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# How Variation in Analytic Choices Can Affect Normalization Parameters and Proton Radius Extractions From Electron Scattering Data

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In order to make use of prior knowledge, such as analytic behavior or a known value at a kinematic endpoint, regressions often make use of floating normalization parameters to allow the fit to shift the data to the known physical limit. As there is often no unique way to make use of this prior knowledge or apply these shifts, different analysis choices can lead to very different conclusions from the same set of data. In this work, we use the Mainz elastic data set with its 1422 cross section points and 31 normalization parameters to illustrate how a single difference in a subjective analysis decision can dramatically affect the results.

## I. INTRODUCTION

As pointed out by Silberzahn *et al.* [1], there is often little appreciation for how chosen analytic strategies can affect a reported result. In this work we will show how a single logical difference can impact the normalization of the Mainz cross section data [2].

## II. PROTON ELASTIC SCATTERING

There has been a renewed interest in low four momentum transfer, $Q^2$, proton elastic scattering data due to the unexpected Muonic lamb shift proton radius results of 0.84078(39) fm [3, 4] while the recommended values from CODATA-2014 was 0.8751(61) fm [5] For the electron scattering data, the proton charge radius, $r_p$, is extracted from the cross sections by determining the slope of the electric form factor, $G_E$, in the limit of four-moment transfer, $Q^2$, approaching zero:

$$r_p \equiv \left( -6 \left. \frac{dG_E(Q^2)}{dQ^2} \right|_{Q^2=0} \right)^{1/2} . \tag{1}$$

Unfortunately, the scattering data is always measured at finite $Q^2$, require extrapolation to extract the charge radius. Many authors have taken many different approaches to do this extraction with systematically different outcomes [6–14].

To illustrate what exactly is happening, and how a single analysis choice can strongly affect the extracted radius, we use the full set of Mainz data and fit it with an eleventh order polynomial. First we perform an unbounded fit, similar to the Mainz approach. Then we fit a second time adding a requirement that the terms alternate in sign, as one might expect from fitting standard dipole function with a high order polynomial.

## III. CROSS SECTION FORMULAS

The cross section is given by

$$\sigma = \sigma_{\text{Mott}} \times \left[ \frac{G_E^2\left(Q^2\right) + \tau G_M^2\left(Q^2\right)}{1+\tau} + 2\tau G_M^2\left(Q^2\right) \tan^2\frac{\theta}{2} \right] \tag{2}$$

where $\tau = \frac{Q^2}{2m_p(E_{Beam}-E'}$ with $Q^2 = 4E_{Beam}E' \sin^2\frac{\theta}{2}$ where $E_{Beam}$ is the energy of the electron beam, E' is the energy of the outgoing electron and $\theta$ is the scattering angle of the outgoing electron. Following the notation of Bernauer *etal.*, the form factors can be parameterized in terms of a polynomial function

$$G^{E,M}_{\text{polynomial,n}}(Q^2) = 1 + \sum_{i=1}^{n} a_i^{E,M} Q^{2\,i}. \tag{3}$$

## IV. NORMALIZATION PARAMETERS

As noted in the work of Bernauer *etal.*, the absolute knowledge of the cross sections is limited by determination of the absolute luminosity. So in order to bring the normalization uncertainty down, we use prior knowledge of the form factors in the limit of $Q^2$=0. This in fine, but it brings a model dependence to the analysis that isn't easily understood, as there are 31 normalization parameters. These parameters are taken in combinations to link sets of data together as shown in Table I with the final value of each cross section point defined by:

$$\sigma = \sigma_p \cdot \text{norm1} \cdot \text{norm2}. \tag{4}$$

Details of how these parameters connect to each of the 1422 cross section points can be found in the supplemental material of Bernauer *etal.* [2].

To show how a single analytic change can affect these normalization parameters, we first do an unbounded regression where the polynomial parameters are allowed to vary freely and then a second regression where we force the parameters to alternate sign.

TABLE I: Shown are the 34 different combinations of the 31 normalization parameters, $N_j$, found in the Mainz data which link the data set together. Also shown are the number of data points and $Q^2$ range of each data set.

| Energy | Spec. | norm1 | norm2 | Points | $Q^2$ Range [GeV$^2$] |
|---|---|---|---|---|---|
| 180 MeV | B | $N_1$ | $N_3$ | 106 | 0.0038 – 0.0129 |
| | B | $N_1$ | $N_4$ | 41 | 0.0101 – 0.0190 |
| | A | $N_3$ | - | 102 | 0.0112 – 0.0658 |
| | B | $N_1$ | $N_5$ | 19 | 0.0190 – 0.0295 |
| | C | $N_2$ | $N_4$ | 38 | 0.0421 – 0.0740 |
| | C | $N_2$ | $N_5$ | 17 | 0.0740 – 0.0834 |
| 315 MeV | B | $N_6$ | $N_9$ | 104 | 0.0111 – 0.0489 |
| | A | $N_7$ | $N_9$ | 38 | 0.0430 – 0.1391 |
| | A | $N_9$ | - | 40 | 0.0479 – 0.1441 |
| | C | $N_8$ | $N_9$ | 62 | 0.1128 – 0.2131 |
| 450 MeV | B | $N_{10}$ | $N_{13}$ | 77 | 0.0152 – 0.0572 |
| | B | $N_{10}$ | $N_{15}$ | 52 | 0.0572 – 0.1175 |
| | A | $N_{13}$ | - | 42 | 0.0586 – 0.2663 |
| | B | $N_{10}$ | $N_{14}$ | 17 | 0.0589 – 0.0851 |
| | A | $N_{11}$ | $N_{13}$ | 36 | 0.0670 – 0.2744 |
| | C | $N_{12}$ | $N_{15}$ | 50 | 0.2127 – 0.3767 |
| | A | $N_{14}$ | - | 2 | 0.2744 – 0.2744 |
| 585 MeV | B | $N_{16}$ | $N_{18}$ | 41 | 0.0255 – 0.0433 |
| | B | $N_{16}$ | $N_{19}$ | 47 | 0.0433 – 0.1110 |
| | A | $N_{18}$ | - | 27 | 0.0590 – 0.0964 |
| | B | $N_{16}$ | $N_{20}$ | 21 | 0.0920 – 0.1845 |
| | A | $N_{19}$ | - | 37 | 0.0964 – 0.4222 |
| | C | $N_{17}$ | $N_{20}$ | 20 | 0.3340 – 0.5665 |
| 720 MeV | B | $N_{21}$ | $N_{25}$ | 47 | 0.0711 – 0.1564 |
| | A | $N_{25}$ | - | 46 | 0.1835 – 0.6761 |
| | C | $N_{24}$ | $N_{26}$ | 28 | 0.6536 – 0.7603 |
| | A | $N_{23}$ | $N_{26}$ | 27 | 0.2011 – 0.2520 |
| | A | $N_{22}$ | $N_{26}$ | 37 | 0.4729 – 0.7474 |
| | A | $N_{21}$ | $N_{26}$ | 36 | 0.1294 – 0.2435 |
| 855 MeV | B | $N_{27}$ | $N_{31}$ | 35 | 0.3263 – 0.4378 |
| | C | $N_{28}$ | $N_{31}$ | 31 | 0.7300 – 0.9772 |
| | A | $N_{29}$ | $N_{30}$ | 32 | 0.3069 – 0.5011 |
| | A | $N_{29}$ | - | 13 | 0.5274 – 0.7656 |
| | B | $N_{27}$ | $N_{29}$ | 54 | 0.0868 – 0.3263 |

## V. REGRESSION

To find the parameter values, we do a weighted least squares minimization with a $\chi^2$ function defined as follows:

$$\chi^2 = \sum_{p=1}^{1422} \left( \frac{\sigma_{Model}(E_p, \theta_p) - \sigma_p \cdot \mathrm{norm1_p} \cdot \mathrm{norm2_p}}{\Delta\sigma_p \cdot \mathrm{norm1_p} \cdot \mathrm{norm2_p}} \right)^2 \tag{5}$$

where for each data point p there is a cross-section, $\sigma_p$, with an energy $E_p$,angle theta$_p$, and normalization parameters: norm1 and norm2 as shown in Table I. This function is clearly not linear in terms, nor is the uncertainty fixed, thus one should be careful when interpreting the results of the regression [15]. As always, one should make plots to check the fit quality [16] especially since this type of minimization can be very sensitive to outliers.

## VI. RESULTS

In Table II and Table III we show the results of fitting with both the bound and unbound polynomial. Interpreting the $a_1^E$ term as being related to the charge radius of the proton via Eq. 1, one can immediately see two very different radii: 0.882 fm and 0.847 fm. Thus the unbounded regression is close to the expected CODATA value while the function with alternating signs (bounded) is closer to the muonic result. Depending on what the researcher was expecting to see, it is clear that such freedom in analysis choices can make it difficult to avoid confirmation bias in a non-blinded analysis.

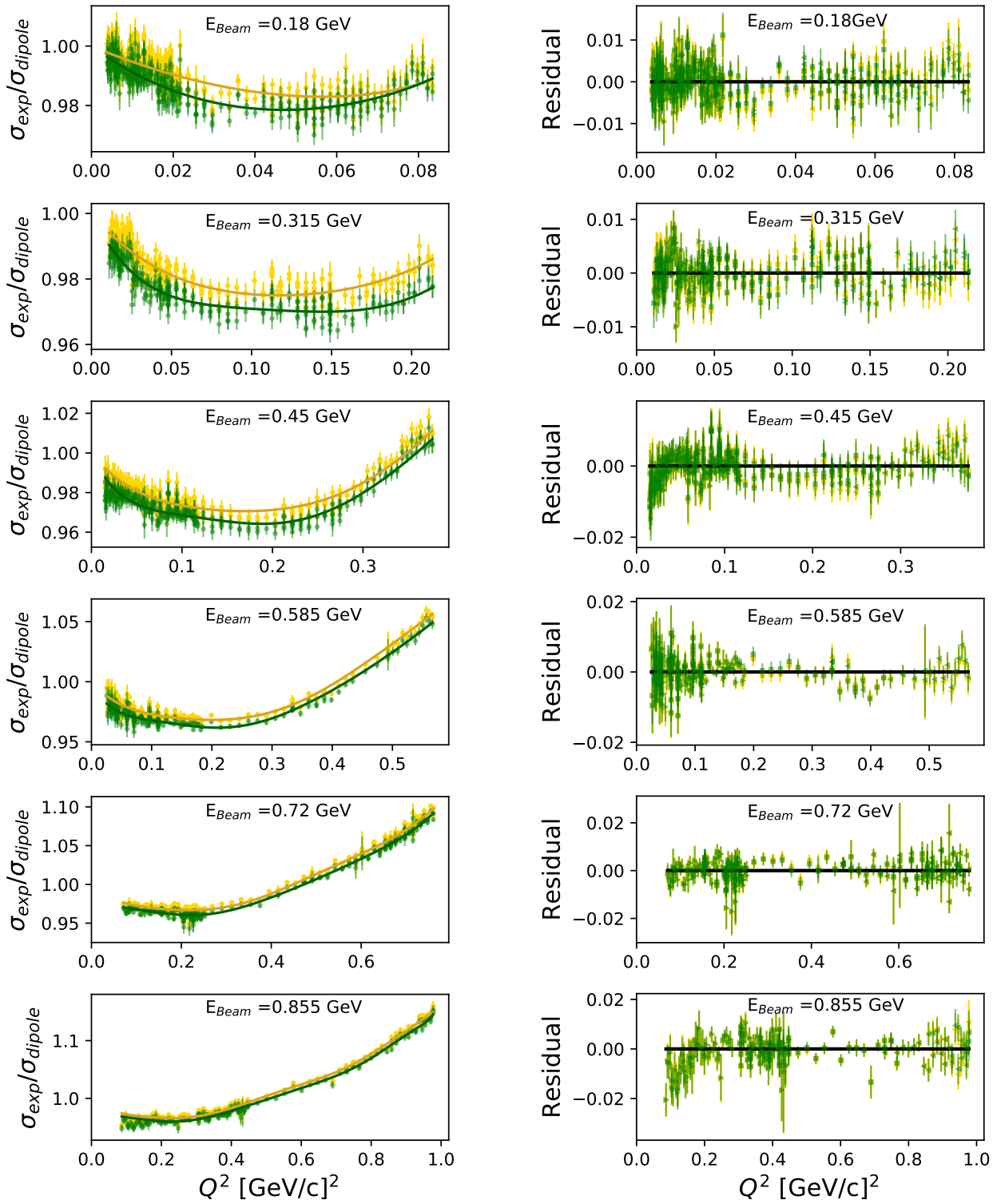
TABLE II: The values of the polynomial terms for the unbounded and bounded fits.

| i | $a_i^E$ | $a_i^M$ | $a_i^E$ | $a_i^M$ |
|---|---|---|---|---|
| 1 | -3.33171 | -2.52364 | -3.07331 | -2.79516 |
| 2 | 13.06315 | -0.70594 | 7.90249 | 5.17258 |
| 3 | -63.80113 | 40.12781 | -19.06093 | -5.72911 |
| 4 | 250.22516 | -176.44192 | 36.42947 | 2.80974 |
| 5 | -661.93288 | 379.37094 | -46.31889 | -0.00000 |
| 6 | 1107.02551 | -390.33782 | 33.71220 | 0.00251 |
| 7 | -996.16448 | 8.34513 | -10.65379 | -0.27676 |
| 8 | 58.48740 | 444.90791 | 0.00003 | 0.00197 |
| 9 | 861.75495 | -492.96968 | -0.00000 | -0.00000 |
| 10 | -818.96268 | 230.24980 | 0.00000 | 0.00000 |
| 11 | 253.05655 | -40.84251 | -0.00000 | -0.00000 |

As should be clear from this example, if one does regressions with so many free parameters, it can be very challenging to understand the regression code and results. Also, confirmation bias can easily lead to improper interpretation of results from a non-frequentist analysis. As Johnny von Neumann is creditting with saying, "With four parameters I can fit an elephant, and with five I can make him wiggle his trunk [17]." Thus, instead of debating which of these two extremely complex regressions (if either) has merit, want we really want are physical models, such as Lorenz *etal.* [18] or Alarcø'n and Weiss [19] and/or radius limits that don't depend on the normalization parameters [20].

## ACKNOWLEDGMENTS

Many thanks to Franziska Hagelstein and Vladimir Pascalutsa for their questions about normalization pa-




(a) Fits

(b) Residuals

FIG. 1: Shown are 1422 point of data analyzed with two different analytic choices. The green points were analyzed using an unbounded eleventh order polynomial in $G_E$ and $G_M$ while the gold points used a bounded polynomial. The systematic difference in the location of the points is due to how the 31 normalization parameters are affected by the choice of a function. While the means are different, the residuals of the fits to the respective functions are rather similar.

TABLE III: Shown are the normalization parameters for the unbounded and bounded regressions along with the difference between them. Compared to the uncertainty of the initial normalizaitons of a few percent, these charges are all small. More interesting, since these affect the data in combinations, many of the changes are simply flips. For example, for the lowest $Q^2$ data, the normalization is given by $N_1 \cdot N_4$ which is $1.00033 \cdot 0.99856$ vs. $0.99794 \cdot 1.00719$ for the unbounded and bounded fits respectively.

| | Unbounded | Bounded | Difference |
|---|---|---|---|
| $N_1$ | 1.00033 | 0.99794 | 0.00239 |
| $N_2$ | 1.00047 | 0.99572 | 0.00475 |
| $N_3$ | 0.99891 | 1.00488 | -0.00597 |
| $N_4$ | 0.99856 | 1.00719 | -0.00863 |
| $N_5$ | 0.99852 | 1.00645 | -0.00793 |
| $N_6$ | 0.99988 | 1.00004 | -0.00016 |
| $N_7$ | 1.00000 | 0.99989 | 0.00011 |
| $N_8$ | 1.00019 | 1.00057 | -0.00038 |
| $N_9$ | 0.99900 | 1.00608 | -0.00708 |
| $N_{10}$ | 1.00014 | 1.00108 | -0.00094 |
| $N_{11}$ | 1.00004 | 1.00004 | -0.00001 |
| $N_{12}$ | 0.99984 | 1.00119 | -0.00135 |
| $N_{13}$ | 0.99878 | 1.00563 | -0.00685 |
| $N_{14}$ | 0.99892 | 1.00540 | -0.00648 |
| $N_{15}$ | 0.99882 | 1.00402 | -0.00520 |
| $N_{16}$ | 1.00010 | 1.00052 | -0.00042 |
| $N_{17}$ | 1.00015 | 1.00107 | -0.00092 |
| $N_{18}$ | 0.99883 | 1.00632 | -0.00750 |
| $N_{19}$ | 0.99885 | 1.00511 | -0.00626 |
| $N_{20}$ | 0.99865 | 1.00435 | -0.00570 |
| $N_{21}$ | 1.00001 | 0.99919 | 0.00081 |
| $N_{22}$ | 0.99949 | 0.99870 | 0.00079 |
| $N_{23}$ | 1.00014 | 0.99990 | 0.00024 |
| $N_{24}$ | 0.99936 | 0.99829 | 0.00107 |
| $N_{25}$ | 0.99875 | 1.00567 | -0.00692 |
| $N_{26}$ | 0.99902 | 1.00690 | -0.00789 |
| $N_{27}$ | 1.00059 | 1.00003 | 0.00056 |
| $N_{28}$ | 0.99961 | 1.00004 | -0.00042 |
| $N_{29}$ | 0.99839 | 1.00596 | -0.00757 |
| $N_{30}$ | 1.00040 | 0.99895 | 0.00144 |
| $N_{31}$ | 0.99869 | 1.00507 | -0.00639 |

rameters that prompted this work and thanks to the many useful comments and suggestions from David Meekins. This work was supported by the U.S. Department of Energy contract DE-AC05-06OR23177 under which Jefferson Science Associates operates the Thomas Jefferson National Accelerator Facility.

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