Lattice QCD exploration of pseudo-PDFs

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We demonstrate a new method of extracting parton distributions from lattice calculations. The starting idea is to treat the generic equal-time matrix element $\mathcal{M}(Pz_3, z_3^2)$ as a function of the Ioffe time $\nu = Pz_3$ and the distance z_3. The next step is to divide $\mathcal{M}(Pz_3, z_3^2)$ by the rest-frame density $\mathcal{M}(0, z_3^2)$. Our lattice calculation shows a linear exponential z_3-dependence in the rest-frame function, expected from the $Z(z_3^2)$ factor generated by the gauge link. Still, we observe that the ratio $\mathcal{M}(Pz_3, z_3^2)/\mathcal{M}(0, z_3^2)$ has a Gaussian-type behavior with respect to z_3 for 6 values of P used in the calculation. This means that $Z(z_3^2)$ factor was canceled in the ratio. When plotted as a function of ν and z_3, the data are very close to z_3-independent functions. This phenomenon corresponds to factorization of the x- and $k_\perp$-dependence for the TMD $\mathcal{F}(x, k_\perp^2)$. For small $z_3 \le 4a$, the residual z_3-dependence is explained by perturbative evolution, with $\alpha_s/\pi = 0.1$.

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I. INTRODUCTION

Extraction of parton distribution functions (PDFs) $f(x)$ [1] on the lattice is a challenging problem attracting a lot of attention. The usual method to approach PDFs on the lattice is to calculate their moments. However, recently, X. Ji [2] suggested a method allowing a calculation of PDFs as functions of x.

Since the PDFs are related to matrix elements of bilocal operators on the light cone $z^2 = 0$, this was a stumbling block preventing a direct calculation of these functions in the lattice gauge theory formulated in Euclidean space.

To overcome this difficulty, X. Ji proposes to use purely space-like separations $z = (0, 0, 0, z_3)$. The functions in this case are quasi-PDFs $Q(y, p_3)$ describing the distribution of the p_3 hadron momentum component. The key point is that quasi-PDFs $Q(y, p_3)$ tend to usual PDFs $f(y)$ in the $p_3 \to \infty$ limit. The same method can be applied to distribution amplitudes (DAs). The results of quasi-PDF calculations on the lattice were reported in Refs. [3–5] and of the pion quasi-DA in Ref. [6].

Recent papers [7, 8] by one of the authors (AR) contain an investigation of the nonperturbative p_3-evolution of quasi-PDFs and quasi-DAs. This study is based on the formalism of virtuality distribution functions [9, 10]. The approach developed in Refs. [7, 8] has established a connection between the quasi-PDFs and the "straight-link" transverse momentum dependent distributions (TMDs) $\mathcal{F}(x, k_\perp^2)$. Starting from simple models for TMDs, models were built for the nonperturbative evolution of quasi-PDFs. The derived curves agree qualitatively with the patterns of p_3-evolution produced by lattice simulations.

The structure of quasi-PDFs was further studied in Ref. [11]. It was shown that, when a hadron is moving, the parton k_3 momentum may be treated as coming from two sources. The hadron's motion as a whole yields the xp_3 part, which is governed by the dependence of the TMD $\mathcal{F}(x, \kappa^2)$ on its first argument namely x. The residual part $k_3 - xp_3$ is controlled by the way that the TMD depends on its second argument, κ^2, which dictates the shape of the primordial rest-frame momentum distribution. Quasi-PDFs due to their convolution nature possess a rather involved pattern of their p_3-evolution, making mandatory relatively big values $p_3 \gtrsim 3$ GeV in order to safely approach the PDF limit.

To accelerate the convergence, a different approach for the PDF extraction from lattice calculations was proposed [11]. It is based on the concept of *pseudo-PDFs* $\mathcal{P}(x, z_3^2)$. They generalize the light-cone PDFs $f(x)$ onto spacelike intervals like $z = (0, 0, 0, z_3)$. The pseudo-PDFs are Fourier transforms of the *Ioffe-time* [12] *distributions* [13] $\mathcal{M}(\nu, z_3^2)$ which are generically given by matrix elements $\langle p|\phi(0)\phi(z)|p\rangle$ written as functions of $\nu = p_3 z_3$ and z_3^2. In contrast to quasi-PDFs, the pseudo-PDFs have the "canonical" $-1 \le x \le 1$ support for all values of z_3^2. In the limit $z_3 \to 0$ they tend to PDFs, showing, in this limit, a typical perturbative evolution with the scale $1/z_3$ being the parameter of evolution.

As discussed in [7, 8], the fast nonperturbative decrease with z_3^3 of the the pseudo-PDFs $\mathcal{P}(x, z_3^2)$ or the Ioffe-time distribution $\mathcal{M}(\nu, z_3^2)$, is responsible for delaying the approach of quasi-PDFs $Q(y, p_3)$ to the PDF $f(y)$. An important observation is that one can strongly reduce the z_3^2-dependence by simply dividing the Ioffe-time distribution $\mathcal{M}(\nu, z_3^2)$ by an appropriate factor $D(z_3^2)$ satisfying $D(0) = 1$ and having the z_3^2-dependence close (on average) to that of $\mathcal{M}(\nu, z_3^2)$. The absence of the ν-dependence in this factor and its $D(0) = 1$ normalization guarantees that the ratio $\mathcal{M}(\nu, z_3^2)/D(z_3^2)$ taken in the $z_3^2 \to 0$ limit will produce the same PDF as the original function $\mathcal{M}(\nu, z_3^2)$ taken in the same limit.

The choice for $D(z_3^2)$ advocated in Ref. [11], is to take it to be equal to the rest-frame function $\mathcal{M}(0, z_3^2)$. An additional advantage of this choice is that both $\mathcal{M}(\nu, z_3^2)$ and $\mathcal{M}(0, z_3^2)$ contain the same multiplicative factor $Z(z_3^2)$ generated by the renormalization of the gauge link. In the ratio, it should cancel out.

Our goal in the present work is an exploratory lattice calculation of the *u*-*d* proton PDF using the strategy outlined in Ref. [11]. To make this article self-contained, we reproduce in Sections II and III the main ideas of Ref. [11]. The description of the method used for the lattice extraction of the reduced Ioffe-time distribution is given in Section IV. The data analysis and interpretation is discussed in Section V. The summary of the paper is given in Section VI.

II. PARTON DISTRIBUTIONS

A. Generic matrix element and Lorentz invariance

The basic object for defining parton distributions is a matrix element of a bilocal operator that (skipping inessential details of its spin structure) may be written generically like $\langle p|\phi(0)\phi(z)|p\rangle$. Due to invariance under Lorentz transformations, it is given by a function of two scalars, (pz) (which will be denoted by $-\nu$) and z^2 (or $-z^2$, in order to have a positive value for spacelike z)

$$\langle p|\phi(0)\phi(z)|p\rangle = \mathcal{M}(-(pz), -z^2) = \mathcal{M}(\nu, -z^2) \, . \qquad (1)$$

One can demonstrate [7, 14] that, for all relevant Feynman diagrams, its Fourier transform $\mathcal{P}(x, -z^2)$ with respect to (pz) has $-1 \leq x \leq 1$ as support, i.e.,

$$\mathcal{M}(-(pz), -z^2) = \int_{-1}^{1} dx \, e^{-ix(pz)} \, \mathcal{P}(x, -z^2) \; . \qquad (2)$$

We want to point out, that Eq. (2) serves as a covariant definition of x. In this definition of x, one does not need to assume that $p^2 = 0$ or $z^2 = 0$.

B. Collinear distributions and pseudo-PDFs

Selecting some particular cases of z and p, one can obtain expressions for various parton distributions, all of them being expressed in terms of the same function $\mathcal{M}(-(pz), -z^2)$. More specifically, by choosing a lightlike z, e.g., having solely the light-front component z_-, we parametrize the matrix element by $f(x)$, the twist-2 parton distribution

$$\mathcal{M}(-p_+ z_-, 0) = \int_{-1}^{1} dx \, f(x) \, e^{-ixp_+ z_-} \quad . \qquad (3)$$

The function $f(x)$ has the standard probabilistic interpretation, in which x is the fraction of the target momentum component p_+ carried by the parton. One can rewrite this definition as

$$\mathcal{M}(\nu, 0) = \int_{-1}^{1} dx \, f(x) \, e^{ix\nu} \quad . \qquad (4)$$

The inverse relation is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \, e^{-ix\nu} \, \mathcal{M}(\nu, 0) = \mathcal{P}(x, 0) \; . \qquad (5)$$

Due to the fact that $f(x) = \mathcal{P}(x, 0)$, the function $\mathcal{P}(x, -z^2)$ provides a generalization of the concept of PDFs onto non-lightlike intervals z^2 (in principle, z^2 may be even timelike). Following [11] , we will be referring to it as the *pseudo-PDF*. The variable $(pz) = -\nu$ is called often the *Ioffe time* [12], and consequently $\mathcal{M}(\nu, -z^2)$ is the *Ioffe-time distribution* [13].

It is well known that in renormalizable theories (including QCD), the function $\mathcal{M}(\nu, -z^2)$ has logarithmic $\sim \ln(-z^2)$ singularities which generate the perturbative evolution of parton densities. In the approach based on the operator product expansion (OPE), the standard procedure is to remove these singularities with the help of some prescription. The most popular of them is the $\overline{\text{MS}}$ scheme based on dimensional regularization. Consequently the resulting PDFs have a dependence on the renormalization scale μ, and therefore one should write the PDFs as $f(x, \mu^2)$.

At small spacelike z^2 and at the leading logarithm level, the pseudo-PDFs are related to the $\overline{\text{MS}}$ distributions by a simple rescaling of their second arguments. In particular, when $z^2 = -z_3^2$, one has

$$\mathcal{P}(x, z_3^2) = f\left(x, (2e^{-\gamma_E}/z_3)^2\right) \; , \qquad (6)$$

where γ_E is the Euler's constant. The rescaling factor between μ and $1/z_3$ is very close to 1, since $2e^{-\gamma_E} = 1.12$.

C. Transverse momentum dependent distributions

Treating the target momentum p as longitudinal, $p = (E, \mathbf{0}_\perp, P)$, one can introduce transverse degrees of freedom. In particular, taking z that has z_- and $z_\perp = \{z_1, z_2\}$ components only, one defines the *TMD* $\mathcal{F}(x, k_\perp^2)$ as follows

$$\mathcal{P}(x, z_\perp^2) = \int d^2\mathbf{k}_\perp e^{i(\mathbf{k}_\perp \mathbf{z}_\perp)} \mathcal{F}(x, k_\perp^2) \; . \qquad (7)$$

In this context, the pseudo-PDFs $\mathcal{P}(x, z_\perp^2)$ actually coincide with the *impact parameter distributions*, a familiar object used in many TMD studies.

The logarithmic $\sim \ln z_\perp^2$ terms in $\mathcal{P}(x, z_\perp^2)$ come from the $\sim 1/k_\perp^2$ hard tail of $\mathcal{F}(x, k_\perp^2)$. Because of this observation, it makes sense to treat $\mathcal{P}(x, z_\perp^2)$ as a sum of a soft part $\mathcal{P}^{\text{soft}}(x, z_\perp^2)$, that is finite as $z_\perp^2$ tends to zero, and of a hard part which reflects the evolution. For the case of TMDs, the soft part decreases faster than $1/k_\perp^2$, for example, like a Gaussian $e^{-k_\perp^2/\Lambda^2}$. In the space of $z_\perp$, the distributions are then concentrated in the region $z_\perp \lesssim 1/\Lambda$.

III. QUASI-DISTRIBUTIONS

A. Definition and relation to TMDs

Since one cannot arrange light-like separations on the lattice, it was proposed [2] to consider equal-time space-like separations $z = (0, 0, 0, z_3)$ (or, for brevity, $z = z_3$). Then, in the $p = (E, 0_\perp, P)$ frame, one can introduce the quasi-PDF $Q(y, P)$ through a parametrization

$$\langle p|\phi(0)\phi(z_3)|p\rangle = \int_{-\infty}^{\infty} dy\, Q(y, P)\, e^{iyPz_3} \ . \tag{8}$$

According to this definition, the quasi-PDF $Q(y, P)$ describes the probability that the parton carries the fraction y of the parent hadron's third momentum component P. Returning to the idea of treating the matrix element as a function of the variables ν and $-z^2$ (which in this case are given by Pz_3 and z_3^2), we have

$$\mathcal{M}(\nu, z_3^2) = \int_{-\infty}^{\infty} dy\, Q(y, P)\, e^{iy\nu} \ . \tag{9}$$

Since $z_3^2 = \nu^2/P^2$, the inverse Fourier transformation may be written as

$$Q(y, P) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\ e^{-iy\nu}\, \mathcal{M}(\nu, \nu^2/P^2) \ . \tag{10}$$

It shows that $Q(y, P)$ tends to $f(y)$ in the $P \to \infty$ limit, since formally $\mathcal{M}(\nu, \nu^2/P^2) \to \mathcal{M}(\nu, 0)$ when $P \to \infty$.

Therefore, the deviation of the quasi-PDF $Q(y, P)$ from the PDF $f(y)$ is controlled by the dependence of $\mathcal{M}(\nu, z_3^2)$ on its second argument. By virtue of Eq. (7), this dependence is related to the dependence of the TMD $\mathcal{F}(x, \kappa^2)$ on κ^2 (its second arguement). Consequently, the difference between $Q(y, P)$ and $f(y)$ may be described by the transverse momentum dependence of the TMDs.

The explicit relation was derived in Ref. [7]

$$Q(y, P)/P = \int_{-\infty}^{\infty} dk_1 \int_{-1}^{1} dx\, \mathcal{F}(x, k_1^2 + (y - x)^2P^2) \ . \tag{11}$$

While being a mere result of Lorentz invariance, it tells us that the distribution of the parton k_3 momentum is influenced by the same physics that generates the $k_\perp$-dependence of the TMDs!

B. Quantum chromodynamics (QCD) case

The formulas that were derived previously can be directly applied to the non-singlet parton densities of QCD. Here, one is considering matrix elements having the following structure

$$\mathcal{M}^\alpha(z, p) \equiv \langle p|\bar\psi(0)\,\gamma^\alpha\, \hat{E}(0, z; A)\psi(z)|p\rangle \ , \tag{12}$$

where $\hat{E}(0, z; A)$ is the standard $0 \to z$ straight-line gauge link in the quark (fundamental) representation. By Lorentz invariance, these matrix elements can be decomposed into p^α and z^α parts

$$\mathcal{M}^\alpha(z, p) = 2p^\alpha\mathcal{M}_p(-(zp), -z^2) + z^\alpha\mathcal{M}_z(-(zp), -z^2) \ . \tag{13}$$

While the $\mathcal{M}_p(-(zp), -z^2)$ part gives the twist-2 distribution when $z^2 \to 0$, the $\mathcal{M}_z(-(zp), -z^2)$ is a purely higher-twist contamination. Thus, one may wish to make an effort to eliminate it from definitions of TMDs and quasi-PDFs.

Introducing TMDs, one takes $z = (z_-, z_\perp)$ and the $\alpha = +$ component of $\mathcal{M}^\alpha$. Hence, the z^α-part drops out, and one gets a TMD $\mathcal{F}(x, k_\perp^2)$ that is related to $\mathcal{M}_p(\nu, z_\perp^2)$ by the scalar formulas (2), (7). After that, $\mathcal{M}_p(\nu, z_\perp^2)$ is the only surviving part of $\mathcal{M}^\alpha(z, p)$, and in the remaining discussion we use the short hand notation of $\mathcal{M} \equiv \mathcal{M}_p$.

In case of quasi-distributions $Q(y, P)$, we can avoid the z^α contamination by considering the time component of $\mathcal{M}^\alpha(z = z_3, p)$ and defining

$$\mathcal{M}^0(z_3, p) = 2p^0 \int_{-1}^{1} dy\, Q(y, P)\ e^{iyPz_3} \ . \tag{14}$$

Then, the scalar formula (11) connects the quasi-PDF $Q(y, P)$ and the TMD $\mathcal{F}(x, k_\perp^2)$.

It should be emphasized that the operator defining $\mathcal{M}^\alpha(z, p)$ includes a $0 \to z$ *straight-line* link instead of a stapled link that is used in the definitions of TMDs which appear as part of the description of semi-inclusive DIS and Drell-Yan processes. It is well known that the stapled links reflect initial or final state interactions specific to these processes.

The "straight-link" TMDs, in this sense, describe the structure of a hadron when it is in its non-disturbed or "primordial" state. One may argue that such a TMD cannot be directly measured in a scattering experiment. However, it is a well-defined object in quantum field theory, and its study on the lattice could be *per se*, an exciting endeavor.

C. Factorized models

The structure of the quasi-PDFs may be illustrated on the example of the simplest models in which the nonperturbative (or soft) part of the TMDs $\mathcal{F}(x, k_\perp^2)$ is represented by a product

$$\mathcal{F}^{\text{soft}}(x, k_\perp^2) = f(x)K(k_\perp^2) \tag{15}$$

of the collinear parton distribution $f(x)$ and a $k_\perp^2$-dependent factor $K(k_\perp^2)$, usually modeled by a Gaussian. As we shall see, the quasi-PDFs have a rather complicated structure, even when they are built from these simple factorized models.

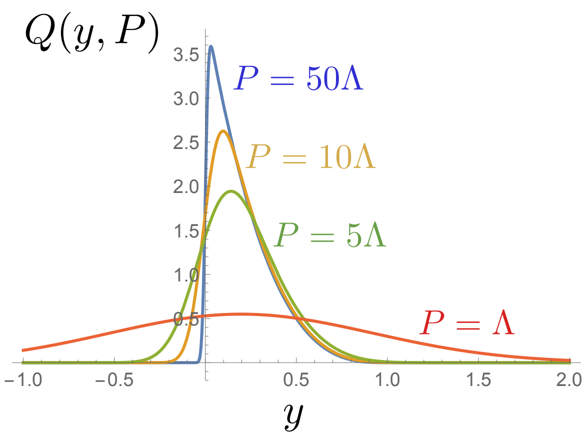

FIG. 1. Evolution of quasi-PDF $Q(y,P)$ in the factorized Gaussian model for $P/\Lambda = 1, 5, 10, 50$.

For the Ioffe-time distribution $\mathcal{M}(\nu, -z^2)$, this Ansatz corresponds to the factorization assumption

$$\mathcal{M}^{\rm soft}(\nu, z_3^2) = \mathcal{M}^{\rm soft}(\nu, 0)\mathcal{M}(0, z_3^2) \tag{16}$$

for its soft part. Still, even if the soft TMD factorizes, the soft part of the quasi-PDF has the convolution structure of Eq. (11). Taking, for example, a Gaussian form

$$K_G(k_\perp^2) = \frac{1}{\pi\Lambda^2} e^{-k_\perp^2/\Lambda^2} \ , \tag{17}$$

one gets the following model for the quasi-PDF

$$Q_G(y,P) = \frac{P}{\Lambda\sqrt{\pi}} \int_{-1}^{1} dx\, f(x)\, e^{-(x-y)^2P^2/\Lambda^2} \ . \tag{18}$$

Choosing for $f(x)$ a simple toy PDF resembling the nucleon valence densities $f(x) = 4(1-x)^3\theta(0 \le x \le 1)$, one gets the curves shown in Fig. 1. For large P, the quasi-PDF clearly tends to the $f(y)$ PDF form. In particular, using a momentum $P \sim 10\Lambda$ one gets a quasi-PDF that is rather close to the $P \to \infty$ limiting shape. Still, since $\Lambda \sim \langle k_\perp \rangle$, assuming the folklore value $\langle k_\perp \rangle \sim 300$ MeV one translates the $P \sim 10\Lambda$ estimate into $P \sim 3$ GeV, which is rather large. Thus, a natural question is how to improve the convergence.

D. Pseudo-PDFs

The involved structure of a quasi-PDF $Q(y,P)$ can be attributed to the formal fact that it is given by the Fourier ν-transform of the function $\mathcal{M}(\nu, \nu^2/P^2)$, in which ν appears both in the first and second argument of the Ioffe-time distribution. Due to this complication, to get close to the PDF limit, one should take P-values that are sufficiently large to neglect the ν-dependence coming from the second argument.

Another way [11] is to try to *eliminate* the z_3^2-dependence induced by $\mathcal{M}(\nu, z_3^2)$. The main idea is based on the observation that if one takes the ν-Fourier transform of the modified function $\mathcal{M}(\nu, z_3^2)/D(z_3^2)$, the $z_3 \to 0$ limit will give the same PDF as the original Ioffe-time distribution, provided that $D(z_3^2)$ is a function of z_3^2 only (but not of ν) and is equal to 1 for $z_3^2 = 0$.

Thus, the strategy is to find a function $D(z_3^2)$ whose z_3^2-dependence would compensate, as much as possible, the z_3^2-dependence of $\mathcal{M}(\nu, z_3^2)$. The next step is to fit the residual polynomial z_3^2-dependence by polynomials of z_3^2 (they may be different for different values of ν), and in this way extrapolate the data to $z_3^2 = 0$ limit. The Fourier transform of the resulting function would correspond to the same PDF as the z_3^2 limit of the original Ioffe distribution $\mathcal{M}(\nu, z_3^2)$.

In the most lucky situation, the ratio $\mathcal{M}(\nu, z_3^2)/D(z_3^2)$ would have no polynomial z_3^2-dependence (or just a very mild one). In particular, when $\mathcal{M}(\nu, z_3^2)$ factorizes, i.e., $\mathcal{M}(\nu, z_3^2) = \mathcal{M}(\nu, 0)\mathcal{M}(0, z_3^2)$, one should take $D(z_3^2) = \mathcal{M}(0, z_3^2)$. In this case, the reduced function

$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)} \tag{19}$$

is equal to $\mathcal{M}(\nu, 0)$, and the task of obtaining the $z_3 \to 0$ limit is accomplished.

While there is no "first principle" reason for such a factorization, one may expect that the functions $\mathcal{M}(\nu, z_3^2)$ for different ν have more or less similar dependence on z_3, basically reflecting the finite size of the nucleon.

As we mentioned already, the soft part of $\mathcal{M}(\nu, z_3^2)$ factorizes if the soft part of TMD $\mathcal{F}(x, k_\perp^2)$ factorizes. That this happens, is a standard assumption of the TMD practitioners (see, e.g., Ref. [15]). So, there are good chances that this part of the z_3^2-dependence of $\mathcal{M}(\nu, z_3^2)$ will be canceled or strongly reduced by the rest-frame function $\mathcal{M}(0, z_3^2)$.

On the lattice, there is another (and troublesome, see, e.g., Ref. [16]) source of z_3-dependence: the $Z(z_3^2)$ factor generated by the renormalization of the gauge link $\hat{E}(0, z_3; A)$. Fortunately, this problematic factor $Z(z_3^2)$ does not depend on ν and is the same for the numerator and denominator of the ratio $\mathfrak{M}(\nu, z_3^2)$. This provides another motivation for using $\mathcal{M}(0, z_3^2)$ as a factor $D(z_3^2)$.

Thus, the proposal is to perform a lattice study of the reduced Ioffe-time function $\mathfrak{M}(\nu, z_3^2)$. Even if it would have a residual polynomial z_3^2-dependence, it should be much easier to extrapolate this dependence to $z_3 = 0$, than the z_3-dependence of the original Ioffe-time distribution $\mathcal{M}(\nu, z_3^2)$.

Furthermore, if one observes that the ratio $\mathfrak{M}(\nu, z_3^2)$ does not have z_3-dependence, one should conclude that $\mathcal{M}(\nu, z_3^2)$ factorizes. In fact, such a factorization has been already observed several years ago in the pioneering study [17] of the transverse momentum distributions in lattice QCD.

Still, there is an unavoidable source of factorization breaking. When z_3 is small, $\mathcal{M}(\nu, z_3^2)$ has logarithmic $\ln z_3^2$ singularities generating the perturbative evolution of PDFs. As we discussed, $1/z_3$ is analogous then to

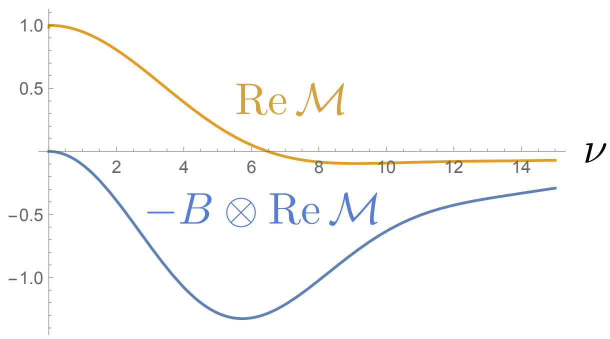

FIG. 2. Real part of model distribution $\mathcal{M}(\nu)$ and the function $-B \otimes \mathrm{Re}\,\mathcal{M}$ that governs its evolution (the minus sign here is for convenience of placing two curves on one figure).

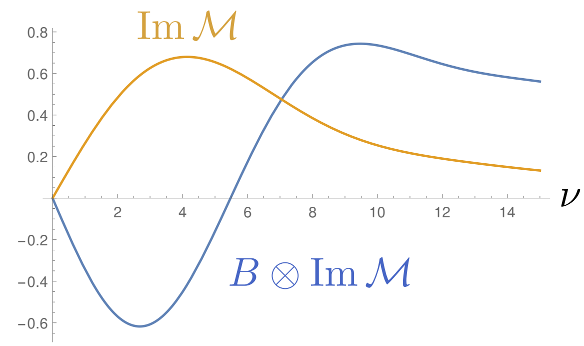

FIG. 3. Imaginary part of model Ioffe-time distribution $\mathcal{M}(\nu)$ and the function $B \otimes \mathrm{Im}\,\mathcal{M}$ that governs its evolution.

the renormalization parameter μ of the scale-dependent PDFs $f(x, \mu^2)$ within the standard OPE approach.

More specifically, for small values of z_3, the pseudo-PDF $\mathcal{P}(x, z_3^2)$ satisfies a leading-order evolution equation with respect to $1/z_3$ that is identical to the evolution equation for $f(x, \mu^2)$ with respect to μ. The evolution equation for the Ioffe-time distribution $\mathcal{M}(\nu, z_3^2)$ can also be written [13],

$$\frac{d}{d \ln z_3^2} \mathcal{M}(\nu, z_3^2) = -\frac{\alpha_s}{2\pi} C_F \int_0^1 du\, B(u) \mathcal{M}(u\nu, z_3^2), \tag{20}$$

where $C_F = 4/3$, and the leading-order evolution kernel $B(u)$ for the non-singlet quark case is given [13] by

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+ , \tag{21}$$

where $[\ldots]_+$ denotes the conventional "plus" prescription, i.e.

$$\int_0^1 du \left[\frac{1+u^2}{1-u}\right]_+ \mathcal{M}(u\nu) = \int_0^1 du\, \frac{1+u^2}{1-u} [\mathcal{M}(u\nu) - \mathcal{M}(\nu)]. \tag{22}$$

Note that being a Fourier transform,

$$\mathcal{M}(\nu) = \int_{-1}^1 dx\, f(x)\, e^{ix\nu} , \tag{23}$$

the Ioffe-time distribution has real and imaginary parts even if the function $f(x)$ is real (which is the case with parton distributions). In particular,

$$\mathrm{Re}\,\mathcal{M}(\nu) = \int_{-1}^1 dx\, f(x) \cos(x\nu) , \tag{24}$$

and

$$\mathrm{Im}\,\mathcal{M}(\nu) = \int_{-1}^1 dx\, f(x) \sin(x\nu) . \tag{25}$$

In Fig. 2, we show the function $\mathrm{Re}\,\mathcal{M}(\nu)$ for a model PDF

$$q(x) = \frac{315}{32} \sqrt{x}(1-x)^3 \theta(0 \le x \le 1) . \tag{26}$$

Its integral is normalized to 1, and it is nonzero for positive x only, which corresponds to the absence of antiquarks. As we shall see, this particular form appears in the description of actual lattice data. In Fig. 3, we show the function $\mathrm{Im}\,\mathcal{M}(\nu)$ for the same model PDF.

We also show in these figures the convolution integrals governing the evolution, namely $-B \otimes \mathrm{Re}\,\mathcal{M}(\nu)$ and $B \otimes \mathrm{Im}\,\mathcal{M}(\nu)$. The reader can notice that, $B \otimes \mathcal{M}(\nu)$ is zero for $\nu = 0$, the fact resulting from the vector current conservation. As a consequence, the perturbative evolution leaves the rest-frame density $\mathcal{M}(0, z_3^2)$ (which is always real) unaffected. In other words, the $\ln z_3^2$ terms are present only in the numerator $\mathcal{M}(\nu, z_3^2)$ of the $\mathfrak{M}(\nu, z_3^2)$ ratio, but not in its $\mathcal{M}(0, z_3^2)$ denominator.

Note also that the evolution of the real part always leads to a *decrease* of $\mathrm{Re}\,\mathcal{M}(\nu, z_3^2)$ when z_3^2 increases. For the imaginary part, the evolution pattern is more complicated. Namely, below $\nu \sim 5.5$, the function $\mathrm{Im}\,\mathcal{M}(\nu, z_3^2)$ *increases* when z_3^2 increases. Only above $\nu \sim 5.5$, the evolution leads to a decrease of $\mathrm{Im}\,\mathcal{M}(u\nu, z_3^2)$ with z_3^2, and the evolution pattern becomes similar to that of the real part.

IV. NUMERICAL INVESTIGATION

In order to check numerically the ideas discussed above we performed lattice QCD calculations in the quenched approximation at $\beta = 6.0$ on $32^3 \times 64$ lattices (lattice spacing $a = 0.093$ fm). We used the non-perturbatively

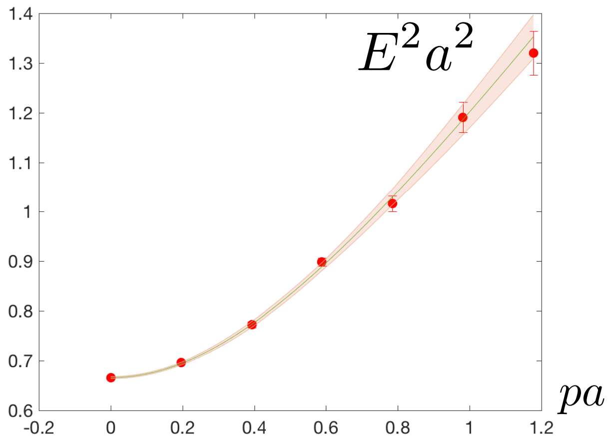

FIG. 4. Nucleon dispersion relation. Energies and momenta are in lattice units. The solid line is the continuum dispersion relation (not a fit) while the errorband is an indication of the statistical error of the lattice nucleon energies.

tuned clover fermion action with the clover coefficients computed by the Alpha collaboration [18].

We used a total of 500 configurations separated by 1000 updates each one consisting of four over-relaxation and one heatbath sweeps. On each configuration we computed correlation functions from 6 randomly selected point sources. The pion and nucleon masses in this setup were determined to be 601(1) MeV and 1411(4)MeV respectively. Conversion to physical energy units was performed used the Alpha collaboration scale setting for quenched QCD [19].

Our nucleon states were boosted up to a total momentum of 2.5 GeV (corresponding to the 6th lattice momentum). Inside this momentum range, the continuum dispersion relation for the nucleon was satisfied within the errors of the calculation, indicating small lattice artifacts of $\mathcal{O}(aP)$. In Fig. 4 we plot the nucleon energy as a function of momentum along with the continuum dispersion relation corresponding to our lattice nucleon zero momentum energy.

The computation of the matrix elements was performed using the methodology described in [20] with an operator insertion given by Eq. (12). Taking the time component of the current we can isolate $\mathcal{M}_p(-z\cdot p, -z^2)$ which as discussed above is directly related to PDFs.

Following [20] we need to compute two types of correlation functions. The first is a regular nucleon two point function given by

$$C_P(t) = \langle \mathcal{N}_P(t)\overline{\mathcal{N}}_P(0)\rangle \,, \tag{27}$$

where $\mathcal{N}_P(t)$ is a helicity averaged, non-relativistic nucleon interpolating field with momentum p. The quark fields in $\mathcal{N}_p(t)$ are smeared with a gauge invariant Gaussian smearing. This choice of an interpolation field is known to couple well to the nucleon ground state (see discussion in [20]). The quark smearing width was optimized to give good overlap with the nucleon ground state

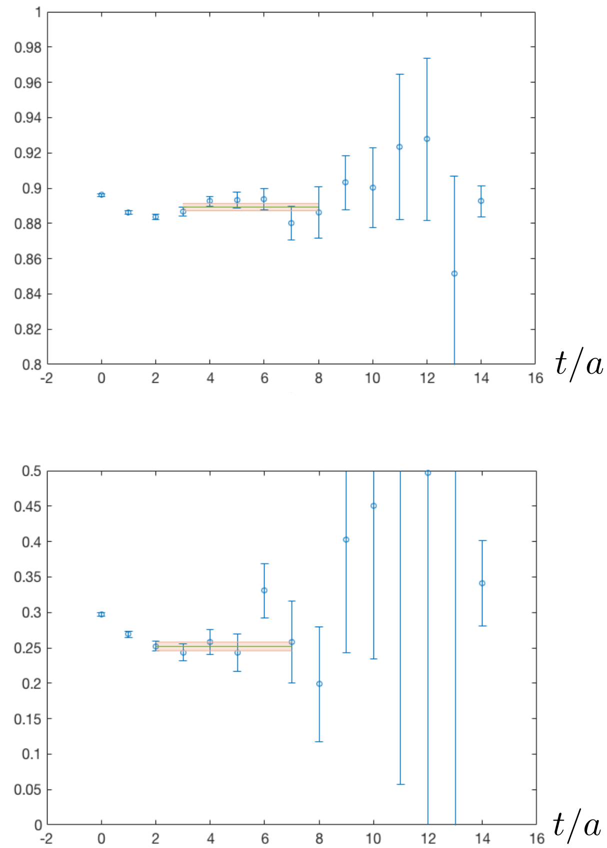

FIG. 5. Typical fits used to extract the reduced matrix element. The upper panel corresponds to $p = 2\pi/L\cdot 2$ and $z = 4$ and the lower panel to $p = 2\pi/L \cdot 3$ and $z = 8$, where momentum and position are in lattice units.

within the range of momenta in our calculation. The second correlator is given by

$$C_P^{\mathcal{O}^0(z)}(t) = \langle \mathcal{N}_P(t)\mathcal{O}^0(z)\overline{\mathcal{N}}_P(0)\rangle \,, \tag{28}$$

where

$$\mathcal{O}^0(z) = \overline{\psi}(0)\gamma^0\tau_3\hat{E}(0,z;A)\psi(z)\,, \tag{29}$$

with τ_3 being the flavor Pauli matrix. The proton momentum and the displacement of the quark fields were both taken along the $\hat{z}$ axis ($\vec{z} = z_3\hat{z}$ and $\vec{p} = P\hat{z}$). We define the effective matrix element as

$$\mathcal{M}_{\rm eff}(z_3P, z_3^2; t) = \frac{C_P^{\mathcal{O}^0(z)}(t+1)}{C_P(t+1)} - \frac{C_P^{\mathcal{O}^0(z)}(t)}{C_P(t)} \;. \tag{30}$$

As it was shown in [20], our matrix element can then be extracted at the large Euclidean time separation as

$$\mathcal{M}(z_3P, z_3^2) = \lim_{t\to\infty} \mathcal{M}_{\rm eff}(z_3P, z_3^2; t) \;. \tag{31}$$

This method of extracting the matrix element, contrary to the traditional sequential source approach, allows for the computation of the matrix element using all source-sink separations for the nucleon creation and annihilation operators.

The resulting effective matrix element has contaminations from excited states that scale as $e^{-t\Delta E}$, where t is the Euclidean time separation of the nucleon creation and annihilation operators, and ΔE is the mass gap to the first excited state of the nucleon. Furthermore, it allows for the computation of all nucleon matrix elements that correspond to different nucleon momentum spin polarization and flavor structure without additional computational cost.

As a result, the total computational cost of this approach is less than the equivalent cost of performing the calculations with the sequential source method, especially because in our approach we put emphasis on having as many nucleon momentum states as possible. This approach has recently been successfully used for both single and multi-nucleon matrix element calculations [21–23].

In order to normalize our lattice matrix elements we note that, for $z_3 = 0$, the matrix element $\mathcal{M}(z_3P, z_3^2)$ corresponds to a local vector (iso-vector) current, and therefore should be equal to 1. However, on the lattice this is not the case due to lattice artifacts. Therefore we introduce a renormalization constant

$$Z_P = \frac{1}{\mathcal{M}(z_3P, z_3^2)|_{z_3=0}} \,. \tag{32}$$

The factor Z_P has to be independent from P. However, again due to lattice artifacts or potential fitting systematics, this is not the case. For this reason, we renormalize the matrix element for each momentum with its own Z_P factor taking this way advantage of maximal statistical correlations to reduce statistical errors, as well as the cancellation of lattice artifacts in the ratio. Therefore, our matrix element is extracted using the ratio

$$\mathcal{M}(z_3P, z_3^2) = \lim_{t\to\infty} \frac{\mathcal{M}_{\rm eff}(z_3P, z_3^2; t)}{\mathcal{M}_{\rm eff}(z_3P, z_3^2; t)|_{z_3=0}} \,. \tag{33}$$

In order to determine the reduced matrix element $\mathfrak{M}(\nu, z_3^2)$ we introduce the double ratio

$$\mathfrak{M}(\nu, z_3^2) = \lim_{t\to\infty} \frac{\mathcal{M}_{\rm eff}(z_3P, z_3^2; t)}{\mathcal{M}_{\rm eff}(z_3P, z_3^2; t)|_{z_3=0}} \times \frac{\mathcal{M}_{\rm eff}(z_3P, z_3^2; t)\big|_{P=0,z_3=0}}{\mathcal{M}_{\rm eff}(z_3P, z_3^2; t)\big|_{P=0}} \,. \tag{34}$$

In practice, the infinite t limit is obtained with a fit to a constant for a suitable choice of a fitting range. In all cases we studied, the average χ^2 per degree of freedom was $\mathcal{O}(1)$. Typical fits used to extract the reduced matrix element are presented in Fig. 5. All fits are performed with the full covariance matrix and the error bars are determined with the jackknife method.

In this calculation we used momenta up to $6 \cdot 2\pi/L$ along the z-axis. This corresponds to a physical momentum of about 2.5 GeV.

V. DISCUSSION OF RESULTS

A. Rest-frame density and Z factor

An important object is the rest-frame density $\mathcal{M}(0, z_3^2)$. It is produced by data at $P = 0$. The results for its imaginary part are compatible with zero, as required. The real part, shown in Fig. 6, is a symmetric function of z_3, and has a clearly visible linear component in its fall-off with $|z_3|$ for small and middle values of $|z_3|$. In fact, a linear exponential factor $Z(z_3^2) \sim e^{-c|z_3|/a}$ is expected as a manifestation of the nonperturbative effects generated by the straight-line gauge link.

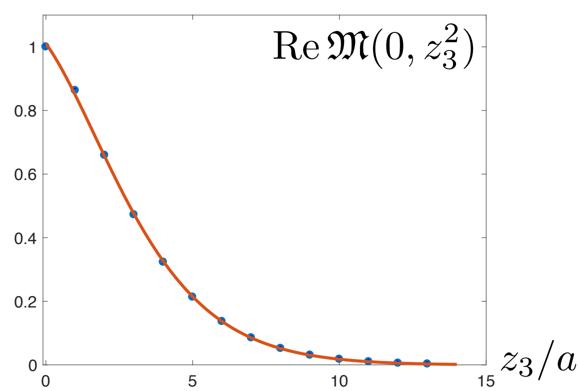

FIG. 6. Real part of the rest-frame density $\mathcal{M}(0, z_3^2)$

B. Reduced Ioffe-time distributions

In Fig. 7, we plot the results for the real part of the ratio $\mathcal{M}(Pz_3, z_3^2)/\mathcal{M}(0, z_3^2)$ as a function of z_3 taken at six fixed values of the momentum P. One can see that all the curves have a Gaussian-like shape. Thus, the $Z(z_3^2)$ link renormalization factor has been canceled in the ratio, as expected.

Furthermore, the curves look similar to each other, differing only by a decreasing width with P. In Fig. 8 , we plot the same data, but change the axis to $\nu = Pz_3$. As one can see, now the data practically fall on the same curve. For the imaginary part, the situation is similar.

This phenomenon corresponds to factorization of the x- and $k_\perp$-dependence for the soft TMD $\mathcal{F}(x, k_\perp^2)$, as discussed in previous sections.

C. Quark-antiquark decomposition

The real part of the Ioffe-time distribution is obtained from the cosine Fourier transform

$$\mathcal{M}_R(\nu) \equiv \int_0^1 dx\, \cos(\nu x)\, q_v(x) \tag{35}$$

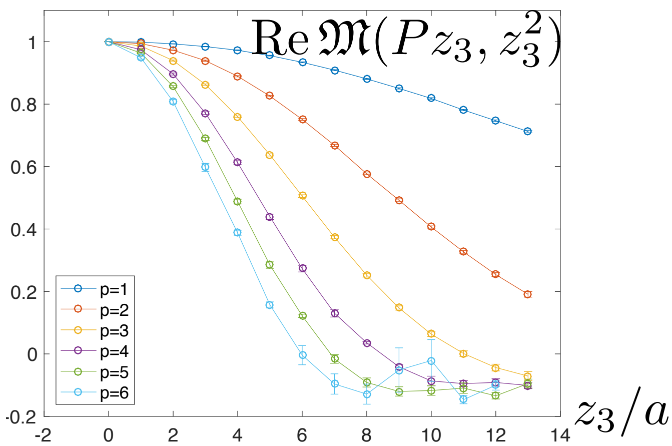

FIG. 7. Real part of the reduced distribution $\mathfrak{M}(Pz_3, z_3^2)$ plotted as a function of z_3. Here, $P = 2\pi p/L$.

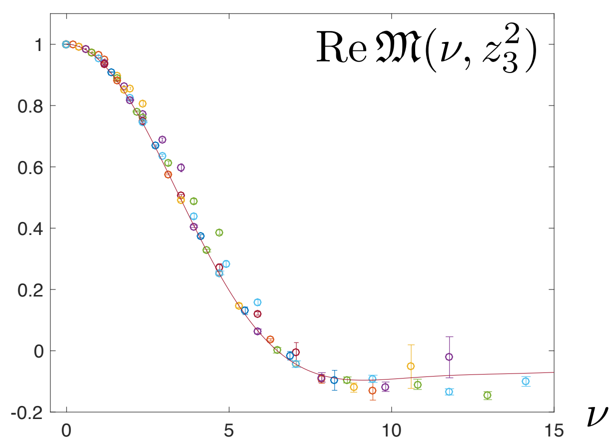

FIG. 8. Real part of $\mathfrak{M}(\nu, z_3^2)$ plotted as a function of $\nu = Pz_3$ and compared to the curve given by Eqs. (35), (36).

of the function $q_v(x)$ given by the difference $q_v(x) = q(x) - \bar{q}(x)$ of quark and antiquark distributions. In our case, q is $u - d$ and $\bar{q} = \bar{u} - \bar{d}$. The x-integral of $u - \bar{u}$ equals to the number of u-quarks in the proton, which is 2, while the x-integral of $d - \bar{d}$ equals 1. Thus, the x-integral of $q_v(x)$ should be equal to 1.

We found that our data for the real part are well described if one chooses the function

$$q_v(x) = \frac{315}{32}\sqrt{x}(1-x)^3 \,, \tag{36}$$

whose x-integral is normalized to 1. To get it, we formed cosine Fourier transforms $\mathcal{M}(\nu; a, b)$ of the normalized $x^a(1-x)^b$-type functions and found the parameters a, b by fitting our data.

The comparison of the data with the curve based on Eqs. (35), (36) is shown in Fig. 8. We also compare our $q_v(x)$ with three global fits for the difference

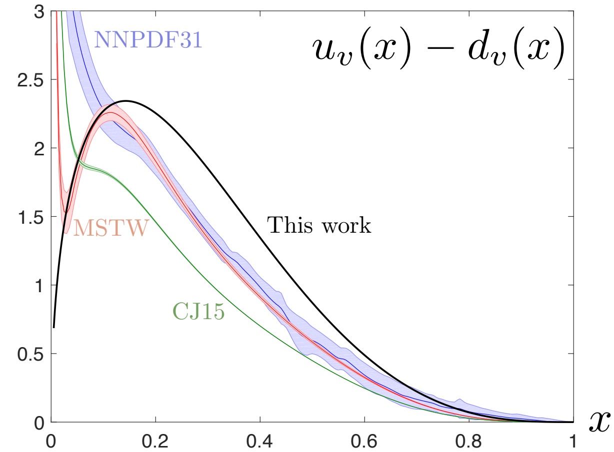

FIG. 9. Valence distribution $q_v(x)$ as given by Eq. (36) compared with the $Q^2 = 1$ GeV^2 NNLO global fits NNPDF31_nnlo_pch_as_0118_mc_164 [24] and MSTW2008nnlo68cl_nf4 [25]; and the NLO global fit CJ15nlo [26], all extracted using the LHAPD6 library [27]. The bands around the global fits indicate their experimental and systematic uncertainties.

$u_v(x) - d_v(x)$ of the valence distributions, see Fig. 9. One can see a reasonable agreement between our curve and NNPDF31 [24] NNLO fit down to $x = 0.1$ and with MSTW [25] NNLO fit down to $x = 0.05$. We also show the NLO fit CJ15 [26].

Since the areas under each curve are equal to 1, our curve compensates the strong deficiency in the $x < 0.1$ region by exceeding the NNLO curves at $x > 0.1$ values. In other words, if our curve would better describe data in the $x < 0.1$ region, it would necessarily be smaller in the $x > 0.1$ region.

The sine Fourier transform

$$\mathcal{M}_I(\nu) \equiv \int_0^1 dx\, \sin(\nu x)\, q_+(x) \tag{37}$$

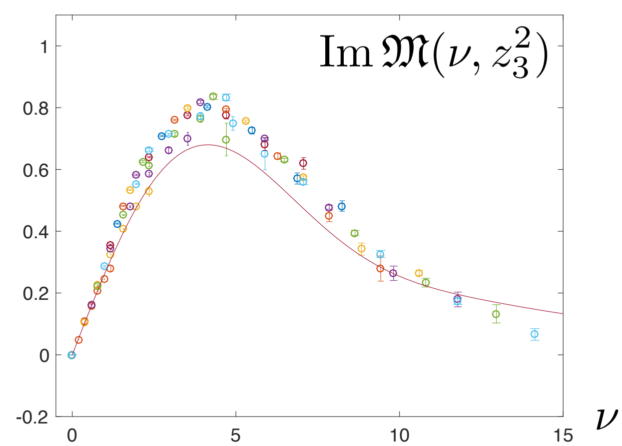

FIG. 10. Imaginary part of $\mathfrak{M}(\nu, z_3^2)$ compared to the curve based on $\bar{q}(x) = 0$.

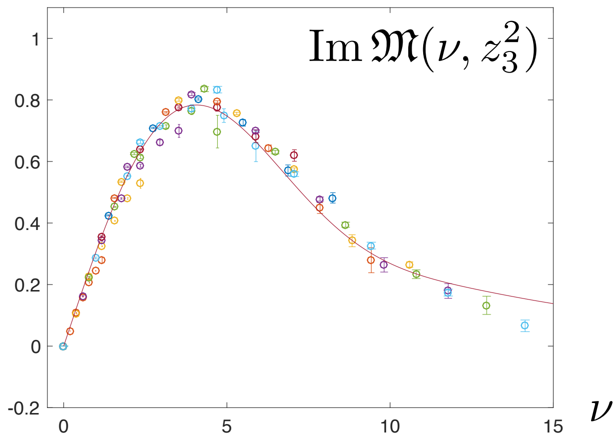

FIG. 11. Imaginary part of $\mathfrak{M}(\nu, z_3^2)$ compared to the curve based on $\bar{q}(x)$ given by Eq. (38).

is built from the function $q_+(x) = q(x)+\bar{q}(x)$, which may be also represented as $q_+(x) = q_v(x) + 2\bar{q}(x)$. If we neglect the antiquark contribution and use $q_+(x) = q_v(x)$, we get the curve shown in Fig. 10. The agreement with the data is strongly improved if we use a non-vanishing antiquark contribution, namely

$$\bar{q}(x) = \bar{u}(x) - \bar{d}(x) = 0.07\left[20\,x(1-x)^3\right] , \tag{38}$$

see Fig. 11. This result corresponds to

$$\int_0^1 dx\,[\bar{u}(x) - \bar{d}(x)] = 0.07 \ . \tag{39}$$

The combined distribution

$$\begin{aligned} q(x) &= u(x) - d(x) \\ &= [q_v(x) + \bar{q}(x)]\,\theta(x>0) - \bar{q}(-x)\,\theta(x<0) \end{aligned} \tag{40}$$

defined on the $-1 \le x \le 1$ interval is shown in Fig. 12.

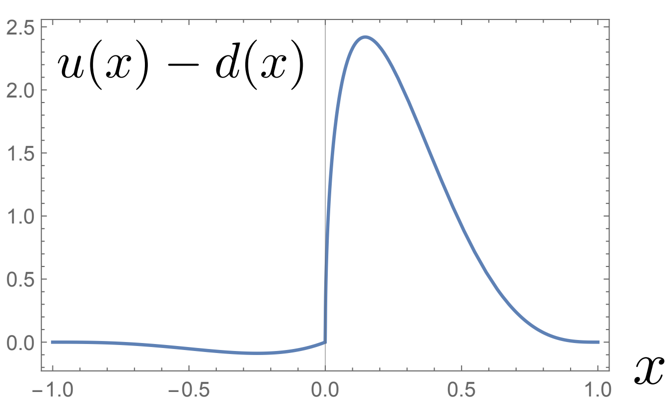

FIG. 12. Overall distribution $q(x)$ as defined by Eq. (40).

D. Evolution

While an overall agreement of the data with a z_3-independent curve looks satisfactory, one can easily notice a residual z_3-dependence in the data. It is especially visible when, for a particular ν, there are several data points corresponding to different values of z_3. It is interesting to check if this dependence corresponds to perturbative evolution.

To begin with, the evolution of the real part should lead to its *decrease* when z_3^2 increases. On the other hand, as pointed out at the end of section III, the function $\mathrm{Im}\,\mathcal{M}(\nu, z_3^2)$ *increases* when z_3^2 *increases* as long as $\nu \lesssim 5.5$. Our data follow these patterns.

As we discussed, the evolution corresponds to $\ln z_3^2$ singularities of the Ioffe-time distributions for small z_3^2. Thus, a natural idea is to check if the data corresponding to small z_3' and z_3 may be related by

$$\mathfrak{M}(\nu, {z'}_3^2){=}\mathfrak{M}(\nu, z_3^2) \ - \frac{2}{3}\frac{\alpha_s}{\pi}\ln({z_3'}^2/z_3^2) B\otimes\mathfrak{M}\,(\nu, z_3^2) \tag{41}$$

for some value of α_s. Here B is the evolution kernel (21). In our case,

$$B\otimes\mathfrak{M}\,(\nu) = \int_0^1 du\,\frac{1+u^2}{1-u}\left[\mathfrak{M}(u\nu) - \mathfrak{M}(\nu)\right]. \tag{42}$$

More specifically, we fix the point z_3' at the value $z_0 = 2a$ corresponding to the $\overline{\mathrm{MS}}$-scheme scale $\mu_0 = 1$ GeV and build the function

$$\widetilde{\mathfrak{M}}(\nu, z_0^2){\equiv}\mathfrak{M}(\nu, z_3^2) \ - \frac{2}{3}\frac{\alpha_s}{\pi}\ln(z_0^2/z_3^2) B\otimes\mathfrak{M}\,(\nu, z_3^2) \tag{43}$$

from the data points for $\mathfrak{M}\,(\nu, z_3^2)$ using various values for α_s.

Since the perturbative evolution is expected for small z_3, we include in this analysis the data with z_3 up to 4 lattice spacings, which corresponds to energy scales $\mu = 2, 1, 0.7$ and 0.5 GeV.

For the real part, these data points are shown in Fig. 13. As one can see, there is a visible scatter of the data points. Using $\alpha_s/\pi = 0.1$, we calculate the "evolved" data points corresponding to the function $\widetilde{\mathfrak{M}}(\nu, z_0^2)$. The results are shown in Fig. 14. The evolved data points are now very close to a universal curve.

In Fig. 15, we show the initial data points for the imaginary part. The evolved data points constructed using the same $\alpha_s/\pi = 0.1$ value are shown in Fig. 16. Again, they are close to a universal curve. This analysis indicates that the residual z_3^2-dependence of $\mathfrak{M}\,(\nu, z_3^2)$ at fixed ν is compatible with the expected logarithmic evolution at small z_3^2. Clearly this is an important feature of our calculation which needs to be further studied as it will play an essential role in reliable extraction of renormalized PDFs from this type of lattice calculations.

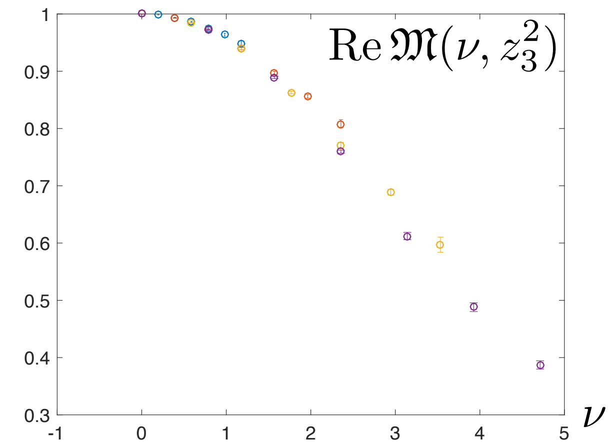

FIG. 13. Real part of $\mathfrak{M}(\nu, z_3^2)$ for $z_3/a = 1, 2, 3,$ and 4.

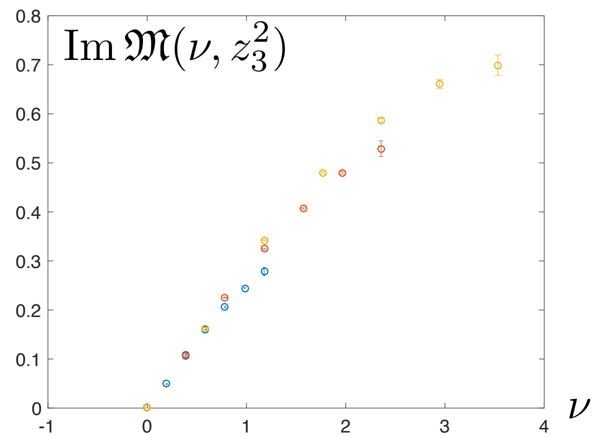

FIG. 15. Imaginary part of $\mathfrak{M}(\nu, z_3^2)$ for $z_3/a = 1, 2, 3,$ and 4.

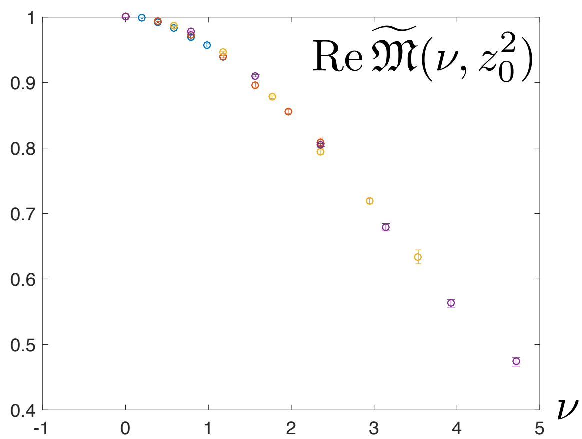

FIG. 14. Evolved data points for the real part.

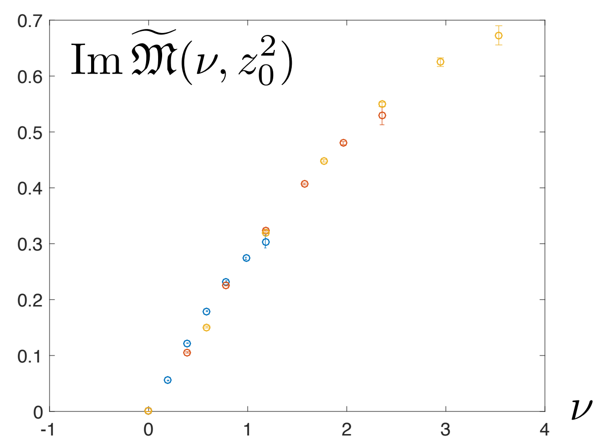

FIG. 16. Evolved data points for the imaginary part.

VI. SUMMARY

In this paper, we demonstrated a new method of extracting parton distributions from lattice calculations. It is based on the ideas, formulated in Ref. [11].

First, we treat the generic equal-time matrix element as a function $\mathcal{M}(\nu, z_3^2)$ of the Ioffe time $\nu = Pz_3$ and the distance z_3. The next idea is to form the ratio $\mathfrak{M}(\nu, z_3^2) \equiv \mathcal{M}(\nu, z_3^2)/\mathcal{M}(0, z_3^2)$ of the Ioffe-time distribution $\mathcal{M}(\nu, z_3^2)$ and the rest-frame density given by $\mathcal{M}(0, z_3^2)$.

Our lattice calculation clearly shows the presence of a linear component in the z_3-dependence of the rest-frame function, that may be attributed to the expected $Z(z_3^2) \sim e^{-c|z_3|/a}$ behavior generated by the gauge link. On the next step, we observe that the ratio $\mathcal{M}(Pz_3, z_3^2)/\mathcal{M}(0, z_3^2)$ has a Gaussian-type behavior with respect to z_3 for all 6 values of P that were used in the calculation. This means that $Z(z_3^2)$ factors entering into the numerator and denominator of the $\mathfrak{M}(Pz_3, z_3^2)$ ratio have been canceled, as expected.

Still, there is no *a priori* principle predicting that the remaining non-logarithmic z_3^2-dependence cancels between the numerator and the denominator of the ratio $\mathcal{M}(\nu, z_3^2)/\mathcal{M}(0, z_3^2)$. Such a z_3^2-dependence can be removed if needed with a systematic fitting procedure from which the Ioffe time PDF will be extracted in the $z_3^2 = 0$ limit.

However, we found that when plotted as a function of ν and z_3, the data both for the real and imaginary parts of $\mathfrak{M}(\nu, z_3^2)$ are very close to the respective universal functions. This observation indicates that the soft part of the z_3^2-dependence of $\mathcal{M}(\nu, z_3^2)$ has been canceled by the rest-frame density $\mathcal{M}(0, z_3^2)$. This phenomenon corresponds to factorization of the x- and $k_\perp$-dependence for the TMD $\mathcal{F}(x, k_\perp^2)$.

While this evidence in favor of the factorization property is an important result on its own, we want to stress that our approach *is not based* on the factorization. It is based on the use of the ratio $\mathcal{M}(\nu, z_3^2)/\mathcal{M}(0, z_3^2)$. Its residual soft z_3^2-dependence may be systematically analyzed and fitted, so that the z_3^2-limit may be taken in a controllable way.

Luckily, the data do not show a visible polynomial dependence on z_3^2 within our current statistical and systematic errors. In future work we intend to carefully study the residual polynomial z_3^2 effects and incorporate them in the extraction of PDFs using the lattice methodology introduced here.

In addition, we have checked that, for small $z_3 \leq 4a$, the residual z_3-dependence may be explained by perturbative evolution, with the α_s value corresponding to $\alpha_s/\pi = 0.1$. We have evolved these small-z_3 data points to the $z_3 = 2a$ scale, which corresponds to $\mu^2 = 1$ GeV^2. The evolved data better approximate universal curves both for real and imaginary parts of $\mathcal{M}$, supporting the argument that perturbative evolution is observed.

Thus, these $\nu \lesssim 4$ parts of the universal curves may be treated as corresponding to the $\mu^2 = 1$ GeV^2 scale. Other data points correspond to $z_3 > 4a$ values, and formally should be treated as corresponding to scales $\mu^2 \lesssim 0.25$ GeV^2. We do not think it is reasonable to use perturbative evolution from such scales to 1 GeV^2. Still, all these data points basically lie on the same universal curve that looks like a smooth continuation of the $\nu \leq 4$ curve. This indicates that evolution stops at such scales.

Hence, the $\nu \geq 4$ part of the universal curve giving an overall description of the data may be treated as corresponding to PDFs "at low normalization point", below which evolution stops. It may be considered as the starting condition for evolution, just like the $\mu^2 = 1$ GeV^2 curve is treated in the MSTW parametrization.

Our curve (36) for the valence $u_v(x) - d_v(x)$ distribution rather closely follows the NNPDF31 and, especially, MSTW NNLO global fits down to very small x values, showing the $(1-x)^3$ behavior for $x \to 1$ in accord with usual expectations.

Still, it strongly deviates from the global fits for $x < 0.1$ in the NNPDF31 case and for $x < 0.05$ in the MSTW case. Thus, our curve does not reproduce the experimentally established $\sim 1/\sqrt{x}$ Regge growth of the valence PDFs. It is quite possible that this outcome is caused by the short-distance cut-off imposed by discretization. To check if this is true, we need to repeat our calculation using a smaller lattice spacing.

The data also indicate a nonzero *positive* antiquark distribution $\bar{q}(x) = \bar{u}(x) - \bar{d}(x)$. It changes the x-integral of $q(x)$ by 7% and has $\sim x(1-x)^3$ behavior. Since we are using the quenched approximation, these antiquarks come from "connected diagrams". Hence, one should expect that the ratio $\bar{u}/\bar{d}$ *must* follow the flavor content of the proton, i.e. $\bar{u}/\bar{d} \sim 2$ and $\bar{u} > \bar{d}$. Our data agree with this expectation.

The present study has an exploratory nature, and its main goal was to develop techniques for lattice extraction of PDFs based on the ideas of Ref. [11]. Our results indicate that the basic method we put forward has a strong potential for obtaining reliable PDFs from lattice QCD. In future work we will refine our methods for incorporating evolution and controlling residual polynomial z_3^2 effects in the extraction of the Ioffe time distributions.

To achieve this, it is evident that smaller lattice spacings are required as well as a larger range of nucleon momenta. Furthermore, we need to study finite volume effects as well as to incorporate dynamical fermions with pion masses closer to the physical point. We plan to address all these issues in our future work.

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