Baryon masses and σ terms in SU(3) BChPT $\times$ $1/\mathrm{N_c}$

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Abstract

Baryon masses and nucleon σ terms are studied with the effective theory that combines the chiral and $1/N_c$ expansions for three flavors. In particular the connection between the deviation of the Gell-Mann-Okubo relation and the σ term associated with the scalar density $\bar{u}u + \bar{d}d - 2\bar{s}s$ is emphasized. The latter is at lowest order related to a mass combination whose low value has given rise to a σ term puzzle. It is shown that while the nucleon σ terms have a well behaved low energy expansion, that mass combination is affected by large higher order corrections non-analytic in quark masses. Adding to the analysis lattice QCD baryon masses, it is found that $\sigma_{\pi N} = 69(10)$ MeV and σ_s has natural magnitude within its relatively large uncertainty.

Keywords: Sigma terms, nucleon mass, baryon masses, Gell-Mann-Okubo mass formula

1. Introduction

Baryon mass dependencies on quark masses, quantified by the different σ-terms, are among the fundamental observables in baryon chiral dynamics. In particular, they give information on the baryon matrix elements of scalar quark densities, for which there is no alternative way for their determination. The definition of σ terms is through the Feynman-Hellmann theorem[1], which, for three flavors, through the physical baryon masses gives access to only two such terms, namely those associated with the SU(3) octet quark mass combinations $m_3 = m_u - m_d$ and $m_8 = \frac{1}{\sqrt{3}}(\hat{m} - m_s)$, where $\hat{m}$ is the average of the u and d quark masses. The σ terms associated with the singlet component $m_0 = \frac{1}{3}(2\hat{m} + m_s)$ require knowledge of baryon masses for unphysical quark masses, which is made possible through lattice QCD (LQCD) calculations. On the other hand, the pion-nucleon σ term $\sigma_{\pi N} \equiv \frac{\hat{m}}{2m_N}\langle N \mid \bar{u}u + \bar{d}d \mid N\rangle$ is accessible through its connection to pion-nucleon scattering via a low energy theorem [1, 2, 3]. Such a determination of $\sigma_{\pi N}$ had a long evolution through the availability of increasingly accurate data and the development of combined methods of dispersion theory and chiral perturbation theory [4, 5, 6, 7, 8, 9, 10, 11]. The values obtained for $\sigma_{\pi N}$ range from 45 MeV [4, 5, 6] to 64 MeV [7, 8, 9, 10, 11, 12], where the difference between the results of the different dispersive analyses resides mostly in the different values of the S-wave πN scattering lengths $a^{1/2,3/2}$ used in the

[1]The following notation will be used: $\sigma_i(B) = m_i \frac{\partial}{\partial m_i} m_B$, where m_i indicates a quark mass ($i = u, d, s$) or combination thereof ($i = 0, 3, 8$), and B is a given baryon. When B is not explicitly indicated it is assumed to be a nucleon.

subtractions, cf. [12]. In addition to the results from the analyses of πN scattering, LQCD calculations extrapolated to or at the physical point obtain different results, with values consistent with the recent πN results [13] and smaller, $\sigma_{\pi N} \approx 40$ MeV [14, 15, 16, 17]. The relatively large range of values obtained for $\sigma_{\pi N}$ keeps it as an active topic of study, and in part motivates the present work. An additional motivation is the relevance of scalar quark operator matrix elements, quantities that are relevant in studies of direct dark matter detection [18, 19, 20], and of lepton flavor violation through $\mu - e$ conversion in scattering with nuclei [21].

A puzzle that has been emphasized for a long time [22] is the relation between $\sigma_{\pi N}$ in the isospin symmetry limit and the nucleon's $\hat{\sigma} \equiv \sqrt{3}\frac{\hat{m}}{m_8}\sigma_8$, namely $\sigma_{\pi N} = \hat{\sigma} + 2\frac{\hat{m}}{m_s}\sigma_s$, which for a natural size value of σ_s should give $\sigma_{\pi N} \sim \hat{\sigma}$. The origin of the puzzle is the relation: $\sigma_8 = \frac{1}{3}(2m_N - m_\Sigma - m_\Xi)$ (or other combinations related via the Gell-Mann-Okubo (GMO) relation) valid at linear order in quark masses, which gives $\hat{\sigma} \sim 25$ MeV. If that relation is a reasonable approximation to the value of $\hat{\sigma}$, the implication is that, contrary to expectations, m_s must give a very large contribution to the nucleon mass even for the smaller values of $\sigma_{\pi N}$. The puzzle is particularly striking for the larger values that have been obtained for $\sigma_{\pi N}$, which would imply $\sigma_s \sim 500$ MeV. Indeed, this is clearly impossible if one considers that σ_s is OZI suppressed with respect to $\sigma_{\pi N}$.

This work analyzes the σ terms through the octet and decuplet baryon masses in the combined chiral and $1/N_c$ expansions BChPT $\times$ $1/\mathrm{N_c}$. The emphasis is in that the effective theory can give at NNLO (one chiral loop) a natural description of baryon masses, including LQCD results, along with the axial couplings which have been obtained in LQCD at different quark masses. In particular, the resolution of the σ term puzzle is explained by the fact that $\Delta\sigma_8 \equiv \sigma_8 - \frac{1}{3}(2m_N - m_\Sigma - m_\Xi)$ receives large non-analytic in quark mass corrections dominated by m_s. It will also be shown that σ_8 itself, and thus $\hat{\sigma}$, has a natural low energy expansion and therefore the origin of the puzzle resides in the large non-analytic correction to the mass combination $\frac{1}{3}(2m_N - m_\Sigma - m_\Xi)$. In fact, a big part of that large correction stems from the contribution of decuplet baryons in the loop, as it was found in Refs. [13, 23]. By analyzing LQCD baryon masses [24], it is found that as expected $\sigma_{\pi N} \sim \hat{\sigma}$, with the results $\sigma_{\pi N} = 69(8)(6)$ MeV, where the errors are respectively the statistical and theoretical (expected NNNLO corrections) ones, and $|\sigma_s| \lesssim 50$ MeV. The connection between the deviation from the GMO relation, $\Delta_{GMO} \equiv 3m_\Lambda + m_\Sigma - 2(m_N + m_\Xi)$, and $\Delta\sigma_8$, both calculable at NNLO and given solely in terms of non-analytic loop contributions, is of particular importance in the present work.

2. BChPT × 1/N$_\mathrm{c}$ analysis of masses and σ terms

The combined BChPT $\times\, 1/\mathrm{N_c}$ [25, 26, 27, 28, 29] implements the consistency of the effective theory with both the approximate chiral symmetry and the expansion in $1/N_c$ of QCD. The expansion requires a link between the chiral and the $1/N_c$ expansions: in practice the natural link is the ξ expansion where $\mathcal{O}(p) = \mathcal{O}(1/N_c) = \mathcal{O}(\xi)$, which is closely related to the so called small scale expansion [30, 31] even when that one did not strictly implement the constraints of the $1/N_c$ expansion. Consistency with $1/N_c$ power counting demands the imposition of a dynamical SU(6) spin-flavor symmetry, which is broken by sub-leading corrections in $1/N_c$ and requires the inclusion of the higher spin baryons (the decuplet in the case $N_c = 3$) and relates low energy constants (LECs) in the chiral Lagrangian. The details on the calculations of baryon masses concerning the present work can be found in [29].

The chiral Lagrangian to $\mathcal{O}\left(\xi^3\right)$, including electromagnetic corrections to the baryon masses is given by [29]:

$$\mathcal{L}_B = \mathbf{B}^\dagger \Bigg(iD_0 + \mathring{g}_A u^{ia} G^{ia} - \frac{C_{HF}}{N_c}\hat{S}^2 - \frac{1}{2\Lambda} c_2 \hat{\chi}_+ + \frac{c_3}{N_c\,\Lambda^3}\,\hat{\chi}_+^2 \\ + \frac{h_1}{N_c^3}\hat{S}^4 + \frac{h_2}{N_c^2\Lambda}\hat{\chi}_+\hat{S}^2 + \frac{h_3}{N_c\Lambda}\chi_+^0\hat{S}^2 + \frac{h_4}{N_c\,\Lambda}\,\chi_+^a\{S^i, G^{ia}\} + \alpha\hat{Q} + \beta\hat{Q}^2 \Bigg)\mathbf{B}. \tag{1}$$

where terms not directly relevant to the baryon masses have been omitted. $M_0 = \mathcal{O}(N_c)$ is the spin-flavor singlet piece of the baryon mass that provides the large mass expansion parameter for HBChPT. In addition to the well known chiral building blocks, $\mathbf{B}$ represents the baryon spin-flavor multiplet field, $\hat{S}^2$ is the square of the baryon spin operator, G^{ia} are the spin-flavor generators of SU(6), and $\hat{Q}$ is the electric charge operator. No baryon-spin dependent electromagnetic effects are included. The term proportional to C_{HF} gives the leading order mass splitting between the spin 1/2 and 3/2 baryons. $\mathring{g}_A$ is identified with $\frac{6}{5}g_A^N$ at the LO, whose physical value is $g_A^N = 1.2723(23)$. The term h_1 is only relevant if baryons with higher spin than 3/2 appear, which requires $N_c \geq 5$. The rest of the terms describe the quark mass effects. The combination $\hat{\chi}_+ = N_c\,\chi_+^0 + \tilde{\chi}_+$, where $\chi_+^0 = \frac{1}{3}\mathrm{Tr}\,\chi_+$ and $\tilde{\chi}_+$ is the traceless piece of χ_+, assures that the nucleon mass dependency on m_s is at most $\mathcal{O}\left(N_c^0\right)$ (OZI). Λ is an arbitrary scale, which is conveniently chosen to be m_ρ. The baryon mass formula then reads (neglecting isospin breaking for now)[29]:

$$m_B = M_0 + \frac{C_{HF}}{N_c}\hat{S}^2 - \frac{c_1}{\Lambda}2B_0(\sqrt{3}m_8 Y + N_c m_0) - \frac{c_2}{\Lambda}4B_0 m_0 - \frac{c_3}{N_c\Lambda^3}\left(4B_0(\sqrt{3}m_8 Y + N_c m_0)\right)^2 \\ - \frac{h_1}{N_c^2\Lambda}\hat{S}^4 - \frac{h_2}{N_c\Lambda}4B_0(\sqrt{3}m_8 Y + N_c m_0)\hat{S}^2 - \frac{h_3}{N_c\Lambda}4B_0 m_0\hat{S}^2 \\ - \frac{h_4}{N_c\Lambda}\frac{4B_0 m_8}{\sqrt{3}}\left(3\hat{I}^2 - \hat{S}^2 - \frac{1}{12}N_c(N_c+6) + \frac{1}{2}(N_c+3)Y - \frac{3}{4}Y^2\right) + \delta m_B^{\mathrm{loop}}, \tag{2}$$

where $\delta m_B^{\mathrm{loop}}$ can be obtained with some work using the results in [29], where the details on the mass renormalization and results for general N_c can be found.

Setting $c_3 = 0$ [2], the terms analytic in quark masses in Eqn. (2) lead to the exact GMO and Equal Spacing mass relations, which are unchanged at generic N_c. On the other hand at generic N_c the mass relation for σ_8 at tree level reads:

$$\Delta\sigma_8 = \sigma_8 - \frac{1}{9}\left(\frac{5N_c - 3}{2}m_N - (2N_c - 3)m_\Sigma - \frac{N_c + 3}{2}m_\Xi\right), \tag{3}$$

The dominant contributions to Δ_{GMO} and $\Delta\sigma_8$ are calculable non-analytic contributions. Δ_{GMO} is $\mathcal{O}\left(\xi^4\right)$ and in large N_c limit it is $\mathcal{O}(1/N_c)$. On the other hand, σ_8 is $\mathcal{O}(\xi)$ and it has a prefactor N_c, and $\Delta\sigma_8$ is $\mathcal{O}\left(\xi^2\right)$ also with a prefactor N_c. c_3 gives a contribution to the Δ_{GMO} which is $\mathcal{O}\left(\xi^5\right)$, and to $\Delta\sigma_8$ at $\mathcal{O}\left(\xi^4\right)$, both being beyond the accuracy of the

[2] The 27-plet SU(3) breaking produced by this term is $\mathcal{O}\left(\xi^5\right)$, and thus for the current purposes it can be neglected

present work. Δ_{GMO}[3] and $\Delta\sigma_8$ are thus determined by the meson masses and by the LECs $\mathring{g}_A/F_\pi$, and C_{HF}. Δ_{GMO} depends rather smoothly on C_{HF}, and drives to a large extent the determination of $\mathring{g}_A/F_\pi$. One finds the interesting fact that the ratio $\Delta\sigma_8/\Delta_{\text{GMO}}$, which is independent of $\mathring{g}_A/F_\pi$, is also almost entirely independent of the value of C_{HF} in a very wide range around its actual value. For $N_c = 3$, $\sigma_8/\Delta_{\text{GMO}} \sim -13.5$, which translates into $\Delta\hat{\sigma}/\Delta_{\text{GMO}} \sim 1.68$.

The analysis of the physical octet and decuplet baryon masses suffice to make the main point of this work. In this case, the LECs c_2, c_3 and h_1 are set to vanish, because at the order of the calculation they are redundant (actually h_1 is altogether irrelevant unless $N_c \geq 5$). A fit is carried out including strong and electromagnetic isospin breaking. This requires using the meson masses with isospin breaking, which include $\eta - \pi^0$ mixing (required to have a consistent renormalization of the baryon masses) and the electromagnetic mass shifts where Dashen's theorem is used, which should be sufficient for the current application. The electromagnetic addition to Δ_{GMO} is equal to $-\frac{4}{3}\beta$, while the strong isospin breaking has negligible effect, and the electromagnetic contribution to the p-n mass difference is equal to $\alpha + \beta$. The result of the fit to physical masses is shown in Table (1), Fit 1.

Fit	$\frac{\mathring{g}_A}{F_\pi}$ MeV^{-1}	$\frac{M_0}{N_c}$ MeV	C_{HF} MeV	c_1	c_2	h_2	h_3	h_4	α MeV	β MeV
1	0.0126(2)	364(1)	166(23)	−1.48(4)	0	0	0.67(9)	0.56(2)	−1.63(24)	2.16(22)
2	0.0126(3)	213(1)	179(20)	−1.49(4)	−1.02(5)	−0.018(20)	0.69(7)	0.56(2)	−1.62(24)	2.14(22)
3	0.0126*	262(30)	147(52)	−1.55(3)	−0.67(8)	0	0.64(3)	0.63(3)	−1.63*	2.14*

Fit	$\Delta^{\text{phys}}_{GMO}$ MeV	σ_8 MeV	$\Delta\sigma_8$ MeV	$\hat{\sigma}$ MeV	$\sigma_{\pi N}$ MeV	σ_s MeV	σ_3 MeV	$\sigma_{u+d}(p-n)$ MeV
1	25.6(1.1)	−583(24)	−382(13)	70(3)(6)	–	–	−1.0(3)	−1.6(6)
2	25.5(1.5)	−582(55)	−381(20)	70(7)(6)	69(8)(6)	−3(32)	−1.0(4)	−1.6(8)
3	25.8*	−615(80)	−384(2)	74(1)(6)	65(15)(6)	−121(15)	–	–

Table 1: Results from fits to baryon masses. Fit 1 uses only the physical octet and decuplet masses, Fit 2 uses the physical and the LQCD masses from Ref. [24] with $M_\pi \lesssim 300$ MeV, and Fit 3 uses only those LQCD masses and imposes the value of $\Delta^{\text{phys}}_{GMO}$ determined by the physical masses. The renormalization scale μ and the scale Λ are taken to be equal to m_ρ. * indicates an input. An estimated theoretical error of 6 MeV is indicated for $\hat{\sigma}$ and $\sigma_{\pi N}$.

The information given by LQCD, where the baryon masses have been obtained with M_K approximately constant and varying $m_u = m_d$ in a range where 213 MeV $< M_\pi <$ 430 MeV [24], is very useful for testing the effective theory, and necessary for calculating $\sigma_{\pi N}$. Two different fits that include LQCD baryon masses were performed, shown in Table (1). One fit combines the physical and LQCD masses, up to $M_\pi \sim 300$ MeV, and the other uses only LQCD

[3] Δ_{GMO} corresponds to having removed the EM corrections, otherwise it is denoted by $\Delta^{\text{phys}}_{\text{GMO}}$

and the physical value of Δ_{GMO}, which is important for controlling the value of $\mathring{g}_A/F_\pi$. In these fits the LEC c_2 which gives the baryon mass dependencies on the singlet quark mass component m_0 becomes significant, and its presence is responsible for the significant change in M_0 compared to the physical fit. M_0 is very precisely determined by the physical masses; Fit 3 shows that it is much less precise if only LQCD masses are used. The constant β can be estimated by the relation $2\beta = m_p - m_n - (m_{\Xi^0} - m_{\Xi^-})$, valid to LO in quark masses, which gives $\beta = 2.78 \pm 0.1$ MeV. The fit indicates that higher order terms in quark masses affect the extraction of β. The theoretical error for $\hat{\sigma}$ and $\sigma_{\pi N}$ accounting for higher order corrections was estimated by explicitly expanding in ξ and identifying the size of the contributions; the magnitude of the theoretical error was then estimated to be $\sim 1/3$ the size of the last term in the expansion.

The observations derived from the effective theory and from the fits are the following:

i) The value of $\mathring{g}_A/F_\pi$ is to a large extent fixed by Δ_{GMO}, and it corresponds to a value of g_A^N at LO which is roughly a factor 0.75 of the physical one; this agrees with what is observed in the analysis of the axial vector couplings [29] provided by LQCD calculations at different values of quark masses [32].

ii) The octet baryons contribute 43% of Δ_{GMO}, and 33% of $\Delta\sigma_8$, which shows the importance of the decuplet contributions.

iv) The first fit determines σ_8. Using the natural renormalization scale $\mu = m_\rho$, the different contributions to σ_8 are primarily given by the terms c_1 (~ -870 MeV), h_4 (~ 110 MeV) and the loop contributions (~ 190 MeV), where the latter two are the NLO contributions. This seems to be a well behaved expansion. On the other hand the mass combination on the RHS of Eqn. (3) has the corresponding pattern -870 MeV, 110 MeV and 570 MeV, the latter loop contribution given by the addition of $\Delta\sigma_8 \sim 380$ MeV. The NLO terms in the mass combination are very large and tend to cancel the LO one.

v) The correction $\Delta\sigma_8$ becomes quite large for $M_K > 350$ MeV, being about 70% of σ_8 for the physical M_K. As mentioned earlier, $\Delta\sigma_8$ and Δ_{GMO} are determined only in terms of $\mathring{g}_A/F_\pi$, C_{HF} and the meson masses. The ratio $\Delta\sigma_8/\Delta_{GMO}$ does not depend on $\mathring{g}_A/F_\pi$, and has virtually no dependence on C_{HF}. The ratio is also modestly dependent on M_K, going from ~ -11 to ~ -14 when M_K is increased from 200 to 600 MeV.

vi) The combined fit of physical and LQCD masses, Fit 2, is compatible with Fit 1; this implies that the chiral extrapolation of the LQCD results to the physical case is consistent.

vii) The fit to only LQCD masses and imposing the physical Δ_{GMO}, Fit 3, serves for a consistency check, which turns out to be quite reasonable. The LQCD masses do not describe correctly the hyperfine mass shifts between the octet and decuplet, which is shown in Fig. 1 where the Δ mass is systematically large, and this is the reason the resulting C_{HF} has some difference with the other fits. The extrapolation to the physical case turns out to be from 20 to 50 MeV larger than the physical octet masses, but less accurate for the decuplet ones where the Δ mass, which is the worst case, comes out to be about 100 MeV larger than the physical one.

viii) It is observed that $\hat{\sigma}$ and $\sigma_{\pi N}$ have both a small and approximately linear dependency on M_K in a very wide range. This in particular indicates that $\hat{m}\,\sigma_s/m_s$ must remain relatively small throughout.

ix) σ_s is poorly determined in the present study because the LQCD results are at approximately fixed m_s. Its range of values is however in line with the natural expectations. A LQCD calculation performed with smaller M_K than the physical one is necessary to obtain σ_s with better precision and also for understanding the effective theory in general.
x) The results obtained for $\sigma_{\pi N}$ are consistent with the larger values obtained from πN analyses [7, 8, 9, 10, 11]. Note however that a more reliable value would require some more accurate and extensive LQCD results. Fig. 1 depicts the result for $\sigma_{\pi N}$ from Fit 2 and its comparison with other results.
xi) The analysis also gives an estimate of the isospin-breaking σ terms σ_3 and $\sigma_{u+d}(p-n)$. In addition one can extract the separate contributions $\sigma_q(N)$, $q = u, d$, $N = p, n$. The results are the following: $\sigma_u(p) = 26.23$ MeV, $\sigma_d(p) = 42.42$ MeV, $\sigma_u(n) = 23.82$ MeV, $\sigma_d(n) = 46.48$ MeV, which checks with $\sigma_{\pi N} = \hat{m}(\sigma_u/m_u + \sigma_d/m_d)$. The relation $\sigma_u(p) = \sigma_d(n)$ in the isospin symmetry limit is of course satisfied, but the naive quark model relation in the isospin limit $\sigma_u(p) = 2\sigma_d(p)$ is significantly violated due to contributions by the SU(2) singlet component of the quark masses.
xii) Obviously, the discussion can be extended to the rest of the σ terms for the different baryons and their various relations [29].
xiii) One can compare with an analysis in ordinary HBChPT without the decuplet. In that case Δ_{GMO} requires $\mathring{g}_A/F_\pi$ to be significantly larger (corresponding to $g_A^N = 1.48$ at LO), which despite the lack of the decuplet contributions leads to values of the σ terms which are not very different but somewhat larger than the ones obtained here ($\hat{\sigma} \sim 83$ MeV, $\sigma_{\pi N} \sim 76$ MeV). The difference lies in the fact that in ordinary HBChPT the corrections to the axial currents couplings have large N_c power violating contributions, which compounded with the larger value of $\mathring{g}_A/F_\pi$ required by Δ_{GMO} lead to a failure in describing the axial couplings obtained in LQCD at different quark masses [32], in particular their observed small quark mass dependencies.
xiv) Although the approach followed in recent work [33] should be expected to give a result for $\sigma_{\pi N}$ similar to the one obtained here, it is actually much smaller. It is not clear to the authors whether this may be entirely due to the different set of LQCD data. However, since $\hat{\sigma}$ is accurately obtained with only the physical masses, the result of [33] would require a large negative σ_s, which seems to be unlikely within the present framework.

3. Summary

The σ terms of nucleons were calculated using SU(3) BChPT $\times$ $1/\mathrm{N_c}$. From the physical octet and decuplet baryon masses a value of $\hat{\sigma}$ is obtained which is much larger than the one predicted by a tree level baryon mass combination, in agreement with similar observations in calculations that included the decuplet baryons as explicit degrees of freedom. The "σ term puzzle" is understood as the result of large non-analytic contributions to that mass combination, while the higher order corrections to the σ terms have natural magnitude. The intermediate spin 3/2 baryons play an important role in enhancing $\hat{\sigma}$ and thus $\sigma_{\pi N}$. The analysis carried out here shows that there is compatibility in the description of

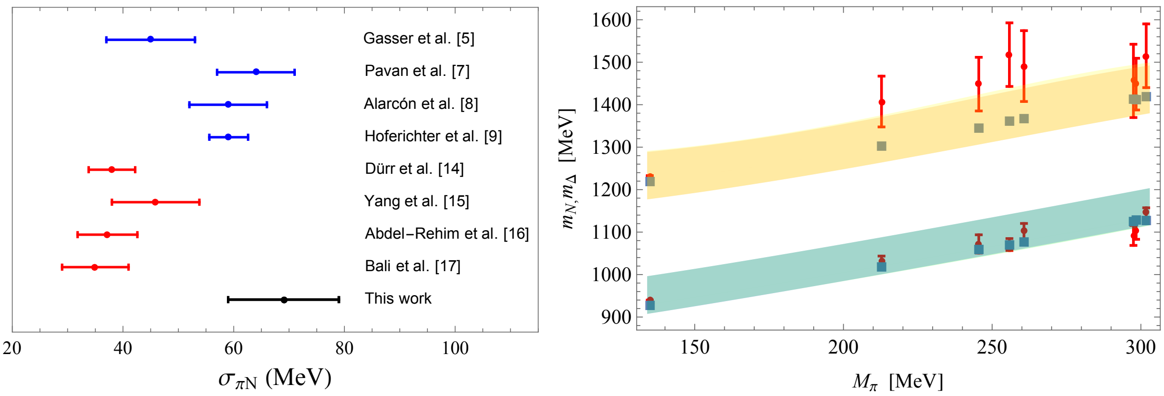

Figure 1: Left panel: summary of the determinations of $\sigma_{\pi N}$ from πN scattering (blue), from LQCD (red), and from this work showing the combined fit and theoretical error. Right panel: N and Δ masses from Fit 2 of Table (1): physical and LQCD masses from [32]. The squares are the results from the fit and the error bands correspond to 68% confidence interval.

Δ_{GMO} and the nucleon σ terms. The value of $\sigma_{\pi N} = 69 \pm 10$ MeV obtained here from including LQCD baryon masses agrees with the more recent results from πN analyses, where the increase in value with respect to previous analyses has been understood as a result of the values of the input scattering lengths, and strongly disfavor the values from recent LQCD evaluations. The tension between results, which includes LQCD, remains as an important problem to which the present approach can hopefully contribute with useful insights. The resolution of that tension will in turn provide a validation test of the approach.

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