. The formulas derived above are directly applicable for non-singlet parton densities in QCD. In that case, one deals with matrix elements of

$$\mathcal{M}^\alpha(z,p) \equiv \langle p|\bar\psi(0)\,\gamma^\alpha\, \hat E(0,z;A)\psi(z)|p\rangle \tag{9}$$

type, where $\hat E(0,z;A)$ is the standard $0 \to z$ straight-line gauge link in the quark (adjoint) representation. These matrix elements may be decomposed into p^α and z^α parts: $\mathcal{M}^\alpha(z,p) = p^\alpha \mathcal{M}_p((zp),-z^2) + z^\alpha \mathcal{M}_z((zp),-z^2)$. The $\mathcal{M}_p((zp),-z^2)$ part gives the twist-2 distribution when $z^2 \to 0$, while $\mathcal{M}_z((zp),-z^2)$ is a purely higher-twist contamination, and it is better to get rid of it.

If one takes $z = (z_-, z_\perp)$ in the $\alpha = +$ component of $\mathcal{O}^\alpha$, the z^α-part drops out, and one can introduce a TMD $\mathcal{F}(x,k_\perp^2)$ that is related to $\mathcal{M}_p(\nu,z_\perp^2)$ by the scalar formula (5). For quasi-distributions, the easiest way to remove the z^α contamination is to take the time component of $\mathcal{M}^\alpha(z=z_3,p)$ and define

$$\mathcal{M}^0(z_3,p) = 2p^0 \int_{-1}^{1} dy\, Q(y,P)\ e^{iyPz_3} \ . \tag{10}$$

Then the connection between $Q(y,P)$ and $\mathcal{F}(x,k_\perp^2)$ is given by the scalar formula (8).

Momentum distributions. The quasi-PDFs describe the distribution in the fraction $y \equiv k_3/P$ of the third component k_3 of the parton momentum to that of the hadron. One can introduce distributions in k_3 itself: $R(k_3,P) \equiv Q(k_3/P)/P$. Then

$$R(k_3,P) = \int_{-1}^{1} dx\, \mathcal{R}(x, k_3 - xP) \ , \tag{11}$$

where

$$\mathcal{R}(x,k_3) \equiv \int_{-\infty}^{\infty} dk_1 \mathcal{F}(x, k_1^2+k_3^2) \tag{12}$$

is the TMD $\mathcal{F}(x,\kappa^2)$ integrated over the k_1 component of the two-dimensional vector $\kappa = \{k_1, k_3\}$.

For a hadron at rest, we have

$$R(k_3, P=0) \equiv r(k_3) = \int_{-1}^{1} dx\, \mathcal{R}(x,k_3) \ . \tag{13}$$

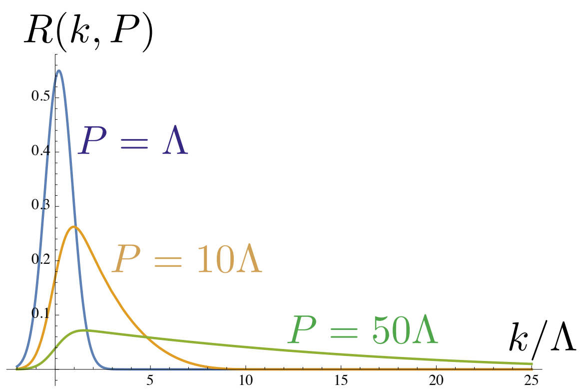

FIG. 1. Momentum distributions $R(k, P)$ in the factorized Gaussian model for $P/\Lambda = 1, 10, 50$.

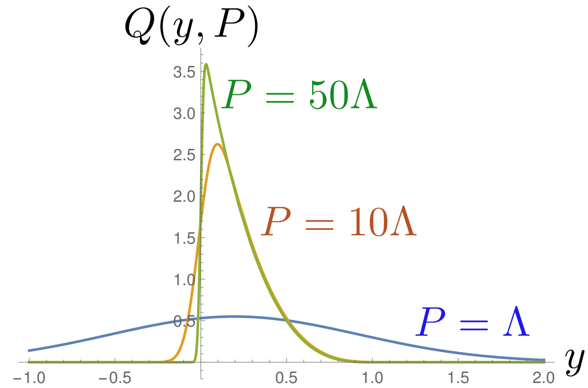

FIG. 2. Evolution of quasi-PDF $Q(y, P)$ in the factorized Gaussian model for $P/\Lambda = 1, 10, 50$.

This one-dimensional distribution may be directly obtained through a parameterization of the density

$$\rho(z_3^2) \equiv \mathcal{M}(0, z_3^2) = \int_{-\infty}^{\infty} dk_3\, r(k_3)\, e^{ik_3 z_3} \tag{14}$$

given by $\langle p|\phi(0)\phi(z_3)|p\rangle|_{\mathbf{p}=\mathbf{0}}$. Thus, $r(k_3)$ describes a primordial distribution of k_3 in a rest-frame hadron.

Factorized models. When the hadron is moving, the parton k_3 momentum, according to Eq. (11), comes from two sources. The first part, xP comes from the motion of the hadron as a whole, and the probability to get xP is governed by the dependence of the TMD $\mathcal{F}(x, \kappa^2)$ on its first argument, x. On the other hand, the probability to get the remaining part $k_3 - xP$ is governed by the dependence of the TMD on its second argument, κ^2, governing the primordial rest-frame momentum distribution.

Since these two sources of k_3 look like rather independent, it is natural to try a factorized model $\mathcal{R}(x, k_3 - xP) = f(x)r(k_3 - xP)$ (the x integral of $f(x)$ is normalized to 1). For original $\mathcal{M}(\nu, -z^2)$ function, this Ansatz corresponds to the factorization assumption $\mathcal{M}(\nu, -z^2) = \mathcal{M}(\nu, 0)\mathcal{M}(0, -z^2)$.

For illustration, we take a Gaussian form $\rho_G(z_3^2) = e^{-z_3^2\Lambda^2/4}$ for the rest-frame density. It corresponds to

$$r_G(k_3) = \frac{1}{\sqrt{\pi}\Lambda} e^{-k_3^2/\Lambda^2} \ . \tag{15}$$

For $f(x)$, we take a simple PDF resembling nucleon valence densities $f(x) = 4(1-x)^3\theta(0 \le x \le 1)$. As one can see from Fig. 1, the curve for $R(k, P)$ changes from a Gaussian shape for small P to a shape resembling stretched PDF for large P. Rescaling to $y = k/P$ variable gives the quasi-PDF $Q(y, P)$ shown in Fig. 2. For large P, it clearly tends to the $f(y)$ PDF form. In particular, using a momentum $P \sim 10\Lambda$ one gets a quasi-PDF that is rather close to the $P \to \infty$ limiting shape. Still, since $\Lambda \sim \langle k_\perp \rangle$, assuming the folklore value $\langle k_\perp \rangle \sim 300$ MeV one translates the $P \sim 10\Lambda$ estimate into $P \sim 3$ GeV, which is uncomfortably large. Thus, a natural question is how to improve the convergence.

Pseudo-PDFs. A formal reason for the complicated structure of a quasi-PDF $Q(y, P)$ is the fact that it is obtained by the ν-integral of $\mathcal{M}(\nu, z_3^2)e^{i\nu y}$ along a non-horizontal line $z_3 = \nu/P$ in the (ν, z_3) plane (see Eq. (7)). With increasing P, its slope decreases, the line becomes more horizontal, and quasi-PDFs convert into PDFs.

In contrast, pseudo-PDFs $\mathcal{P}(x, z_3^2)$, by definition, are given by integration of $\mathcal{M}(\nu, z_3^2)e^{i\nu x}$ over horizontal lines $z_3 =$ const. A very attractive feature of the pseudo-PDFs is that they have the $-1 \le x \le 1$ support for all z_3 values. For small z_3, they convert into PDFs.

More precisely, when z_3 is small, z_3 is analogous to the renormalization parameter μ of scale-dependent PDFs $f(x, \mu^2)$ of the standard OPE approach. In particular, for small z_3, the pseudo-PDF $\mathcal{P}(x, z_3^2)$ satisfies a leading-order evolution equation with respect to $1/z_3$ that coincides with the evolution equation for $f(x, \mu^2)$ with respect to μ. One can also write the evolution equation [12] for the Ioffe-time distribution $\mathcal{M}(\nu, z_3^2)$,

$$\frac{d}{d \ln z_3^2} \mathcal{M}(\nu, z_3^2) = \frac{\alpha_s}{2\pi} C_F \int_0^1 du\, B(u)\mathcal{M}(u\nu, z_3^2), \tag{16}$$

where the leading-order evolution kernel $B(u)$ for the non-singlet quark case is given [12] by

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+ \ , \tag{17}$$

with $[\ldots]_+$ denoting the standard "plus" prescription.

For the model used above (and $x \to -x$ symmetrized, as required for non-singlet PDFs), we have $\mathcal{M}(\nu, 0) = 12\left[\nu^2 - 4\sin^2(\nu/2)\right]/\nu^4$. The shapes of this function and of the convolution integral $B \otimes \mathcal{M}(\nu)$ are shown in Fig. 3. As one can see, $B \otimes \mathcal{M}(\nu)$ vanishes for $\nu = 0$, which reflects conservation of the vector current. Thus, the rest-frame density $\mathcal{M}(0, z_3^2)$ is not affected by perturbative evolution.

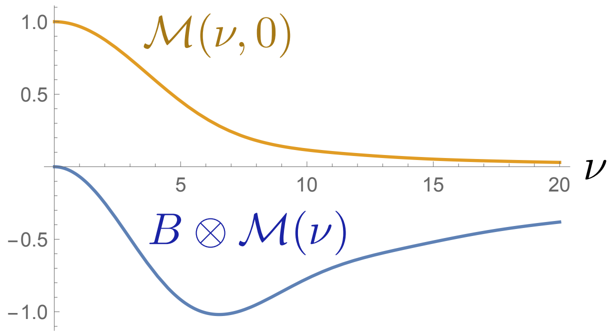

FIG. 3. Model Ioffe-time distribution $\mathcal{M}(\nu, 0)$ and the function $B \otimes \mathcal{M}$ governing its evolution.

Lattice implementation. A possible way to find the Ioffe-time distribution on the lattice (suggested by K. Orginos) is to calculate $\mathcal{M}(Pz_3, z_3^2)$ for several values of P, and then to fit the results by a function of ν and z_3^2.

Recalling our discussion of two apparently independent sources of obtaining k_3 for a moving hadron, one may hope that $\mathcal{M}(\nu, z_3^2)$ factorizes, i.e., $\mathcal{M}(\nu, z_3^2) = \mathcal{M}(\nu, 0)\mathcal{M}(0, z_3^2)$. Then the reduced function

$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)} \tag{18}$$

is equal to $\mathcal{M}(\nu, 0)$, and the goal of obtaining $\mathcal{M}(\nu, 0)$ is reached. What remains is just to take its Fourier transform to get the PDF $f(x)$.

A serious disadvantage of quasi-PDFs is that they have the x-convolution structure (8) even if the TMD [and $\mathcal{M}(\nu, z_3^2)$] factorizes. On the other hand, using pseudo-PDFs in the form of the ratio $\mathfrak{M}(\nu, z_3^2)$, one divides out the z_3^2-dependence of the primordial distribution without affecting the ν-dependence that dictates the shape of PDF. A further advantage of using the ratio (pointed out by K. Orginos) is the cancellation of the z_3^2-dependence generated on the lattice by the gauge link $\hat{E}(0, z_3; A)$.

In reality, $\mathfrak{M}(\nu, z_3^2)$ will have a residual z_3^2-dependence. It comes both from a possible violation of factorization for the soft part and from unavoidable perturbative evolution. For nonzero ν, the latter should be visible as a $\ln(1/z_3^2\Lambda^2)$ spike for small z_3^2.

Hence, a proposed strategy is to extrapolate $\mathfrak{M}(\nu, z_3^2)$ to $z_3^2 = 0$ from not too small values of z_3^2, say, from those above 1 fm^2. The resulting function $\mathcal{M}^{\rm soft}(\nu, 0)$ may be treated as the Ioffe-time distribution producing the PDF $f_0(x)$ "at low normalization point". The remaining $\ln(1/z_3^2\Lambda^2)$ spikes at small z_3 will generate its evolution. Of course, an actual technical implementation of this program should be discussed when the lattice data on $\mathcal{M}(\nu, z_3^2)$ will become available.

Summary. In this paper, we showed that quasi-PDFs may be seen as hybrids of PDFs and the primordial rest-frame momentum distributions of partons. In this context, the parton's k_3 momentum comes from the motion of the hadron as a whole and from the primordial rest-frame momentum distribution. The complicated convolution nature of quasi-PDFs necessitates using $p_3 \gtrsim$ 3 GeV to wipe out the primordial momentum distribution effects and get reasonably close to the PDF limit.

As an alternative approach, we propose to use pseudo-PDFs $\mathcal{P}(x, z_3^2)$ that generalize the light-front PDFs onto spacelike intervals. By a Fourier transform, they are related to the Ioffe-time distributions $\mathcal{M}(\nu, z_3^2)$ given by generic matrix elements written as functions of $\nu = p_3 z_3$ and z_3^2. The advantageous features of pseudo-PDFs are that they, first, have the same $-1 \leq x \leq 1$ support as PDFs, and second, their z_3^2-dependence for small z_3^2 is governed by a usual evolution equation.

Forming the ratio $\mathcal{M}(\nu, z_3^2)/\mathcal{M}(0, z_3^2)$ of Ioffe-time distributions one divides out the bulk of z_3^2 dependence generated by the primordial rest-frame distribution. Furthermore, taking this ratio one can exclude the z_3^2-dependent factor coming from the $\hat{E}(0, z_3; A)$ link creating difficulties (see, e.g., [14]) for lattice calculations of quasi-PDFs.

Testing the efficiency of using pseudo-PDFs for lattice extractions of PDFs is a challenge for future studies.

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[1] R. P. Feynman, *Photon-hadron interactions,* Reading 1972, 282p
[2] X. Ji, Phys. Rev. Lett. **110**, 262002 (2013).
[3] H. W. Lin, J. W. Chen, S. D. Cohen and X. Ji, Phys. Rev. D **91**, 054510 (2015).
[4] J. W. Chen, S. D. Cohen, X. Ji, H. W. Lin and J. H. Zhang, Nucl. Phys. B **911**, 246 (2016).
[5] C. Alexandrou, *et al.*, Phys. Rev. D **92**, 014502 (2015).
[6] J. H. Zhang, J. W. Chen, X. Ji, L. Jin and H. W. Lin, arXiv:1702.00008 [hep-lat].
[7] A. Radyushkin, Phys. Lett. B **767**, 314 (2017).
[8] A. V. Radyushkin, Phys. Rev. D **95**, no. 5, 056020 (2017).
[9] A. V. Radyushkin, Phys. Lett. B **735**, 417 (2014).
[10] A. V. Radyushkin, Phys. Rev. D **93**, no. 5, 056002 (2016).
[11] B. L. Ioffe, Phys. Lett. **30B**, 123 (1969).
[12] V. Braun, P. Gornicki and L. Mankiewicz, Phys. Rev. D **51**, 6036 (1995).
[13] A. V. Radyushkin, Phys. Lett. **131B**, 179 (1983).
[14] T. Ishikawa, Y. Q. Ma, J. W. Qiu and S. Yoshida, arXiv:1609.02018 [hep-lat].