A complete matching for quasi-distribution functions in large momentum effective theory

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We complete the procedure of extracting parton distribution functions (PDFs) using large momentum effective theory (LaMET) at leading power accuracy in the hadron momentum. We derive a general factorization formula for the quasi parton distribution functions in the presence of mixing, and give the corresponding hard matching kernel both for the unpolarized and for the polarized quark and gluon quasi-PDFs at $\mathcal{O}(\alpha_s)$. Our calculation is performed in a regularization-independent momentum subtraction scheme. The results allow us to match the nonperturbatively renormalized quasi-PDFs to normal PDFs in the presence of mixing, and therefore can be used to extract flavor-singlet quark PDFs as well as the gluon PDFs from lattice simulations.

I. INTRODUCTION

Understanding the internal structure of hadrons from quarks and gluons — the fundamental degrees of freedom of QCD Lagrangian — has been a key goal in hadron physics. However, this is profoundly difficult because it requires solving QCD at large distance scales and thus at strong coupling. In high energy collisions, the hadron and/or the probe moves nearly at the speed of light, the hadron structure greatly simplifies and can be characterized by certain parton observables such as the parton distribution functions (PDFs), lightcone distribution amplitudes (DAs) etc. The parton observables are defined as the expectation value of lightcone correlations in the hadron state and therefore can not be readily computed on a Euclidean lattice. Currently, the most widely used approach to determine them is to assume a smoothly parametrized form and fit the unknown parameters to a large variety of experimental data (for a recent review, see e.g. Ref. [1]). Lattice efforts on determining parton observables have been mainly focused on the computation of their moments, which are matrix elements of local operators. The parton observables can be reconstructed in principle if all their moments are known. However, to date only the first few moments can be calculated in lattice QCD [2–5] due to power divergent mixings between different moments operators and increasing stochastic noise for high moments operators.

In the past few years, a breakthrough has been made to circumvent the above difficulty, which has now been formulated as large momentum effective theory (LaMET) [6, 7]. According to LaMET, a parton observable can be directly accessed from lattice QCD using the following procedure: 1) Construct an appropriate static-operator matrix element (quasi-observable) that approaches the parton observable in the infinite momentum limit of the external hadron. The quasi-observable constructed in this way is usually hadron-momentum-dependent but time-independent, and thus can be readily computed on the lattice. 2) Calculate the quasi-observable on the lattice and renormalize it non-perturbatively in an appropriate scheme. 3) Match the renormalized quasi-observable to the parton observable through a factorization formula accurate up to power corrections that are suppressed by the hadron momentum. The existence of such a factorization is ensured by construction; for a proof in the case of isovector quark distribution, see Refs. [8–10].

Since LaMET was proposed, much progress has been achieved both in the theoretical understanding of the formalism [10–61] and in the direct calculation of PDFs from lattice QCD [25, 26, 31, 32, 34, 62–73]. In particular, multiplicative renormalization of both the quark [20, 29, 30] and the gluon [53, 54] quasi-PDF has been established in coordinate space. Non-perturbative renormalization in the regularization-independent momentum subtraction (RI/MOM) scheme as well as a perturbative matching in the same scheme has been carried out for the isovector quark quasi-PDFs in Refs. [18, 31, 67, 70] (see also [19, 32, 66]). Despite limited volumes and relatively coarse lattice

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spacings, the state-of-the-art nucleon isovector quark PDFs determined from lattice data at the physical point have shown a reasonable agreement [66, 67, 70] with phenomenological results extracted from the experimental data [74–78]. Of course, a careful study of theoretical uncertainties and lattice artifacts is still needed to fully establish the reliability of the results.

So far the lattice calculations of PDFs have been focused on the isovector quark PDFs only, which do not involve mixing with gluon PDFs and therefore are the easiest to calculate. In the past few years, there has been increasing interest in calculating flavor-singlet quark PDFs and gluon PDFs from lattice QCD. Such calculations are possible only if the renormalization and mixing pattern of gluon quasi-PDFs are fully understood. In Ref. [53], we performed a systematic study of the renormalization property of the gluon quasi-PDF operator, and showed that with an appropriate choice it can be multiplicatively renormalizable. We have identified four independent gluon quasi-PDF operators that have an easy implementation on the lattice. Also, a general factorization formula for the gluon as well as the quark quasi-PDF in the presence of mixing has been conjetured.

In this paper, we provide all necessary inputs for extracting both the flavor-singlet quark PDF and the gluon PDF from lattice QCD, thereby completing the procedure of calculating PDFs using LaMET at leading power accuracy in the hadron momentum. We explain how to nonperturbatively renormalize the quark and gluon quasi-PDFs, and derive the general factorization formula for the renormalized quasi-PDFs in the presence of mixing, following the operator product expansion (OPE) method in Refs. [9, 10]. We then present the complete one-loop results for the hard matching kernels that appear in the general factorization of quasi-PDFs.

The rest of the paper is organized as follows: In Sec. II, we briefly review the renormalization and factorization of quark and gluon quasi-PDFs. In Sec. III, we present our one-loop calculation of the matching kernel connecting the RI/MOM renormalized quasi-PDFs to the PDFs in $\overline{\rm MS}$ scheme, with a particular focus on the unpolarized case. Sec. IV is devoted to the calculation in the polarized case. We conclude in Sec. V and give some computational details in the Appendix.

II. RENORMALIZATION AND FACTORIZATION OF QUARK AND GLUON QUASI-PDFS

In this section, we give a brief review of the renormalization and factorization of quark and gluon quasi-PDFs in LaMET.

A. Quasi-PDFs in LaMET

In parton physics, the PDFs are defined as the hadron matrix elements of quark and gluon nonlocal correlators along the lightcone. For example, the unpolarized quark distribution is defined as

$$f_{q_i/H}(x,\mu) \equiv \int \frac{d\xi^-}{4\pi}\, e^{-ixP^+\xi^-} \langle P\big|\bar{q}_i(\xi^-)\gamma^+ W(\xi^-,0) q_i(0)\big|P\rangle \tag{1}$$

for a given flavor i, where $x = k^+/P^+$ is the longitudinal momentum fraction carried by the quark of flavor i, μ is the renormalization scale in the $\overline{\rm MS}$ scheme, $P^\mu = (P^0, 0, 0, P^z)$ is the hadron momentum , $\xi^\pm = (t \pm z)/\sqrt{2}$ are the lightcone coordinates, and

$$W(\xi^-,0) = \exp\bigg(-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-)\bigg) \tag{2}$$

is the Wilson line inserted to ensure gauge invariance of the nonlocal correlator, where $A^+ = A^+_a t^a$ with t^a being the generators in the fundamental representation of color $SU(3)$ group.

Analogously, the unpolarized gluon distribution can be defined as [79]

$$f_{g/H}(x,\mu) = \int \frac{d\xi^-}{2\pi x P^+} e^{-ixP^+\xi^-} \langle P|F_a^{+i}(\xi^-)\mathcal{W}(\xi^-,0)F_a^{+i}(0)|P\rangle, \tag{3}$$

where $F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu$ is the gluon field strength, and i runs over the transverse indices. The Wilson line $\mathcal{W}$ takes a similar form as the quark case, but is defined in the adjoint representation.

The quark and gluon PDFs defined above can not be readily computed on the lattice due to their real-time dependence. However, according to LaMET, they can be extracted from lattice calculations of appropriately constructed quasi-PDFs via a factorization procedure. For the unpolarized quark PDF, a well-suited quasi-PDF candidate is given by

$$\tilde{f}_{q_i/H}(x,\mu,P^z) = N\int \frac{dz}{4\pi} e^{izxP^z}\langle P|\overline{q}_i(z)\Gamma W(z,0)q_i(0)|P\rangle, \tag{4}$$

where z is a spatial direction, $\Gamma = \{\gamma^z, \gamma^t\}$ is a Dirac matrix with the corresponding normalization factor $N = \{1, P^z/P^t\}$, respectively. As shown in Ref. [29], the renormalization of the quark quasi-PDF defined above has a multiplicative form so that the matrix elements at different z do not mix with each other. Moreover, the latter choice with $\Gamma = \gamma^t$ has the advantage of avoiding mixing with the scalar PDF when a non-chiral lattice fermion is used [19, 33]. We will focus on the latter choice in the rest of the paper.

In comparison with the quark case, what is the most appropriate operator to define the gluon quasi-PDF is less obvious. In principle, one can choose

$$O_g^{\mu\nu}(z,0) = F^{\mu\alpha}(z)\mathcal{W}(z,0)F_\alpha^{\ \nu}(0), \tag{5}$$

with $\mu,\nu = \{t,z\}$ and α running either over all Lorentz indices or only over transverse indices. However, such a choice in general mixes with other operators under renormalization. Using the auxiliary field approach [80], we have explicitly shown [53] that different components of $O^{\mu\nu}$ indeed renormalize differently, which complicates the construction of appropriate gluon quasi-PDFs. A detailed description of the formalism used in Ref. [53] as well as [29] will be given in the next subsection. Nevertheless, we have identified four gluon operators [53] that are multiplicatively renormalizable and therefore are suitable for defining the gluon quasi-PDF. These operators are

$$\begin{aligned}
O_g^{(1)}(z,0) &\equiv F^{ti}(z)\mathcal{W}(z,0)F_i^{\ t}(0),\\
O_g^{(2)}(z,0) &\equiv F^{zi}(z)\mathcal{W}(z,0)F_i^{\ z}(0),\\
O_g^{(3)}(z,0) &\equiv F^{ti}(z)\mathcal{W}(z,0)F_i^{\ z}(0),\\
O_g^{(4)}(z,0) &\equiv F^{z\mu}(z)\mathcal{W}(z,0)F_\mu^{\ z}(0),
\end{aligned} \tag{6}$$

where a summation over transverse (all) components is implied for $i\,(\mu)$. The corresponding gluon quasi-PDF is then defined as

$$\tilde{f}^{(n)}_{g/H}(x,\mu,P^z) = N^{(n)}\int \frac{dz}{2\pi x P^z} e^{izxP^z}\langle P|O_g^{(n)}(z,0)|P\rangle. \tag{7}$$

The normalization factors are given by

$$N^{(2)} = N^{(4)} = 1, \quad N^{(1)} = \frac{(P^z)^2}{(P^t)^2}, \quad N^{(3)} = \frac{P^z}{P^t}, \tag{8}$$

so that all partonic PDFs at tree-level are

$$\tilde{f}^{(n,0)}_{g/g}(x,\mu,P^z) = \delta(x-1), \tag{9}$$

with the hadron state H being replaced by a gluon state. Note that in the above result (also in the sections below unless stated otherwise) we have ignored the contribution from the crossed diagrams, which correspond to interchanging the contraction between the two external gluons and gluon fields from the operators $O_g^{(n)}$. These crossed diagrams contribute to $x<0$ and can be easily obtained from $\tilde{f}^{(n)}_{g/H}(x) = -\tilde{f}^{(n)}_{g/H}(-x)$.

The above gluon quasi-PDF operator is defined in terms of an adjoint gauge link. Alternatively, it can be defined using gauge links in the fundamental representation $U(z_2,z_1)$ as [80–85] (taking $O^{(3)}$ as an example)

$$O_g^{(3)}(z_2,z_1) = 2\,\mathrm{Tr}[F^{ti}(z_2)U(z_2,z_1)F_i^{\ z}(z_1)U(z_1,z_2)], \tag{10}$$

where $F^{\mu\nu} = F^a_{\mu\nu}t^a$ and t^a is the generator in the fundamental representation with $\mathrm{tr}[t^at^b] = 1/2\delta^{ab}$. Eq. (5) makes the study of its renormalization property simpler, whereas Eq. (10) is more straightforward to implement on the lattice. In the following, we will mainly focus on the definition Eq. (5), but the results also apply to Eq. (10).

B. Auxiliary Field Approach

The multiplicative renormalizability of quark and gluon quasi-PDFs in coordinate space has been proved using an auxiliary field approach [29, 53]. Other proofs have been available using a similar formalism [20] or using the Feynman diagrammatic approach [30, 54]. In the auxiliary field approach [80], one introduces an auxiliary "heavy quark" field into the QCD Lagrangian such that the Wilson line can be reinterpreted as a two-point function of the auxiliary field. For the quark/gluon quasi-PDF, this auxiliary field is chosen to be in the fundamental/adjoint representation of color $SU(3)$ group, respectively. Moreover, the auxiliary "heavy quark" has trivial spin degrees of freedom. In the following we present, as an example, the auxiliary Lagrangian that can be used to study quark quasi-PDFs. For gluon quasi-PDFs the procedure is completely analogous.

The effective Lagrangian including an auxiliary fundamental "heavy quark" field (denoted as Q) can be written as

$$\mathcal{L} = \mathcal{L}_{\rm QCD} + \overline{Q}(x) in \cdot D Q(x), \tag{11}$$

where $D_\mu = \partial_\mu + igt_a A_{a,\mu}$ is the covariant derivative in the fundamental representation. We will focus on the case with $n^\mu = (0, 0, 0, -1)$, although the discussion may go through for any spacelike n.

In the above theory, we can replace the Wilson line by the product of two auxiliary "heavy quark" fields. This can be seen as following. After integrating out the "heavy quark" field, we have

$$\int \mathcal{D}\overline{Q}\mathcal{D}Q\, Q(x)\overline{Q}(y) e^{i\int d^4x \mathcal{L}} = S_Q(x, y) e^{i\int d^4x \mathcal{L}_{\rm QCD}} \tag{12}$$

up to a determinant $\det(in \cdot D)$ that can be shown to be a constant and absorbed into the normalization of the generating functional [86]. The propagator $S_Q(x, y)$ satisfies

$$n \cdot D\, S_Q(x, y) = \delta^{(4)}(x - y). \tag{13}$$

For $n^\mu = (0, 0, 0, -1)$, its solution is given by [1]

$$\begin{aligned} S_Q(x, y) &= -\theta(x^z - y^z)\delta(x^0 - y^0)\delta^{(2)}(\vec{x}_\perp - \vec{y}_\perp) W(x, y) \\ &= -\theta(x^z - y^z)\delta(x^0 - y^0)\delta^{(2)}(\vec{x}_\perp - \vec{y}_\perp) W(x^z, y^z), \end{aligned} \tag{15}$$

with an appropriate boundary condition. Without loss of generality, we can restrict ourselves to the case $x^z > y^z$. The δ-functions ensure that the time and transverse components of x and y are equal, and therefore generate a spacelike Wilson line along the longitudinal direction.

C. Renormalization of Quasi-PDFs in Auxiliary Field Approach

1. Quark Quasi-PDFs

The above formalism allows one to replace the Wilson line $W(z_2, z_1)$ appearing in the quark quasi-PDF by the product of two auxiliary "heavy quark" fields $Q(z_2)\overline{Q}(z_1)$. The quark bilocal operator

$$O_{q_i}(z_2, z_1) = \bar{q}_i(z_2)\Gamma W(z_2, z_1) q_i(z_1)$$

then reduces to the product of two local composite operators

$$\mathcal{O}_{q_i}(z_2, z_1) = \bar{q}_i(z_2)\Gamma Q(z_2)\overline{Q}(z_1) q_i(z_1) \equiv \bar{j}(z_2) j(z_1), \tag{16}$$

with

$$\bar{j}(z_2) = \bar{q}_i(z_2)\Gamma Q(z_2), \quad j(z_1) = \overline{Q}(z_1) q_i(z_1). \tag{17}$$

In dimensional regularization (DR), the local operators $\bar{j}(z_2), j(z_1)$ are "heavy-to-light" like and are multiplicatively renormalized:

$$\bar{j}(z_2) = Z_{\bar{j}} \bar{j}_R(z_2), \quad j(z_1) = Z_j j_R(z_1), \tag{18}$$

[1] A different solution

$$S_Q(x, y) = \theta(-(x^z - y^z))\delta(x^0 - y^0)\delta^{(2)}(\vec{x}_\perp - \vec{y}_\perp) W(x^z, y^z), \tag{14}$$

will give a different sign for the $d = 4$ poles in Eq. (34).

with ($D = 4 - 2\epsilon$):

$$Z_{\bar{j}} = Z_j = 1 + \frac{\alpha_s}{2\pi\epsilon} + \mathcal{O}(\alpha_s^2). \tag{19}$$

After integrating out the auxiliary field, the nonlocal operator renormalizes as [29]

$$O_{q_i,R}(z_2, z_1) = Z_{\bar{j}}^{-1} Z_j^{-1} \bar{q}_i(z_2) \Gamma W(z_2, z_1) q_i(z_1). \tag{20}$$

In a cutoff regularization such as the lattice regularization, when going beyond leading-order perturbation theory, the self-energy of the heavy quark introduces a linear divergence which has to be absorbed into an effective mass counterterm,

$$\delta\mathcal{L}_m = -\delta m \overline{Q} Q, \tag{21}$$

where $\delta m \sim \mathcal{O}(1/a)$ with a being the lattice spacing [87]. As shown in Ref. [29], apart from the structures in the Lagrangian Eq. (11), this is the only possible renormalizable counterterm one can write down that is consistent with the symmetry of the theory. Although lattice regularization breaks Lorentz symmetry and introduces operator mixing, it is not a concern for $O_q(z_2, z_1)$ because there is no lower-dimensional operator that can mix with it in lattice QCD. Moreover, Becchi-Rouet-Stora-Tyutin (BRST) invariance requires a dependence of δm on the signature of n in Eq. (11) [80], which yields a vanishing δm for a lightlike n^μ. For a spacelike n^μ, δm in Eq. (21) is imaginary, we can therefore write $\delta m = i\delta\bar{m}$.

Including the effective mass term Eq. (21) into the Lagrangian and integrating out the auxiliary "heavy quark", we then have the following renormalization for the non-local quark bilocal operator [29]

$$O_{q_i,R}(z_2, z_1) = Z_{\bar{j}}^{-1} Z_j^{-1} e^{\delta\bar{m}|z_2 - z_1|} \bar{q}_i(z_2) \Gamma W(z_2, z_1) q_i(z_1). \tag{22}$$

2. Gluon Quasi-PDFs

For the renormalization of non-local gluon quasi-PDF operators, the desired auxiliary Lagrangian has exactly the same form as that for the quark, except that now the auxiliary "heavy quark" and the covariant derivative are in the adjoint representation. To distinguish the adjoint auxiliary "heavy quark" from the fundamental one used in the previous subsection, we denote the former as $\mathcal{Q}$ below.

Given the auxiliary adjoint "heavy quark", we can replace the non-local gluonic operator in Eq. (6) by the product of two local composite operators. For example,

$$\mathcal{O}_g^{(3)}(z_2, z_1) = J_1^{ti}(z_2) \overline{J}_{1,i}^{\;\;z}(z_1), \tag{23}$$

where

$$J_1^{ti}(z_2) = F_a^{ti}(z_2) \mathcal{Q}_a(z_2), \; \overline{J}_{1,i}^{\;\;z}(z_1) = \overline{\mathcal{Q}}_b(z_1) F_{b,i}^{\;\;z}(z_1). \tag{24}$$

In the following, we will consider the renormalization of $J_1^{\mu\nu}$ in general.

In contrast to the quark case, extra complications arise when renormalizing the local gluon composite operators $J_1, \overline{J}_1$ above. Let us first consider the perturbative renormalization in DR in a covariant gauge. In such a scheme, gauge-invariant local composite operators can, in general, mix with operators of the same or lower mass dimension under renormalization. It is well-known that the mixing operators can be of the following three types: 1) gauge-invariant operators, 2) BRST exact operators or operators that are the BRST variation of some operators, 3) operators that vanish by equation of motion (see e.g. [88]). Therefore, one also needs counterterms from the mixing operators to fully cancel the ultraviolet (UV) divergences of the original operators.

When the mass of the auxiliary "heavy quark" $m = 0$, it has been shown in Refs. [80, 81] that, the only operators that are allowed by BRST symmetry to mix with $J_1^{\mu\nu}$ in DR are

$$\begin{aligned} J_2^{\mu\nu} &= n_\rho (F_a^{\mu\rho} n^\nu - F_a^{\nu\rho} n^\mu) \mathcal{Q}_a / n^2, \\ J_3^{\mu\nu} &= (-in^\mu A_a^\nu + in^\nu A_a^\mu)(in \cdot D\mathcal{Q})_a / n^2, \end{aligned} \tag{25}$$

where $J_2^{\mu\nu}$ is gauge invariant, and $J_3^{\mu\nu}$ is proportional to the massless equation of motion of $\mathcal{Q}$ and therefore vanishes in a physical matrix element. When $m \neq 0$, the mixing involves one more operator

$$J_4^{\mu\nu} = (-in^\mu A_a^\nu + in^\nu A_a^\mu) m \mathcal{Q}_a / n^2. \tag{26}$$

As shown in Ref. [53], it is necessary to include this term, as it gets renormalized in other renormalization schemes. It turns out that this term always combines with $J_3^{\mu\nu}$ such that the resulting operator is proportional to the massive equation of motion of $\mathcal{Q}$. In other words, instead of $J_3^{\mu\nu}$ in Eq. (25) we now have

$$J_3^{\mu\nu} = (-in^\mu A_a^\nu + in^\nu A_a^\mu)((in\cdot D - m)\mathcal{Q})_a/n^2. \tag{27}$$

This is consistent with the discussion of the operator mixing pattern above. Our one-loop computation in Ref. [53] indeed verified this.

We can write the renormalization of the above composite operators in the following form

$$\begin{pmatrix} J_{1,R}^{\mu\nu} \\ J_{2,R}^{\mu\nu} \\ J_{3,R}^{\mu\nu} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ 0 & Z_{22} & Z_{23} \\ 0 & 0 & Z_{33} \end{pmatrix} \begin{pmatrix} J_1^{\mu\nu} \\ J_2^{\mu\nu} \\ J_3^{\mu\nu} \end{pmatrix}, \tag{28}$$

with a triangular mixing matrix.

The renormalization constants in Eq. (28) are not all independent. From the equations above, it is easy to see that $J_2^{z\mu}$ is degenerate with $J_1^{z\mu}$. This has important implications for the renormalization matrix. From Eq. (28), we have

$$\begin{aligned} J_{1,R}^{z\mu} &= Z_{11}J_1^{z\mu} + Z_{12}J_2^{z\mu} + Z_{13}J_3^{z\mu} = (Z_{11}+Z_{12})J_2^{z\mu} + Z_{13}J_3^{z\mu}, \\ J_{2,R}^{z\mu} &= Z_{22}J_2^{z\mu} + Z_{23}J_3^{z\mu}. \end{aligned} \tag{29}$$

Since $J_1^{z\mu}$ and $J_2^{z\mu}$ are identical, their renormalization must be the same. Therefore one has

$$Z_{11} + Z_{12} = Z_{22}, \quad Z_{13} = Z_{23}, \tag{30}$$

which is validated by the explicit one-loop calculation in Ref. [81]. One can therefore rewrite the mixing matrix in Eq. (28) in the following form

$$\begin{pmatrix} J_{1,R}^{\mu\nu} \\ J_{2,R}^{\mu\nu} \\ J_{3,R}^{\mu\nu} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{22}-Z_{11} & Z_{13} \\ 0 & Z_{22} & Z_{13} \\ 0 & 0 & Z_{33} \end{pmatrix} \begin{pmatrix} J_1^{\mu\nu} \\ J_2^{\mu\nu} \\ J_3^{\mu\nu} \end{pmatrix}. \tag{31}$$

The above mixing pattern indicates that the $J_1^{z\mu}$ and $J_1^{ti}(i=1,2)$ renormalize independently with the simplified renormalization equations:

$$\begin{pmatrix} J_{1,R}^{z\mu} \\ J_{3,R}^{z\mu} \end{pmatrix} = \begin{pmatrix} Z_{22} & Z_{13} \\ 0 & Z_{33} \end{pmatrix} \begin{pmatrix} J_1^{z\mu} \\ J_3^{z\mu} \end{pmatrix}; \qquad J_{1,R}^{ti} = Z_{11}J_1^{ti}. \tag{32}$$

Moreover the J_1^{ij} shares the same renormalization with J_1^{ti}. The reason that (J_1^{ti}, J_1^{ij}) and $J_1^{z\mu}$ have different renormalizations is due to the breaking of symmetry in the presence of a four-vector n^μ along the z direction. Since $J_2^{\mu i}$ are not independent, we will neglect $J_2^{\mu i}$ henceforth.

In Ref. [80], it was also shown that in the $\overline{\text{MS}}$ scheme the renormalization constants fulfill further relations

$$Z_{11} + Z_{12} = 1, \qquad\qquad Z_{13} + Z_{23} + Z_{33} = 1. \tag{33}$$

However, these relations might not be necessarily true in a general renormalization scheme.

Now let us consider the UV divergences, in particular power divergences, in $J_1^{\mu\nu}$ and how to renormalize them. The power divergence structure of this operator has been considered in Refs. [22, 23] using a simple cutoff regularization. One has to be cautious when dealing with power divergences in such a cutoff scheme. The cutoff regularization in general explicitly breaks gauge invariance in QCD (except for the lattice cutoff which preserves gauge invariance), and might obscure the structure of genuine power divergences of the theory. To avoid this, we will work in DR and kept track of the linear divergences by expanding the results around $d=3$. The linear divergences appear as poles at $d=3$. In this way, one can extract the linear divergences in a gauge invariant manner.

Fig. 1 shows the one-loop diagrams that give rise to linearly divergent contributions to the operator $J_1^{\mu\nu}$. There are other one-loop diagrams, which contribute with at most logarithmic divergences. Our calculation in coordinate space yields the following result for the two diagrams in Fig. 1,

$$\begin{aligned} I_1^{\rho\nu} &= -\frac{\alpha_s C_A}{\pi}\Big\{\frac{1}{d-4}(A_a^\nu n^\rho - A_a^\rho n^\nu)n\cdot\partial\mathcal{Q}_a/n^2 - \frac{\pi\mu}{d-3}\big(n^\rho A_a^\nu - n^\nu A_a^\rho\big)\mathcal{Q}_a + reg.\Big\}, \\ I_2^{\rho\nu} &= -\frac{\alpha_s C_A}{\pi}\Big\{\frac{1}{d-4}\Big[\frac{1}{4}F_a^{\rho\nu}\mathcal{Q}_a + \frac{1}{2}\big(F_a^{\rho\sigma}n_\nu n_\sigma - F_a^{\nu\sigma}n_\rho n_\sigma\big)\mathcal{Q}_a/n^2 + \frac{1}{2}(A_a^\rho n^\nu - A_a^\nu n^\rho)n\cdot\partial\mathcal{Q}_a/n^2\Big] \\ &\quad + \frac{\pi\rho}{d-3}\big(n^\rho A_a^\nu - n^\nu A_a^\rho\big)\mathcal{Q}_a + reg.\Big\}, \end{aligned} \tag{34}$$

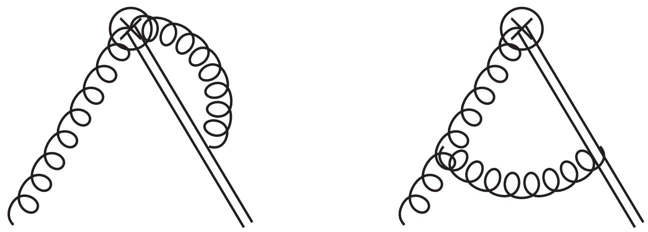

FIG. 1. One-loop corrections to the operator $J_1^{\mu\nu}$ that contain linear divergences. The double line represents an auxiliary adjoint "heavy quark" field $\mathcal{Q}$.

where *reg.* denotes terms that are regular both at $d=4$ and $d=3$, μ is the regularization scale. Combining the two diagrams, the linear divergences cancel. Our results show an identical mixing pattern as in Ref. [81] (note that the normalization of the direction vector there is $\dot{x}^2=1$, whereas we have $n^2=-1$). The only linear divergence that remains in the theory comes from the self-energy of the auxiliary field. Actually, the same conclusion can also be reached in Ref. [23] if a gauge-invariant regulator is used. We have explicitly checked that this is the case by redoing the calculation of Ref. [23] in DR and keeping track of the linear divergences as poles around $d=3$. All linear divergences cancel in diagrams without Wilson line self-energy corrections. Only the self-energy of the Wilson line gives non-vanishing linear divergences, which can be removed by the mass renormalization to be discussed below.

The above discussion has been focused on renormalization in DR. In a cutoff regularization such as the lattice regularization, the mass term of the auxiliary field is not forbidden by the symmetry of the theory, and it can appear beyond leading order in perturbation theory even if it does not exist at leading order. This is indeed what happens here. In perturbation theory, $m=\delta m$ starts from $O(\alpha_s)$. Such a mass term serves the purpose of absorbing the power divergences arising from the Wilson line self energy. Apart from this, there is no other power divergence in the theory. Moreover, for dimensional reasons, there is no other antisymmetric operator that can mix with $J_i^{\mu\nu}$ discussed in the previous subsection. Therefore in a gauge-invariant cutoff scheme, the operator renormalization in Eq. (31) remains the same except that the $J_3^{\mu\nu}$ contains a mass as shown in Eq. (27).

The mixing pattern can be used to derive the building blocks for constructing an appropriate gluon quasi-PDF. To this end, we may identify one of the indices in $J_1^{\mu\nu}$ with z or t and let the other run either over all Lorentz components or over the transverse components only. It is worthwhile to point out at this stage that the operator $J_3^{\mu\nu}$ only yields contact terms when integrating out the "heavy quark" field. This can be seen from the fact that the equation of motion operator acting on the "heavy quark" propagator yields a δ-function. The contact terms do not vanish only when $z_2=z_1$, which indicates that an extra renormalization is required when the distance between two local composite operators shrinks to zero. For the renormalization of the non-local gluon quasi-PDF operator, the operator $J_3^{\mu\nu}$ is irrelevant and can be ignored.

With the above building blocks, $J_{1,R}^{zi}$, $J_{1,R}^{ti}$, $J_{1,R}^{z\mu}$, and their conjugate counterparts, four multiplicatively renormalizable gluon quasi-PDF operators for unpolarized PDFs can be constructed, as given in Sec. II A. One option is to use $J_{1,R}^{ti}$ and $\overline{J}_{1,R}^{ti}$:

$$\mathcal{O}_{g,R}^{(3)}(z_2,z_1)\equiv J_{1,R}^{ti}(z_2)\overline{J}_{1,R,i}{}^{z}(z_1). \tag{35}$$

After integrating out the auxiliary "heavy quark" field, this operator renormalizes multiplicatively as ($\delta m=i\overline{\delta m}$)

$$O_{g,R}^{(3)}(z_2,z_1)=(F^{ti}(z_2)\mathcal{W}(z_2,z_1)F_i{}^{z}(z_1))_R=Z_{11}Z_{22}e^{\overline{\delta m}|z_2-z_1|}F^{ti}(z_2)\mathcal{W}(z_2,z_1)F_i{}^{z}(z_1). \tag{36}$$

The renormalization of other operators can be written down analogously with different renormalization factors.

In the large momentum limit, the operators $\mathcal{O}_{g,R}^{(i)}(i=1,2,3,4)$ differ from each other only by power corrections. Therefore they belong to the same universality class defining the gluon quasi-PDF operator. With the above four operators, one can use any combinations of them to study the gluon quasi-PDF, however such combinations are usually not multiplicatively renormalizable. A notable example is

$$O_{g,R}^{(5)}(z_2,z_1)\equiv(F^{t\mu}(z_2)\mathcal{W}(z_2,z_1)F^{z}{}_{\mu}(z_1))_R=-O_{g,R}^{(1)}(z_2,z_1)-O_{g,R}^{(2)}(z_2,z_1)-O_{g,R}^{(4)}(z_2,z_1). \tag{37}$$

This operator (minus the trace term) has been used in the recent simulation [71]. Since the renormalizations for $O_{g,R}^{(1)}(z_2,z_1)$ and $O_{g,R}^{(2,4)}(z_2,z_1)$ are different, $O_{g,R}^{(5)}(z_2,z_1)$ is not multiplicatively renormalizable.

D. Renormalization in RI/MOM Scheme and Implementation on Lattice

From the discussions above, it is clear that operators at different z do not mix under renormalization. This allows us to carry out a nonperturbative renormalization of the quasi-PDF in the following ways: 1) Calculate the endpoint renormalization factors and the mass counterterm nonperturbatively. The calculation of the former is rather straightforward, while the latter can be determined by using the static-quark potential for the renormalization of Wilson loops [89]. This has been used in early studies of nucleon PDFs and meson DAs [26, 28, 65]. 2) Calculate the renormalization factors as a whole for each z. This is analogous to the renormalization of local composite operators, which is usually carried out in a RI/MOM scheme [90] on the lattice. In the RI/MOM scheme, the renormalization of local composite operators is done by demanding that the counterterm cancels all loop contributions to their matrix element between off-shell external states at specific momenta [18, 31]. For multiplicatively renormalizable nonlocal correlators such as the quasi-PDFs given above, the renormalization is similar but now one requires calculating the renormalization factors at each z.

The quark and gluon quasi-PDFs can, in general, mix with each other under renormalization. In Ref. [53], we have argued that inserting the gluon quasi-PDF operator into a quark state only yields finite mixing as long as all subdivergences have been renormalized (note that this is different from the quark and gluon lightcone operators defining the PDFs [91, 92]). The mixing effect can, in principle, be deferred to be considered at the factorization stage. Here we find that taking into account the mixing at the renormalization stage will help improve the convergence in the implementation of the matching in the RI/MOM scheme. To this end, it suffices to consider the following mixing of quasi-PDFs

$$\begin{pmatrix} O^{(n)}_{g,R}(z,0) \\ O^{s}_{q,R}(z,0) \end{pmatrix} = \begin{pmatrix} Z_{11}(z) & Z_{12}(z)/z \\ zZ_{21}(z) & Z_{22}(z) \end{pmatrix} \begin{pmatrix} O^{(n)}_{g}(z,0) \\ O^{s}_{q}(z,0) \end{pmatrix}, \tag{38}$$

where $O^s_q(z_1,z_2) = 1/2[\bar{q}_i(z_1)\Gamma W(z_1,z_2) q_i(z_2) - (z_1 \leftrightarrow z_2)]$ is the C-even combination of quark operators, $Z_{ij}(z)$ are dimensionless factors, and z compensates for the different mass dimension between the quark and gluon quasi-PDF operators. In the limit $z \to 0$ (taken after combining the entries of the mixing matrix and bare operators), the above mixing pattern reduces to the mixing pattern of local operators.

The renormalization factors in the above mixing matrix can be determined using the following renormalization conditions

$$\begin{aligned} \frac{\mathrm{Tr}[\Lambda_{22}(p,z)\mathcal{P}]_R}{\mathrm{Tr}[\Lambda_{22}(p,z)\mathcal{P}]_{\text{tree}}}\Bigg|_{\substack{p^2=-\mu_R^2 \\ p_z=p_z^R}} &= 1, \qquad \frac{[P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]_R}{[P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]_{\text{tree}}}\Bigg|_{\substack{p^2=-\mu_R^2 \\ p_z=p_z^R}} = 1, \\ \mathrm{Tr}[\Lambda_{12}(p,z)\mathcal{P}]_R\Bigg|_{\substack{p^2=-\mu_R^2 \\ p_z=p_z^R}} &= 0, \qquad [P^{ab}_{ij}\Lambda^{ab,ij}_{21}(p,z)]_R\Bigg|_{\substack{p^2=-\mu_R^2 \\ p_z=p_z^R}} = 0, \end{aligned} \tag{39}$$

where $\Lambda_{\{11,12\}}$ ($\Lambda_{\{21,22\}}$) denote the amputated Green's functions of $O^{(n)}_g$ (O^s_q) in an offshell gluon and quark state, respectively. $\mathcal{P}$ and P^{ab}_{ij} are projection operators that are associated with the quark and gluon matrix elements and define the RI/MOM renormalization factors. μ_R and p^R_z are unphysical scales introduced in the RI/MOM scheme to specify the subtraction point. b, c are color indices and i, j Lorentz indices. In the non-singlet quark PDF case with $\Gamma = \gamma^t$ [49], the amputated Green's function has the following structure

$$\Lambda(p,z) = \widetilde{f}_t(p,z)\gamma^t + \widetilde{f}_z(p,z)\frac{p^t\gamma^z}{p^z} + \widetilde{f}_p(p,z)\frac{p^t \not{p}}{p^2}, \tag{40}$$

and $\mathcal{P}$ was chosen there in such a way that it projects out the coefficient of γ^t only, which captures all terms in $\Lambda_{\gamma^t}(p,z)$ that lead to ultraviolet (UV) divergences in the local limit. However, in general both the coefficient of γ^t and γ^z can lead to UV divergences in the local limit. This is the case e.g. in the mixing diagram to be considered below. We will need both coefficients to define the RI/MOM counterterm. As for P^{ab}_{ij}, one simple choice is $P^{ab}_{ij} = \delta^{ab} g_{\perp,ij}/(2-D)$, where $g_{\perp,ij}$ denotes the transverse metric tensor and D is the spacetime dimension.

From Eqs. (38) and (39), the renormalization factors can be determined as following:

$$
\begin{aligned}
Z_{11}(z) &= \frac{[P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]_{\rm tree}{\rm Tr}[\Lambda_{22}(p,z)\mathcal{P}]}{([P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]{\rm Tr}[\Lambda_{22}(p,z)\mathcal{P}]-[P^{ab}_{ij}\Lambda^{ab,ij}_{21}(p,z)]{\rm Tr}[\Lambda_{12}(p,z)\mathcal{P}])},\\
Z_{12}(z)/z &= -\frac{[P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]_{\rm tree}{\rm Tr}[\Lambda_{12}(p,z)\mathcal{P}]}{([P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]{\rm Tr}[\Lambda_{22}(p,z)\mathcal{P}]-[P^{ab}_{ij}\Lambda^{ab,ij}_{21}(p,z)]{\rm Tr}[\Lambda_{12}(p,z)\mathcal{P}])},\\
zZ_{21}(z) &= -\frac{[P^{ab}_{ij}\Lambda^{ab,ij}_{21}(p,z)]{\rm Tr}[\Lambda_{22}(p,z)\mathcal{P}]_{\rm tree}}{([P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]{\rm Tr}[\Lambda_{22}(p,z)\mathcal{P}]-[P^{ab}_{ij}\Lambda^{ab,ij}_{21}(p,z)]{\rm Tr}[\Lambda_{12}(p,z)\mathcal{P}])},\\
Z_{22}(z) &= \frac{[P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]{\rm Tr}[\Lambda_{22}(p,z)\mathcal{P}]_{\rm tree}}{([P^{ab}_{ij}\Lambda^{ab,ij}_{11}(p,z)]{\rm Tr}[\Lambda_{22}(p,z)\mathcal{P}]-[P^{ab}_{ij}\Lambda^{ab,ij}_{21}(p,z)]{\rm Tr}[\Lambda_{12}(p,z)\mathcal{P}])}.
\end{aligned}
\tag{41}
$$

The renormalized hadron matrix element of $O_R(z,0)$ are then given by

$$
\begin{aligned}
\langle P|O^{(n)}_{g,R}(z,0)|P\rangle &= Z_{11}(z)\langle P|O^{(n)}_{g}(z,0)|P\rangle + Z_{12}(z)/z\langle P|O^{s}_{q}(z,0)|P\rangle,\\
\langle P|O^{s}_{q,R}(z,0)|P\rangle &= Z_{22}(z)\langle P|O^{s}_{q}(z,0)|P\rangle + zZ_{21}(z)\langle P|O^{(n)}_{g}(z,0)|P\rangle.
\end{aligned}
\tag{42}
$$

The renormalized quasi-PDF $\tilde{q}_R(x,P_z,\mu_R)$ in the RI/MOM scheme can be obtained by a Fourier transform given in Eqs. (4) and (7), respectively. Note that we can take the continuum limit $a\to 0$ in $\tilde{h}_R$ since all terms singular in a have been removed by the renormalization procedure. This means that the factorization of the renormalized matrix element can be studied in the continuum, as will be done in the next subsection. An derivation of the form of the RI/MOM counterterm is given in Appendix A.

E. Factorization

In Ref. [53], we have given a general factorization formula for the quark and gluon quasi-PDFs in the presence of mixing. In this subsection, we give a detailed derivation of it using the operator product expansion (OPE), along the same line as that used for the isovector quark quasi-PDF [10]. For illustration purposes, we choose $\Gamma=\gamma^t$ for the quark quasi-PDF and $O^{(4)}_g$ for the gluon quasi-PDF. The derivation for other operators follows straightforwardly from what is presented below.

The renormalized quark and gluon nonlocal operator matrix elements can be expanded in terms of gauge-invariant local operator matrix elements to the leading-twist approximation as

$$
\begin{aligned}
\tilde{h}_{q_i,R}(z,P^z,\mu) \simeq{}& \frac{1}{2P^t}\sum_{n=1}^{\infty}\frac{(-iz)^{n-1}}{(n-1)!}\Big[C^{(n-1)}_{q_iq_j}(\mu^2z^2)\langle P|n^t_{\mu_1}n_{\mu_2}...n_{\mu_n}O^{\mu_1\cdots\mu_n}_{q_j}(\mu)|P\rangle\\
&+C^{(n-1)}_{qg}(\mu^2z^2)\langle P|n^t_{\mu_1}n_{\mu_2}...n_{\mu_n}O^{\mu_1\cdots\mu_n}_{g}(\mu)|P\rangle\Big],\\
\tilde{h}_{g,R}(z,P^z,\mu) \simeq{}& \frac{1}{(P^z)^2}\sum_{n=2}^{\infty}\frac{(-iz)^{n-2}}{(n-2)!}\Big[C^{(n-2)}_{gg}(\mu^2z^2)\langle P|n_{\mu_1}...n_{\mu_n}O^{\mu_1\cdots\mu_n}_{g}(\mu)|P\rangle\\
&+C^{(n-2)}_{gq}(\mu^2z^2)\langle P|n_{\mu_1}...n_{\mu_n}O^{\mu_1\cdots\mu_n}_{q_j}(\mu)|P\rangle\Big],
\end{aligned}
\tag{43}
$$

where we have introduced extra normalization factors so that the two matrix elements have the same mass dimension. For simplicity, we have also denoted all renormalization scales with μ. $n^t_\mu=(1,0,0,0)$ and $n^\mu=(0,0,0,-1)$, $C^{(n)}_{q_iq_j}=\delta_{ij}+\frac{\alpha_s}{2\pi}C^{(n),1}_{q_iq_j}+\mathcal{O}(\alpha_s^2)$, $C^{(n)}_{\{qg,gq\}}=\frac{\alpha_s}{2\pi}C^{(n),1}_{\{qg,gq\}}+\mathcal{O}(\alpha_s^2)$ and $C^{(n)}_{gg}=1+\frac{\alpha_s}{2\pi}C^{(n),1}_{gg}+\mathcal{O}(\alpha_s^2)$ denote the Wilson coefficients. $O^{\mu_1\cdots\mu_n}_{q_j}$ and $O^{\mu_1\cdots\mu_n}_{g}$ are the renormalized symmetric traceless twist-2 quark and gluon operators

$$
\begin{aligned}
O^{\mu_1\cdots\mu_n}_{q_j} &= Z^n_{q_j}\big[\bar{q}_j(0)\gamma^{\{\mu_1}iD^{\mu_2}\cdots iD^{\mu_n\}}q_j(0)-{\rm trace}\big],\\
O^{\mu_1\cdots\mu_n}_{g} &= Z^n_{g}\big[F^{\{\mu_1\nu}(0)iD^{\mu_2}\cdots iD^{\mu_{n-1}}F_\nu^{\ \mu_n\}}(0)-{\rm trace}\big],
\end{aligned}
\tag{44}
$$

where $\{\cdots\}$ denotes a symmetrization of the enclosed indices. Their matrix elements are related to the moments of quark and gluon PDF, respectively

$$
\begin{aligned}
\langle P|O^{\mu_1\cdots\mu_n}_{q_j}|P\rangle &= 2a_{q_j,n}(\mu)(P^{\mu_1}\cdots P^{\mu_n}-{\rm trace}),\\
\langle P|O^{\mu_1\cdots\mu_n}_{g}|P\rangle &= 2a_{g,n}(\mu)(P^{\mu_1}\cdots P^{\mu_n}-{\rm trace}),
\end{aligned}
\tag{45}
$$

with

$$a_{q_j,n}(\mu) = \int_{-1}^{1} dx\, x^{n-1} f_{q_j/H}(x,\mu), \qquad a_{g,n}(\mu) = \frac{1}{2}\int_{-1}^{1} dx\, x^{n-1} f_{g/H}(x,\mu). \tag{46}$$

Owing to the symmetry of the gluon PDF, $a_{g,n}$ does not vanish only for even n.

Let us first consider $\tilde{h}_{q_i,R}(z,P^z,\mu)$. Ignoring all trace terms, we can write

$$\begin{aligned}
\tilde{h}_{q_i,R}(z,P^z,\mu) &= \sum_{n=1}^{\infty} \frac{(-izP^z)^{n-1}}{(n-1)!}\left[C_{q_iq_j}^{(n-1)}(\mu^2z^2)a_{q_j,n}(\mu) + C_{qg}^{(n-1)}(\mu^2z^2)a_{g,n}(\mu)\right] \\
&= \sum_{n=1}^{\infty} \frac{(-i\nu)^{n-1}}{(n-1)!}\left[C_{q_iq_j}^{(n-1)}(\mu^2z^2)\int_{-1}^{1} dx\, x^{n-1} f_{q_j/H}(x,\mu) + \frac{C_{qg}^{(n-1)}(\mu^2z^2)}{2}\int_{-1}^{1} dx\, x^{n-1} f_{g/H}(x,\mu)\right] \\
&= \sum_{n=1}^{\infty} \frac{(-i\nu)^{n-1}}{(n-1)!} C_{q_iq_j}^{(n-1)}(\mu^2z^2)\int_{-1}^{1} dx\, x^{n-1} \int \frac{d\nu'}{2\pi} e^{ix\nu'} h_{q_i}(\nu',\mu) \\
&+ \sum_{n=2,\text{even}}^{\infty} \frac{(-i\nu)^{n-1}}{(n-1)!} \frac{C_{qg}^{(n-1)}(\mu^2z^2)}{2}\int_{-1}^{1} dx\, x^{n-1} \int \frac{d\nu'}{2\pi} e^{ix\nu'} h_{g}(\nu',\mu),
\end{aligned} \tag{47}$$

where we have introduced the Ioffe-time $\nu = -z\cdot P = zP^z$ and $\nu' = -\xi\cdot P = -P^+\xi^-$, $h_{q_i/g,R}$ denote the coordinate space matrix elements used to define the quark and gluon PDFs at lightlike separation $\xi^2 = 0$. Defining

$$\begin{aligned}
\int \frac{d\nu}{2\pi} e^{iu\nu} \sum_{n=1}^{\infty} \frac{(-i\nu)^{n-1}}{(n-1)!} C_{q_iq_j}^{(n-1)}(\mu^2z^2) &= \mathcal{C}_{q_iq_j}(u,\mu^2z^2), \\
\int \frac{d\nu}{2\pi} e^{iu\nu} \sum_{n=2,\text{even}}^{\infty} \frac{(-i\nu)^{n-1}}{(n-1)!} \frac{C_{qg}^{(n-1)}(\mu^2z^2)}{2} &= \mathcal{C}_{qg}(u,\mu^2z^2),
\end{aligned} \tag{48}$$

with u being in the range $(-1,1)$ [38, 93], we then have

$$\begin{aligned}
\tilde{h}_{q_i,R}(z,P^z,\mu) &= \int_{-1}^{1} dx \int_{-1}^{1} du\, e^{-iux\nu}\left[\mathcal{C}_{q_iq_j}(u,\mu^2z^2)\int \frac{d\nu'}{2\pi} e^{ix\nu'} h_{q_i}(\nu',\mu) + \mathcal{C}_{qg}(u,\mu^2z^2)\int \frac{d\nu'}{2\pi} e^{ix\nu'} h_{g}(\nu',\mu)\right] \\
&= \int_{-1}^{1} du\, \mathcal{C}_{q_iq_j}(u,\mu^2z^2) h_{q_i}(u\nu,\mu) + \int_{-1}^{1} du\, \mathcal{C}_{qg}(u,\mu^2z^2) h_{g}(u\nu,\mu).
\end{aligned} \tag{49}$$

This is the general factorization of the coordinate space matrix element in the presence of mixing. To convert it to the factorization of quasi-PDFs, we need a Fourier transform of the above relation

$$\begin{aligned}
&\tilde{f}_{q_i/H}(x,P^z,\mu) = P^z \int \frac{dz}{2\pi} e^{izxP^z} \tilde{h}_{q_i,R}(z,P^z,\mu) \\
&= P^z \int \frac{dz}{2\pi} e^{izxP^z} \sum_{n=1}^{\infty} \frac{(-izP^z)^{n-1}}{(n-1)!}\left[C_{q_iq_j}^{(n-1)}(\mu^2z^2)\int_{-1}^{1} dy\, y^{n-1} f_{q_j/H}(y) + \frac{C_{qg}^{(n-1)}(\mu^2z^2)}{2}\int_{-1}^{1} dy\, y^{n-1} f_{g/H}(y)\right] \\
&= \int_{-1}^{1} dy \int dz'\,\delta(z'-zy)\int \frac{dzP^z}{2\pi} e^{izxP^z} \sum_{n=1}^{\infty} \frac{(-izyP^z)^{n-1}}{(n-1)!}\left[C_{q_iq_j}^{(n-1)}(\mu^2z^2) f_{q_j/H}(y) + \frac{C_{qg}^{(n-1)}(\mu^2z^2)}{2} f_{g/H}(y)\right] \\
&= \int_{-1}^{1} \frac{dy}{|y|} \int \frac{d\nu'}{2\pi}\, e^{i\nu' x/y} \sum_{n=1}^{\infty} \frac{(-i\nu')^{n-1}}{(n-1)!}\left[C_{q_iq_j}^{(n-1)}\Big(\frac{\mu^2\nu'^2}{y^2(P^z)^2}\Big) f_{q_j/H}(y) + \frac{1}{2}C_{qg}^{(n-1)}\Big(\frac{\mu^2\nu'^2}{y^2(P^z)^2}\Big) f_{g/H}(y)\right] \\
&= \int_{-1}^{1} \frac{dy}{|y|}\left[C_{q_iq_j}\Big(\frac{x}{y},\frac{\mu}{yP^z}\Big) f_{q_j/H}(y) + C_{qg}\Big(\frac{x}{y},\frac{\mu}{yP^z}\Big) f_{g/H}(y)\right],
\end{aligned} \tag{50}$$

where we have defined

$$\begin{aligned}
C_{q_iq_j}\Big(\frac{x}{y},\frac{\mu}{yP^z}\Big) &= \int \frac{d\nu'}{2\pi}\, e^{i\nu' x/y} \sum_{n=1}^{\infty} \frac{(-i\nu')^{n-1}}{(n-1)!} C_{q_iq_j}^{(n-1)}\Big(\frac{\mu^2\nu'^2}{y^2(P^z)^2}\Big), \\
C_{qg}\Big(\frac{x}{y},\frac{\mu}{yP^z}\Big) &= \int \frac{d\nu'}{2\pi}\, e^{i\nu' x/y} \sum_{n=1}^{\infty} \frac{(-i\nu')^{n-1}}{(n-1)!} C_{qg}^{(n-1)}\Big(\frac{\mu^2\nu'^2}{y^2(P^z)^2}\Big)/2.
\end{aligned} \tag{51}$$

Now let us turn to $\tilde{h}_{q_i,R}(z,P^z,\mu)$. By ignoring all trace terms, one can write as before

$$
\begin{aligned}
\tilde{h}_{g,R}(z,P^z,\mu) &= \sum_{n=2}^{\infty}\frac{(-i\nu)^{n-2}}{(n-2)!}\Big[C_{gg}^{(n-2)}(\mu^2z^2)\int_{-1}^{1}dx\,x^{n-1}f_{g/H}(x,\mu)+2C_{gq}^{(n-2)}(\mu^2z^2)\int_{-1}^{1}dx\,x^{n-1}f_{q/H}(x,\mu)\Big]\\
&= \sum_{n=2,\text{even}}^{\infty}\frac{(-i\nu)^{n-2}}{(n-2)!}C_{gg}^{(n-2)}(\mu^2z^2)\int_{-1}^{1}dx\,x^{n-1}\int\frac{d\nu'}{2\pi}e^{ix\nu'}h_g(\nu',\mu)\\
&+\sum_{n=2}^{\infty}\frac{(-i\nu)^{n-2}}{(n-2)!}2C_{gq}^{(n-2)}(\mu^2z^2)\int_{-1}^{1}dx\,x^{n-1}\int\frac{d\nu'}{2\pi}e^{ix\nu'}h_q(\nu',\mu).
\end{aligned}
\tag{52}
$$

Defining

$$
\begin{aligned}
\int\frac{d\nu}{2\pi}e^{iu\nu}\sum_{n=2,\text{even}}^{\infty}\frac{(-i\nu)^{n-1}}{(n-2)!}C_{gg}^{(n-2)}(\mu^2z^2) &= -i\mathcal{C}_{gg}(u,\mu^2z^2),\\
\int\frac{d\nu}{2\pi}e^{iu\nu}\sum_{n=2}^{\infty}\frac{(-i\nu)^{n-1}}{(n-2)!}2C_{gq}^{(n-2)}(\mu^2z^2) &= -i\mathcal{C}_{gq}(u,\mu^2z^2),
\end{aligned}
\tag{53}
$$

we then have the following factorization in coordinate space

$$
\begin{aligned}
\tilde{h}_{g,R}(z,P^z,\mu) &= \int_{-1}^{1}dx\int_{-1}^{1}du\,\frac{e^{-iux\nu}}{\nu}\Big[\mathcal{C}_{gg}(u,\mu^2z^2)\int\frac{d\nu'}{2\pi}e^{ix\nu'}h_g(\nu',\mu)+\mathcal{C}_{gq}(u,\mu^2z^2)\int\frac{d\nu'}{2\pi}e^{ix\nu'}h_q(\nu',\mu)\Big]\\
&= \int_{-1}^{1}du\,\frac{\mathcal{C}_{gg}(u,\mu^2z^2)}{\nu}h_g(u\nu,\mu)+\int_{-1}^{1}du\frac{\mathcal{C}_{gq}(u,\mu^2z^2)}{\nu}h_q(u\nu,\mu).
\end{aligned}
\tag{54}
$$

The factorization in mometum space reads

$$
\begin{aligned}
&\tilde{f}_{g/H}(x,P^z,\mu) = P^z\int\frac{dz}{2\pi x}e^{izxP^z}\tilde{h}_{g,R}(z,P^z,\mu)\\
&= \int\frac{dzP^z}{2\pi x}e^{izxP^z}\sum_{n=2}^{\infty}\frac{(-izP^z)^{n-2}}{(n-2)!}\Big[C_{gg}^{(n-2)}(\mu^2z^2)\int_{-1}^{1}dy\,y^{n-1}f_{g/H}(y)+2C_{gq}^{(n-2)}(\mu^2z^2)\int_{-1}^{1}dy\,y^{n-1}f_{q/H}(y)\Big]\\
&= \int_{-1}^{1}\frac{dy}{|y|}\frac{y}{x}\int\frac{d\nu'}{2\pi}\,e^{i\nu'x/y}\sum_{n=2}^{\infty}\frac{(-i\nu')^{n-2}}{(n-2)!}\Big[C_{gg}^{(n-2)}\Big(\frac{\mu^2\nu'^2}{y^2(P^z)^2}\Big)f_{g/H}(y)+2C_{gq}^{(n-2)}\Big(\frac{\mu^2\nu'^2}{y^2(P^z)^2}\Big)f_{q/H}(y)\Big]\\
&= \int_{-1}^{1}\frac{dy}{|y|}\Big[C_{q_iq_j}\Big(\frac{x}{y},\frac{\mu}{yP^z}\Big)f_{q_j/H}(y)+C_{qg}\Big(\frac{x}{y},\frac{\mu}{yP^z}\Big)f_{g/H}(y)\Big],
\end{aligned}
\tag{55}
$$

where we have defined

$$
\begin{aligned}
C_{gg}\Big(\frac{x}{y},\frac{\mu}{yP^z}\Big) &= \frac{y}{x}\int\frac{d\nu'}{2\pi}\,e^{i\nu'x/y}\sum_{n=2}^{\infty}\frac{(-i\nu')^{n-2}}{(n-2)!}C_{gg}^{(n-2)}\Big(\frac{\mu^2\nu'^2}{y^2(P^z)^2}\Big),\\
C_{gq}\Big(\frac{x}{y},\frac{\mu}{yP^z}\Big) &= \frac{y}{x}\int\frac{d\nu'}{2\pi}\,e^{i\nu'x/y}\sum_{n=2}^{\infty}\frac{(-i\nu')^{n-2}}{(n-2)!}2C_{gq}^{(n-2)}\Big(\frac{\mu^2\nu'^2}{y^2(P^z)^2}\Big).
\end{aligned}
\tag{56}
$$

Restoring all renormalization scales, the general factorization of the quark and gluon quasi-PDFs reads

$$
\begin{aligned}
\tilde{f}_{g/H}^{(n)}(x,P^z,p_z^R,\mu_R) &= \int_{-1}^{1}\frac{dy}{|y|}\Big[C_{gg}\Big(\frac{x}{y},\frac{\mu_R}{p_z^R},\frac{yP^z}{\mu},\frac{yP^z}{p_z^R}\Big)f_{g/H}(y,\mu)+C_{gq_j}\Big(\frac{x}{y},\frac{\mu_R}{p_z^R},\frac{yP^z}{\mu},\frac{yP^z}{p_z^R}\Big)f_{q_j/H}(y,\mu)\Big]\\
&+\mathcal{O}\Big(\frac{M^2}{(P^z)^2},\frac{\Lambda_{\rm QCD}^2}{(P^z)^2}\Big),\\
\tilde{f}_{q_i/H}(x,P^z,p_z^R,\mu_R) &= \int_{-1}^{1}\frac{dy}{|y|}\Big[C_{q_iq_j}\Big(\frac{x}{y},\frac{\mu_R}{p_z^R},\frac{yP^z}{\mu},\frac{yP^z}{p_z^R}\Big)f_{q_j/H}(y,\mu)+C_{qg}\Big(\frac{x}{y},\frac{\mu_R}{p_z^R},\frac{yP^z}{\mu},\frac{yP^z}{p_z^R}\Big)f_{g/H}(y,\mu)\Big]\\
&+\mathcal{O}\Big(\frac{M^2}{(P^z)^2},\frac{\Lambda_{\rm QCD}^2}{(P^z)^2}\Big),
\end{aligned}
\tag{57}
$$

where a summation of j over all quark flavors is implied. The factorization for the polarized quasi-PDFs has the same form as Eq. (57), with all unpolarized distributions being replaced by the polarized ones and also different hard coefficients. It is worthwhile to point out that the higher-twist contributions shall behave like $1/[x^2(1-x)(P^z)^2]$ instead of $1/(P^z)^2$, as demonstrated in Ref. [55].

III. ONE-LOOP MATCHING FOR UNPLOLARIZED QUASI-PDFS IN RI/MOM SCHEME

As shown in the previous section, when the hadron momentum P^z is much larger than the hadronic scale, the highet-twist contributions get suppressed (except for very small/large x), the quasi-PDFs can be factorized into the lightcone PDFs with perturbatively calculable hard matching coefficients. In this section, we present the one-loop calculation of the hard matching coefficients for unpolarized quark and gluon quasi-PDFs in the presence of mixing. The polarized case will be presented in the next section. Our result is obtained in the RI/MOM scheme, which can be used to connect the RI/MOM renormalized quasi-PDFs to the PDFs in $\overline{\text{MS}}$ scheme. Since the matching depends on UV physics only and not on the external state, we can calculate it in quark or gluon external states $|q(p)\rangle$, $|g(p)\rangle$. The infrared (IR) divergences can be regularized using their offshellness.

A. Gluon in Gluon

Let us start with the gluon matrix element of the gluon quasi-PDF operator, which is the most complicated among all calculations. At tree-level one finds:

$$x\tilde{f}^{(n,0)}_{g/g}(x,\rho)=\delta(x-1),\quad yf^{(0)}_{g/g}(y,\mu)=\delta(y-1), \tag{58}$$

with $\rho=-p^2/p_z^2$. As before, we have ignored the crossed terms which can be obtained from $\{\tilde{f},f\}(x)=-\{\tilde{f},f\}(-x)$. Ignoring such terms has no impact on the extraction of the matching coefficient. The above results lead to the following tree-level matching coefficient:

$$C^{(0)}_{gg}(x/y)=\delta(x/y-1). \tag{59}$$

At one-loop level, the partonic quasi-PDF can be written as follows:

$$x\tilde{f}^{(n)}_{g/g}(x,\rho)=\left[x\tilde{f}^{(n)}_{g/g}(x,\rho)\right]_{+}+\tilde{c}^{(n)}\delta(x-1), \tag{60}$$

with $n=1,2,3,4$, and the "+" subscript denotes the usual plus-prescription

$$[f(x)]_{+}=f(x)-\delta(1-x)\int dx'\,f(x'). \tag{61}$$

Integrating Eq. (60) over the momentum fraction, one arrives at

$$\int_0^1 dx\,x\tilde{f}^{(n)}_{g/g}(x)=\tilde{c}^{(n)}, \tag{62}$$

which corresponds to the matrix element of local operators

$$\tilde{c}^{(n)}=\frac{1}{p_z^2}N^{(n)}\langle g(p)|O^{(n)}(0,0)|g(p)\rangle. \tag{63}$$

Before we proceed, let us make a few general remarks on the calculation to follow.

- The above equations apply to bare operator matrix elements. One can write down similar expressions for the renormalized ones. In our calculation of the matching coefficients, the PDF is renormalized in $\overline{\text{MS}}$ scheme while the quasi-PDF is renormalized in the RI/MOM scheme. The renormalized local operator matrix elements in the two schemes differ from each other in general.

- The above matrix element in an offshell gluon state can mix with matrix elements of gauge variant operators. To illustrate this point, it is worthwhile to consider the UV divergence from the matrix element of the local gluon operator $F^{\mu\alpha}(0)F^{\nu\beta}(0)$ in an offshell gluon state

$$
\begin{aligned}
&\langle p,\rho|F^{\mu\alpha}(0)F^{\nu\beta}(0)|p,\sigma\rangle = -\frac{\alpha_s C_A}{4\pi\epsilon}\\
&\times\Bigg\{\frac{1}{12}p^2\Big(9g^{\alpha\nu}g^{\beta\sigma}g^{\mu\rho}-9g^{\alpha\beta}g^{\mu\rho}g^{\nu\sigma}-g^{\alpha\nu}g^{\beta\rho}g^{\mu\sigma}+g^{\alpha\beta}g^{\mu\sigma}g^{\nu\rho}+g^{\alpha\sigma}\left(g^{\beta\rho}g^{\mu\nu}-g^{\beta\mu}g^{\nu\rho}\right)\\
&+g^{\alpha\rho}\left(9g^{\beta\mu}g^{\nu\sigma}-9g^{\beta\sigma}g^{\mu\nu}\right)-2g^{\alpha\nu}g^{\beta\mu}g^{\rho\sigma}+2g^{\alpha\beta}g^{\mu\nu}g^{\rho\sigma}\Big)\\
&+\frac{1}{6}p^\mu p^\nu\left(4g^{\alpha\sigma}g^{\beta\rho}+10g^{\alpha\rho}g^{\beta\sigma}-7g^{\alpha\beta}g^{\rho\sigma}\right)-\frac{1}{6}p^\beta p^\mu\left(4g^{\alpha\sigma}g^{\nu\rho}+10g^{\alpha\rho}g^{\nu\sigma}-7g^{\alpha\nu}g^{\rho\sigma}\right)\\
&-\frac{1}{6}p^\alpha p^\nu\left(10g^{\beta\sigma}g^{\mu\rho}+4g^{\beta\rho}g^{\mu\sigma}-7g^{\beta\mu}g^{\rho\sigma}\right)+\frac{1}{6}p^\alpha p^\beta\left(4g^{\mu\sigma}g^{\nu\rho}+10g^{\mu\rho}g^{\nu\sigma}-7g^{\mu\nu}g^{\rho\sigma}\right)\\
&-\frac{3}{4}p^\mu p^\rho\left(g^{\alpha\nu}g^{\beta\sigma}-g^{\alpha\beta}g^{\nu\sigma}\right)-\frac{3}{4}p^\nu p^\sigma\left(g^{\alpha\rho}g^{\beta\mu}-g^{\alpha\beta}g^{\mu\rho}\right)+\frac{3}{4}p^\alpha p^\rho\left(g^{\beta\sigma}g^{\mu\nu}-g^{\beta\mu}g^{\nu\sigma}\right)\\
&+\frac{3}{4}p^\beta p^\sigma\left(g^{\alpha\rho}g^{\mu\nu}-g^{\alpha\nu}g^{\mu\rho}\right)-\frac{1}{6}p^\mu p^\sigma\left(g^{\alpha\nu}g^{\beta\rho}-g^{\alpha\beta}g^{\nu\rho}\right)-\frac{1}{6}p^\nu p^\rho\left(g^{\alpha\sigma}g^{\beta\mu}-g^{\alpha\beta}g^{\mu\sigma}\right)\\
&+\frac{1}{6}p^\alpha p^\sigma\left(g^{\beta\rho}g^{\mu\nu}-g^{\beta\mu}g^{\nu\rho}\right)+\frac{1}{6}p^\beta p^\rho\left(g^{\alpha\sigma}g^{\mu\nu}-g^{\alpha\nu}g^{\mu\sigma}\right)+\frac{1}{6}p^\rho p^\sigma\left(g^{\alpha\nu}g^{\beta\mu}-g^{\alpha\beta}g^{\mu\nu}\right)\Bigg\}+\mathcal{O}(\epsilon^0), \qquad (64)
\end{aligned}
$$

with the cross-diagrams neglected. This leads to the following contributions to the UV divergences in $\tilde{c}^{(n)}$:

$$
\begin{aligned}
\tilde{c}^{(1,g)} &= \frac{\alpha_s C_A}{12\pi\epsilon}\frac{p^2}{p^2+p_z^2}+\mathcal{O}(\epsilon^0),\\
\tilde{c}^{(2,g)} &= -\frac{\alpha_s C_A}{12\pi\epsilon}\frac{p^2}{p_z^2}+\mathcal{O}(\epsilon^0),\\
\tilde{c}^{(3,g)} &= \mathcal{O}(\epsilon^0),\\
\tilde{c}^{(4,g)} &= \frac{\alpha_s C_A}{3\pi\epsilon}\frac{p^2}{p_z^2}+\mathcal{O}(\epsilon^0),
\end{aligned} \qquad (65)
$$

if a physical projection $P_{ij}^{ab}=\delta^{ab}g_{\perp,ij}/(2-D)$ is employed. As can be seen from the above equations, the UV divergences might depend on the offshellness of external gluons, which is a sign of the potential mixing with gauge variant operators. It is interesting to note that the UV divergence of $\tilde{c}^{(3,g)}$ is independent of p^2. This is because it corresponds to the tz component of the gluon energy momentum tensor for which all gauge variant operators to mix turn out to vanish [94, 95]. As we will see below, such a behavior is consistent with the asymptotic behavior at large x of the quasi-PDF defined with $O^{(3)}_{g,R}(z,0)$, which does not depend on p^2 either. This feature turns out to help achieve a better convergence in the implementation of the matching. Thus, in the following we will focus on $O^{(3)}_{g,R}(z,0)$, and present the one-loop matching calculation for the gluon quasi-PDF defined with this operator. For completeness and comparison purposes, the results for other definitions are also collected in Appendix B.

- In pure Yang-Mills theory, $O^{(3)}(0,0)$ does not renormalize, as shown by the results in Eq. (65) [2]. In QCD, quarks can enter the gluon diagrams relevant for the above calculation, but only through gluon wave function renormalization at one-loop level, and lead to the following contribution to $\tilde{c}^{(3,g)}$ and $c^{(3,g)}$ (the counterpart of $\tilde{c}^{(3,g)}$ for the gluon PDF) after renormalization

$$
\tilde{c}^{(3,g)}_{\text{RI/MOM}}=1-\frac{\alpha_s T_f}{3\pi}\left(-\ln\frac{-p^2}{\mu_R^2}\right),\quad c^{3,g}_{\overline{\text{MS}}}=1-\frac{\alpha_s T_f}{3\pi}\left(-\ln\frac{-p^2}{\mu^2}+\frac{5}{3}\right). \qquad (66)
$$

This will be needed in the calculation of the matching coefficient below.

[2] In general, one should be cautious about offshell gluons, as calculating the matrix element of gluon energy momentum tensor in offshell gluon states and then taking the onshell limit is rather tricky due to the existence of IR divergences [94, 96].

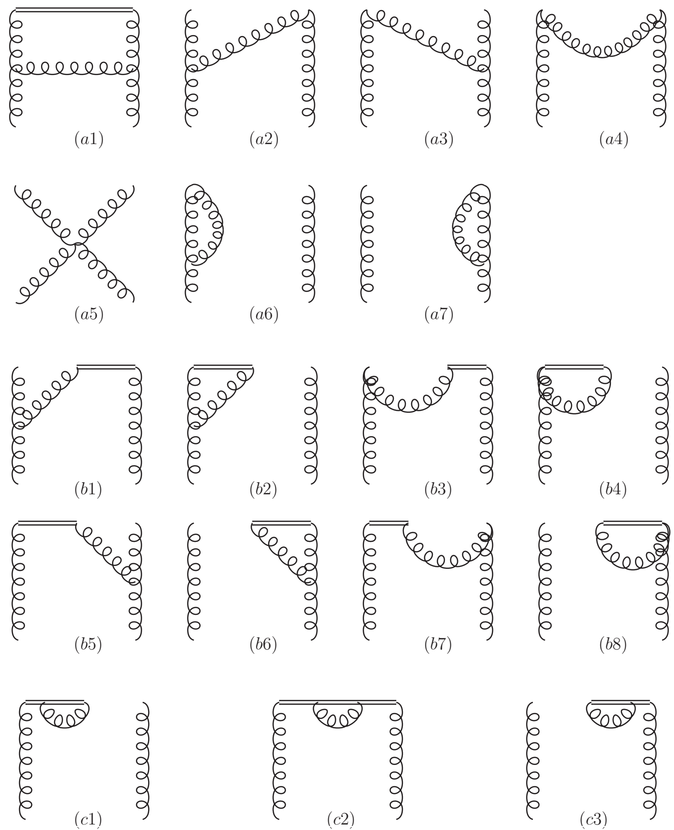

FIG. 2. One-loop diagrams for the gluon quasi-PDF. The gluon self-energy diagrams are not shown.

Now we present the one-loop results for the partonic quasi-PDF and PDF. The calculation is carried out in Landau gauge, and the steps are similar to those presented in Refs. [22, 23]. Given Eqs. (60) and (66), we only present the distribution part, *i.e.* the first term in Eq. (60). To this end, we need to calculate the one-loop matrix element of $O^{(3)}(z,0)$ in an offshell gluon state. The relevant Feynman diagrams are shown in Fig. 2, and the result reads

$$
\begin{aligned}
&\left[x\tilde{f}^{(3,1)}_{g/g}(x,\rho)\right]_+ \\
&=\frac{\alpha_s C_A}{2\pi}\begin{cases}
\Bigg[\frac{-(\rho-4)^2(\rho-1)+8(\rho+2)x^4-16(\rho+2)x^3-2\left(\rho^2+8\rho-24\right)x^2+\left(6\rho^2+20\rho-32\right)x}{8(\rho-1)^2(x-1)}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\
+\frac{4x^3}{(2x-1)(\rho+4x^2-4x)}+\frac{8x^4-16x^3-22x^2+34x-9}{4(\rho-1)(x-1)(2x-1)}-\frac{8x^3(x-1)}{(\rho+4x^2-4x)^2}+\frac{3(2x-1)x}{2(\rho-1)^2}-\frac{4x+1}{4(x-1)}\Bigg]_+, & x>1 \\
\Bigg[\frac{-(\rho-4)^2(\rho-1)+8(\rho+2)x^4-16(\rho+2)x^3-2\left(\rho^2+8\rho-24\right)x^2+\left(6\rho^2+20\rho-32\right)x}{8(\rho-1)^2(x-1)}\frac{1}{\sqrt{1-\rho}}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1-\rho}} \\
+\frac{-30x^2+34x-9}{4(\rho-1)(x-1)}+\frac{3\left(4x^3-4x^2+x\right)}{2(\rho-1)^2}+\frac{6x+1}{4(x-1)}\Bigg]_+, & 0<x<1 \\
\Bigg[-\frac{-(\rho-4)^2(\rho-1)+8(\rho+2)x^4-16(\rho+2)x^3-2\left(\rho^2+8\rho-24\right)x^2+\left(6\rho^2+20\rho-32\right)x}{8(\rho-1)^2(x-1)}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\
-\frac{4x^3}{(2x-1)(\rho+4x^2-4x)}+\frac{-8x^4+16x^3+22x^2-34x+9}{4(\rho-1)(x-1)(2x-1)}+\frac{8x^3(x-1)}{(\rho+4x^2-4x)^2}-\frac{3(2x-1)x}{2(\rho-1)^2}+\frac{4x+1}{4(x-1)}\Bigg]_+, & x<0.
\end{cases}
\end{aligned}
$$

As in the quark case [18, 49], the bare quasi-PDF result is obtained by taking the onshell limit $\rho\to 0$ of the above expression except where it has to be kept as an IR regulator

$$\left[x\tilde{f}^{(3,1)}_{g/g}(x,\rho\to 0)\right]_+ = \frac{\alpha_s C_A}{2\pi}\begin{cases}\left[\frac{2(1-x+x^2)^2}{x-1}\ln\frac{x-1}{x}+\frac{4x^3-6x^2+8x-5}{2(x-1)}\right]_+, & x>1\\ \left[\frac{2(1-x+x^2)^2}{x-1}\ln\frac{\rho}{4}+\frac{12x^4-24x^3+30x^2-17x+5}{2(x-1)}\right]_+, & 0<x<1\\ \left[-\frac{2(1-x+x^2)^2}{x-1}\ln\frac{x-1}{x}-\frac{4x^3-6x^2+8x-5}{2(x-1)}\right]_+, & x<0.\end{cases}$$

The renormalized lightcone PDF can be calculated analogously, and gives

$$\left[xf^{(1)}_{g/g}\left(x,\frac{\mu^2}{-p^2}\right)\right]_+ = \theta(x)\theta(1-x)\left\{\frac{\alpha_s C_A}{2\pi}\left[\frac{2(1-x+x^2)^2}{x-1}\ln\frac{-p^2x(1-x)}{\mu^2}+2x^3-2x^2+3x-2\right]_+ - \frac{\alpha_s C_A}{4\pi}\left[\frac{x}{1-x}\right]_+\right\}, \tag{67}$$

where the result in the first square bracket is derived in Feynman gauge.

The one-loop matching coefficient is given by the difference in the renormalized quasi-PDF and lightcone PDF

$$xC^{(3,1)}_{gg}(x,r,\frac{p^z}{\mu},\frac{p^z}{p^R_z}) = \left[x\tilde{f}^{(n,1)}_{g/g}(x,\rho\to 0) - xf^{(1)}_{g/g}\left(x,\frac{\mu^2}{-p^2}\right) - (x\tilde{f}^{(n,1)}_{g/g})_{C.T.}\right]_+ + \left(\tilde{c}^{(3,g)}_{\mathrm{RI/MOM}} - c^{3,g}_{\overline{\mathrm{MS}}}\right)\delta(x-1), \tag{68}$$

where the $\ln(-p^2)$ dependence in each individual term cancels out in the combination on the r.h.s., and the counterterm in the RI/MOM scheme is determined as (see Appendix A)

$$(x\tilde{f}^{(n,1)}_{g/g})_{C.T.} = \left|\frac{p_z}{p^R_z}\right| x\tilde{f}^{(n,1)}_{g/g}\left(\frac{p_z}{p^R_z}(x-1)+1, r\right) \tag{69}$$

with $r=\mu_R^2/(p^R_z)^2$.

B. Quark in Quark

This case has already been considered at one-loop in a comprehensive way in Ref. [49]. For completeness, we also quote the results here and briefly explain how it was obtained. The corresponding Feynman diagrams are given in Fig. 3.

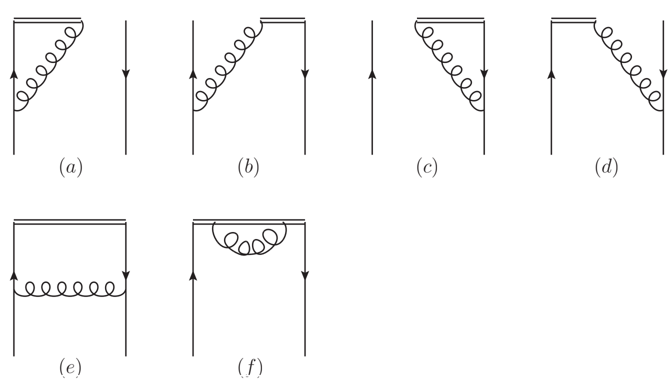

FIG. 3. One loop diagrams for the quark quasi-PDF. The quark self energy diagrams are not shown.

Owing to the offshellness of the external quark, the one-loop quark quasi-PDF contains two more Dirac structures apart from the tree-level one γ^t, and is given by the following projection [49]

$$\mathrm{Tr}\left[\left(\left[\tilde{f}^{(1)}_{q/q,t}(x,\rho)\right]_+\gamma^t + \left[\tilde{f}^{(1)}_{q/q,z}(x,\rho)\right]_+\frac{p^t}{p^z}\gamma^z + \left[\tilde{f}^{(1)}_{q/q,p}(x,\rho)\right]_+\frac{p^t\not{p}}{p^2}\right)\mathcal{P}\right], \tag{70}$$

where the coefficients of γ^t and γ^z read in Landau gauge

$$\tilde{f}^{(1)}_{q/q,t}(x,\rho)=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{2x^2}{(2x-1)(\rho+4x^2-4x)}+\frac{4x-3}{2(\rho-1)(2x-1)}-\frac{3}{2(x-1)}-\frac{\left(3\rho+4x^2+(\rho-8)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}(x-1)}, & x>1\\ \frac{4x-3}{2(\rho-1)}+\frac{3}{2(x-1)}-\frac{\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}}\left(3\rho+4x^2+(\rho-8)x\right)}{4(1-\rho)^{3/2}(x-1)}, & 0<x<1\\ -\frac{2x^2}{(2x-1)(\rho+4x^2-4x)}+\frac{3-4x}{2(\rho-1)(2x-1)}+\frac{3}{2(x-1)}+\frac{\left(3\rho+4x^2+(\rho-8)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}(x-1)}, & x<0\end{cases}$$

$$\tilde{f}^{(1)}_{q/q,z}(x,\rho)=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{\left(2\rho^2+3\rho+4(\rho+2)x^2-(13\rho+8)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}(x-1)}\\ \quad+\frac{2\left(3x^2-2x\right)}{(2x-1)^3(\rho+4x^2-4x)}-\frac{8\left(x^3-x^2\right)}{(2x-1)(\rho+4x^2-4x)^2}+\frac{8x^4-34x^3+40x^2-17x+2}{(\rho-1)(x-1)(2x-1)^3}+\frac{3(4x-3)}{2(\rho-1)^2(2x-1)}, & x>1\\ \frac{\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}}\left(2\rho^2+3\rho+4(\rho+2)x^2-(13\rho+8)x+4\right)}{4(1-\rho)^{5/2}(x-1)}+\frac{2-3x}{(\rho-1)(x-1)}+\frac{3(4x-3)}{2(\rho-1)^2}, & 0<x<1\\ -\frac{\left(2\rho^2+3\rho+4(\rho+2)x^2-(13\rho+8)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}(x-1)}\\ \quad-\frac{2\left(3x^2-2x\right)}{(2x-1)^3(\rho+4x^2-4x)}+\frac{8\left(x^3-x^2\right)}{(2x-1)(\rho+4x^2-4x)^2}+\frac{-8x^4+34x^3-40x^2+17x-2}{(\rho-1)(x-1)(2x-1)^3}-\frac{3(4x-3)}{2(\rho-1)^2(2x-1)}, & x<0.\end{cases} \tag{71}$$

In Ref. [49], a so-called minimal projector has been used, which determines the bare quark quasi-PDF as

$$\left[\tilde{f}^{(1)}_{q/q}(x,\rho\to 0)\right]_+=\left[\tilde{f}^{(1)}_{q/q,t}(x,\rho\to 0)\right]_++\left[\tilde{f}^{(1)}_{q/q,z}(x,\rho\to 0)\right]_+, \tag{72}$$

with the following explicit form

$$\left[\tilde{f}^{(1)}_{q/q}(x,\rho\to 0)\right]_+=\frac{\alpha_s C_F}{2\pi}\begin{cases}\left[\frac{x^2+1}{x-1}\ln\frac{x-1}{x}+1\right]_+, & x>1\\ \left[\frac{x^2+1}{x-1}\ln\frac{\rho}{4}+\frac{8x^2-8x+5}{2(x-1)}\right]_+, & 0<x<1\\ -\left[\frac{x^2+1}{x-1}\ln\frac{x-1}{x}+1\right]_+, & x<0.\end{cases}$$

Note that in this case there is no extra local term like $\tilde{c}^{(3,g)}_{\rm RI/MOM}$ above due to vector current conservation. The renormalized lightcone quark PDF has the following expression

$$\left[f^{(1)}_{q/q}\left(x,\frac{\mu^2}{-p^2}\right)\right]_+=\left\{\frac{\alpha_s C_F}{2\pi}\left[\frac{x^2+1}{x-1}\ln\frac{-p^2(1-x)x}{\mu^2}+\frac{-5+10x-6x^2}{2(1-x)}\right]_+\right\}\theta(x)\theta(1-x), \tag{73}$$

The matching coefficient can then be extracted as

$$C^{(1)}_{qq}(x,r,\frac{p^z}{\mu},\frac{p^z}{p^R_z})=\left[\tilde{f}^{(1)}_{q/q}(x,\rho\to 0)-f^{(1)}_{q/q}\left(x,\frac{\mu^2}{-p^2}\right)-(\tilde{f}^{(n,1)}_{q/q})_{C.T.}\right]_+, \tag{74}$$

where again the $\ln(-p^2)$ dependence cancels out in the combination on the r.h.s., and the counterterm in the RI/MOM scheme is determined as:

$$(\tilde{f}^{(1)}_{q/q})_{C.T.}=\left|\frac{p_z}{p^R_z}\right|\tilde{f}^{(1)}_{q/q,t}\left(\frac{p_z}{p^R_z}(x-1)+1,r\right). \tag{75}$$

C. Gluon in Quark

Now we turn to the mixing contributions. Let us first consider the quark matrix element of the gluon quasi-PDF operator, whose one-loop diagram is given in Fig. 4.

To illustrate the kinematic dependence of the mixing terms, it is useful to begin with the one-loop quark matrix element of the matrix element as a normalization of Fig. 4:

$$\begin{aligned}\langle q|F^{\mu\alpha}(0)F^{\nu\beta}(0)|q\rangle=-\frac{\alpha_s C_F}{12\pi\epsilon}\bar{u}(p)\Big(&-\gamma^\mu p^\beta g^{\alpha\nu}+\gamma^\alpha p^\beta g^{\mu\nu}+\gamma^\beta p^\alpha g^{\mu\nu}-\gamma^\beta p^\mu g^{\alpha\nu}\\ &+\gamma^\nu\left(p^\mu g^{\alpha\beta}-p^\alpha g^{\beta\mu}\right)+\gamma^\mu p^\nu g^{\alpha\beta}-\gamma^\alpha p^\nu g^{\beta\mu}+\not{p}\left(g^{\alpha\nu}g^{\beta\mu}-g^{\alpha\beta}g^{\mu\nu}\right)\Big)u(p).\end{aligned} \tag{76}$$

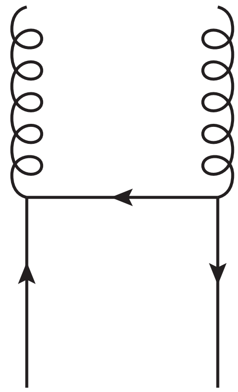

FIG. 4. One-loop diagram for the quark matrix element of the gluon quasi-PDF operator.

For $O^{(3)}_{g,R}(0,0)$, we have

$$\langle q|O^{(3)}_{g,R}(0,0)|q\rangle = \frac{\alpha_s C_F}{6\pi\epsilon}\bar{u}(p)[p^t\gamma^z + p^z\gamma^t]u(p) + \mathcal{O}(\epsilon^0). \tag{77}$$

As the tz component of the gluon energy momentum tensor, $O^{(3)}_{g,R}(0)$ in general mixes with the same component of the quark contribution

$$T^{tz}_q = \frac{1}{2}i\bar{\psi}iD^{(t}\gamma^{z)}\psi + \frac{1}{2}i\bar{\psi}i\overleftarrow{D}^{(t}\gamma^{z)}\psi, \tag{78}$$

where $(\cdots)$ denotes an antisymmetrization of the enclosed indices. The above operator has the same momentum dependence as Eq. (77) when sandwiched in a quark state.

The renormalized mixing contribution from the lightcone gluon PDF has the following form

$$xf^{(1)}_{g/q}\left(x, \frac{\mu^2}{-p^2}\right) = \frac{\alpha_s C_F}{2\pi}\left[(1+(1-x)^2)\ln\frac{\mu^2}{-p^2 x(1-x)} + x(1-x) - 2\right]. \tag{79}$$

For the quasi-PDF, we follow the decomposition as in the quark case:

$$\mathrm{Tr}\left[\left(xf^{(n,1)}_{g/q,t}(x,\rho)\gamma^t + xf^{(n,1)}_{g/q,z}(x,\rho)\frac{p^t}{p^z}\gamma^z + xf^{(n,1)}_{g/q,p}(x,\rho)\frac{p^t\not{p}}{p^2}\right)\mathcal{P}\right], \tag{80}$$

and choose the projector $\mathcal{P}$ such that it projects out the coefficients of both γ^t and γ^z. We therefore have

$$x\tilde{f}^{(n,1)}_{g/q} = xf^{(n,1)}_{g/q,t} + xf^{(n,1)}_{g/q,z}. \tag{81}$$

For the $O^{(3)}_{g,R}$, the result reads

$$x\tilde{f}^{(3,1)}_{g/q}(x,\mu,P^z) = \frac{\alpha_s C_F}{2\pi}\begin{cases} -\frac{5\rho^2-10\rho+(8\rho+4)x^2-4(\rho+2)x+8}{4(\rho-1)^2}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\ -\frac{(\rho-4)\rho+8(2\rho+1)x^3-4\left(\rho^2+2\rho+6\right)x^2+2\left(3\rho^2-2\rho+8\right)x}{2(1-\rho)^2(\rho+4x^2-4x)}, & x>1 \\ -\frac{5\rho^2-10\rho+(8\rho+4)x^2-4(\rho+2)x+8}{4(\rho-1)^2}\frac{1}{\sqrt{1-\rho}}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1-\rho}} \\ -\frac{(2x-1)(\rho+2(\rho+2)x-4)}{2(1-\rho)^2}, & 0<x<1 \\ \frac{5\rho^2-10\rho+(8\rho+4)x^2-4(\rho+2)x+8}{4(\rho-1)^2}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\ +\frac{(\rho-4)\rho+8(2\rho+1)x^3-4\left(\rho^2+2\rho+6\right)x^2+2\left(3\rho^2-2\rho+8\right)x}{2(1-\rho)^2(\rho+4x^2-4x)}, & x<0. \end{cases}$$

In the limit $\rho \to 0$, we have for the bare quasi-PDF

$$x\tilde{f}^{(3,1)}_{g/q}(x,\mu,P^z) = \frac{\alpha_s C_F}{2\pi}\begin{cases} -\left(1+(1-x)^2\right)\ln\frac{x-1}{x} - x + 2, & x>1 \\ -\left(1+(1-x)^2\right)\ln\frac{\rho}{4} - 4x^2 + 6x - 2, & 0<x<1 \\ \left(1+(1-x)^2\right)\ln\frac{x-1}{x} + x - 2, & x<0. \end{cases}, \tag{82}$$

In the limit $x \to \infty$, the above expression behaves asymptotically as

$$x\tilde{f}^{(3,1)}_{g/q}(x,\mu,P^z) \to \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{2}+\frac{4}{3x}\right). \tag{83}$$

If one integrates over the momentum fraction with DR, it is straightforward to find that the above behavior is consistent with the local result in Eq. (77).

As before, the matching coefficient can be extracted as

$$xC^{(3,1)}_{g/q}(x,r,\frac{p^z}{\mu},\frac{p^z}{p^R_z}) = \left[x\tilde{f}^{(3,1)}_{g/q}(x,\rho\to 0) - xf^{(1)}_{g/q}\left(x,\frac{\mu^2}{-p^2}\right) - (x\tilde{f}^{(3,1)}_{g/q})_{C.T.}\right], \tag{84}$$

with the counterterm in the RI/MOM scheme determined as:

$$(x\tilde{f}^{(n,1)}_{g/q})_{C.T.} = \left|\frac{p_z}{p^R_z}\right| x\tilde{f}^{(n,1)}_{g/q}\left(\frac{p_z}{p^R_z}(x-1)+1, r\right). \tag{85}$$

D. Quark in Gluon

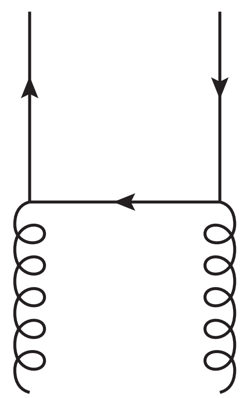

FIG. 5. One-loop diagram for the gluon matrix element of the quark quasi-PDF operator.

We come to the gluon matrix element of quark operator, and we start with the local matrix element, normalization of Fig. 5:

$$\langle g|\bar\psi\gamma^\mu\psi|g\rangle = \epsilon_\sigma\epsilon^*_\rho \frac{\alpha_s\left(-2p^\mu g^{\rho\sigma}+g^{\mu\sigma}p^\rho+g^{\mu\rho}p^\sigma\right)}{12\pi\epsilon}. \tag{86}$$

If $\mu = t$ and physical polarizations are used for the external gluons, one has the result:

$$\langle g|\bar\psi\gamma^t\psi|g\rangle = \frac{\alpha_s\sqrt{p^2+p_z^2}}{6\pi\epsilon}, \tag{87}$$

which has the same momentum dependence with the matrix element of $O^{(3)}_{g,R}$.

For the lightcone PDF, the result of the mixing diagram in Fig. 5 reads

$$f^{(1)}_{q/g}\left(x,\frac{\mu^2}{-p^2}\right) = \frac{\alpha_s T_f}{2\pi}\left[(x^2+(1-x)^2)\ln\frac{\mu^2}{-p^2x(1-x)} - 1\right], \tag{88}$$

while for the quasi-PDF one has:

$$\begin{aligned}
&\tilde{f}^{(1)}_{q/g}(x,\rho) \\
&= \frac{\alpha_s T_f}{2\pi}\begin{cases}
-\frac{\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4}{4(1-\rho)^{3/2}}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} - \frac{(2x-1)\left(-(\rho-4)\rho+4(\rho+2)x^2-4(\rho+2)x\right)}{2(1-\rho)^{3/2}(\rho+4x^2-4x)}, & x>1 \\
-\frac{\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4}{4(1-\rho)^{3/2}}\frac{1}{\sqrt{1-\rho}}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1-\rho}} - \frac{-\rho+12x^2-12x+4}{2(1-\rho)^{3/2}}, & 0<x<1 \\
\frac{\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4}{4(1-\rho)^{3/2}}\frac{1}{\sqrt{1-\rho}}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} - \frac{(2x-1)\left((\rho-4)\rho-4(\rho+2)x^2+4(\rho+2)x\right)}{2(1-\rho)^{3/2}(\rho+4x^2-4x)}, & x<0.
\end{cases}
\end{aligned} \tag{89}$$

Taking $\rho \to 0$ gives the bare quasi-PDF result

$$\tilde{f}^{(1)}_{q/g}(x, \rho \to 0) = \frac{\alpha_s T_f}{2\pi} \begin{cases} -(x^2 + (1-x)^2) \ln \frac{x-1}{x} - 2x + 1, & x > 1 \\ -(x^2 + (1-x)^2) \ln \frac{\rho}{4} - 6x^2 + 6x - 2, & 0 < x < 1 \\ (x^2 + (1-x)^2) \ln \frac{x-1}{x} + 2x - 1, & x < 0. \end{cases}$$

The matching coefficient is then given by

$$C^{(1)}_{qg}(x, r, \frac{p^z}{\mu}, \frac{p^z}{p^R_z}) = \left[\tilde{f}^{(1)}_{q/g}(x, \rho \to 0) - f^{(1)}_{q/g}\left(x, \frac{\mu^2}{-p^2}\right) - (\tilde{f}^{(n,1)}_{q/g})_{C.T.} \right], \tag{90}$$

with

$$(\tilde{f}^{(n,1)}_{q/g})_{C.T.} = \left| \frac{p_z}{p^R_z} \right| \tilde{f}^{(n,1)}_{q/g} \left(\frac{p_z}{p^R_z}(x-1) + 1, r \right). \tag{91}$$

IV. ONE-LOOP MATCHING FOR POLARIZED QUASI-PDF IN RI/MOM SCHEME

A. Gluon in Gluon

Now we turn to the polarized case. The calculation can be done in complete analogy to that presented in the previous section. As demonstrated in Ref. [53], to study the polarized gluon PDF

$$\Delta f_{g/H}(x, \mu) = i\epsilon_{\perp ij} \int \frac{d\xi^-}{2\pi x P^+} e^{-i\xi^- x P^+} \langle P | F^{+i}(\xi^- n_+) \mathcal{W}(\xi^- n_+, 0; L_{n_+}) F^{j+}(0) | P \rangle, \tag{92}$$

we may use the following three operators to define the corresponding quasi-PDF

$$\Delta O^1(z, 0) = i\epsilon_{\perp,ij} F^{ti}(z_2) \mathcal{W}(z_2, z_1) F^{tj}(z_1), \tag{93}$$

$$\Delta O^2(z, 0) = i\epsilon_{\perp,ij} F^{zi}(z_2) \mathcal{W}(z_2, z_1) F^{zj}(z_1), \tag{94}$$

$$\Delta O^3(z, 0) = i\epsilon_{\perp,ij} F^{ti}(z_2) \mathcal{W}(z_2, z_1) F^{zj}(z_1), \tag{95}$$

where $\epsilon_{\perp,ij}$ is the two-dimensional antisymmetric tensor:

$$\epsilon_{\perp,ij} = \epsilon_{\mu\nu ij} n_t^\mu n^\nu, \tag{96}$$

with the convention $\epsilon^{0123} = 1$. $n_t^\mu = (1, 0, 0, 0)$. The projection operator for the polarized gluon quasi-PDF is chosen as:

$$\mathcal{P}_{\perp,ij} = \frac{i}{D-2} \epsilon_{\mu\nu ij} n_t^\mu n^\nu. \tag{97}$$

As before, we decompose the polarized quasi-PDF as

$$x \Delta \tilde{f}^{(n)}_{g/g}(x) = [x \Delta \tilde{f}]_+ + \Delta \tilde{c}^{(n)} \delta(x-1). \tag{98}$$

Integrating over the momentum fraction

$$\int_{-\infty}^{\infty} dx\, x \Delta \tilde{f}^{(n)}_{g/g}(x) = \Delta \tilde{c}^{(n)}, \tag{99}$$

one obtains the matrix element of the corresponding local operators:

$$\Delta \tilde{c}^{(n)} = \frac{1}{(P^z)^2} \Delta N^{(n)} \langle g(P) | \Delta O^{(n)}_R(0, 0) | g(P) \rangle, \tag{100}$$

with

$$\Delta N^{(1)} = \frac{(P^z)^2}{(P^t)^2}, \;\; \Delta N^{(2)} = 1, \;\; \Delta N^{(3)} = \frac{P^z}{P^t}. \tag{101}$$

The local matrix elements have the following divergence structure

$$\Delta\tilde{c}^{(1)} = -\frac{\alpha_s C_A(P^2+6(P^z)^2)}{24\pi\epsilon(P^2+(P^z)^2)}, \tag{102}$$

$$\Delta\tilde{c}^{(2)} = -\frac{\alpha_s C_A(5P^2+6(P^z)^2)}{24\pi\epsilon(P^z)^2}, \tag{103}$$

$$\Delta\tilde{c}^{(3)} = -\frac{\alpha_s C_A}{4\pi\epsilon}, \tag{104}$$

where only the UV divergence of $\Delta\tilde{c}^{(3)}$ does not depend on the external momentum. For the same reason as the unpolarized case, we choose $\Delta O_g^{(3)}$ to define the polarized gluon quasi-PDF and present the corresponding one-loop matching.

Now we present the one-loop results. The light-cone PDF yields the following real contribution

$$x\Delta f^{(1)}_{g/g}(x,\mu) = \frac{\alpha_s C_A}{2\pi}\left\{\frac{x}{x-1}\left[\left(4x^2-6x+4\right)\ln\frac{-p^2(1-x)x}{\mu^2}+8x^2-11x+7+\frac{(1-\xi)}{2}\right]\right\}_+\theta(x)\theta(1-x), \tag{105}$$

whereas the quasi-PDF gives

$$x\Delta\tilde{f}^{(3,1)}_{g/g}(x,\rho) = \frac{\alpha_s C_A}{2\pi}\begin{cases} -\frac{\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x}{8(1-\rho)^{5/2}(x-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\ +\frac{4x^3}{(2x-1)(\rho+4x^2-4x)}+\frac{-8x^3-8x^2+14x-3}{4(\rho-1)(x-1)(2x-1)}-\frac{8\left(x^4-x^3\right)}{(\rho+4x^2-4x)^2}+\frac{3(2x-1)}{2(\rho-1)^2}-\frac{4x+1}{4(x-1)}, & x>1 \\ -\frac{\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x}{8(1-\rho)^{5/2}(x-1)}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}} \\ +\frac{3\left(4x^2-4x+1\right)}{2(\rho-1)^2}+\frac{-16x^3+8x^2+6x-3}{4(\rho-1)(x-1)}+\frac{6x+1}{4(x-1)}, & 0<x<1 \\ \frac{\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x}{8(1-\rho)^{5/2}(x-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} \\ -\frac{4x^3}{(2x-1)(\rho+4x^2-4x)}+\frac{8x^3+8x^2-14x+3}{4(\rho-1)(x-1)(2x-1)}+\frac{8\left(x^4-x^3\right)}{(\rho+4x^2-4x)^2}-\frac{3(2x-1)}{2(\rho-1)^2}+\frac{4x+1}{4(x-1)}, & x<0. \end{cases} \tag{106}$$

In the $\rho\to 0$ limit, the above result gets simplified

$$x\Delta\tilde{f}^{(3,1)}_{g/g}(x,\rho) = \frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{8x^2+4\left(2x^2-3x+2\right)x\ln\frac{x-1}{x}-8x+1}{2(x-1)}, & x>1 \\ \frac{4\left(2x^2-3x+2\right)x\ln\frac{\rho}{4}+20x^3-28x^2+15x-1}{2(x-1)}, & 0<x<1 \\ -\frac{8x^2+4\left(2x^2-3x+2\right)x\ln\frac{x-1}{x}-8x+1}{2(x-1)}, & x<0. \end{cases} \tag{107}$$

The virtual contribution is the same for the unpolarized and polarized gluon quasi-PDF, while the real contribution differs in the asymptotic limit as

$$x\Delta\tilde{f}^{(3,1)}_{g/g}-x\tilde{f}^{(3,1)}_{g/g}\to\frac{\alpha_s C_A}{2\pi}\left(\frac{2}{3}-\frac{1}{2x}\right). \tag{108}$$

Integrating over x in DR, this gives the UV divergence in Eq. (104) as expected.

The matching kernel can be written using the matching kernel for the unpolarized gluon quasi-PDF as

$$\begin{aligned} x\Delta C^{(3,1)}_{gg}(x,r,\frac{p^z}{\mu},\frac{pz}{p^R_z}) = {}& xC^{(3,1)}_{gg}(x,r,\frac{p^z}{\mu},\frac{pz}{p^R_z})+\Bigg[\left(x\Delta\tilde{f}^{(3,1)}_{g/g}(x,\rho\to 0)-x\tilde{f}^{(3,1)}_{g/g}(x,\rho\to 0)\right) \\ & -\left(x\Delta f^{(3,1)}_{g/g}\left(x,\frac{\mu^2}{-p^2}\right)-xf^{(3,1)}_{g/g}\left(x,\frac{\mu^2}{-p^2}\right)\right)-(x\Delta\tilde{f}^{(3,1)}_{g/g})_{C.T.}\Bigg], \end{aligned} \tag{109}$$

where again the $\ln(-p^2)$ dependence in each individual term cancels out in the combination on the r.h.s., and the counterterm in the RI/MOM scheme is determined as:

$$(\Delta x\tilde{f}^{(3,1)}_{g/g})_{C.T.} = \left|\frac{p_z}{p^R_z}\right|\left[x\Delta\tilde{f}^{(3,1)}_{g/g}\left(\frac{p_z}{p^R_z}(x-1)+1,r\right)-x\tilde{f}^{(3,1)}_{g/g}\left(\frac{p_z}{p^R_z}(x-1)+1,r\right)\right]. \tag{110}$$

B. Quark in Quark

For completeness, we also give the result for the polarized quark quasi-PDF and PDF defined as following

$$\Delta\tilde{f}_{q_i/H}(x,\mu,P^z) = \frac{P^t}{P^z}\int\frac{dz}{4\pi}e^{izxP^z}\langle P|\overline{q}_i(z)\gamma^z\gamma_5 W(z,0)q_i(0)|P\rangle, \tag{111}$$

$$\Delta f_{q_i/H}(x,\mu) = \int\frac{d\xi^-}{4\pi}e^{-i\xi^- xP^+}\langle P|\overline{q}_i(\xi^-)\gamma^+\gamma_5 W(\xi^-,0)q_i(0)|P\rangle. \tag{112}$$

The result for the polarized lightcone quark PDF are the same as that for the unpolarized one:

$$\Delta f^{(1)}_{q/q}\left(x,\frac{\mu^2}{-p^2}\right) = f^{(1)}_{q/q}\left(x,\frac{\mu^2}{-p^2}\right). \tag{113}$$

For the quasi-PDF, the one-loop result can be decomposed into:

$$\mathrm{Tr}\left[\left(\left[\Delta\tilde{f}^{(1)}_{q/q,t}\right]_+\gamma^t + \left[\Delta\tilde{f}^{(1)}_{q/q,z}\right]_+\gamma^z + \left[\Delta\tilde{f}^{(1)}_{q/q,p}\right]_+\frac{p^z\not{p}}{p^2}\right)\gamma_5\mathcal{P}\right]. \tag{114}$$

If the Dirac matrix in Eq. (111) is $\gamma^z\gamma^5$, $\Delta\tilde{f}^{(1)}_{q/q,t}$ vanishes, and we have:

$$\Delta\tilde{f}^{(1)}_{q/q,z}(x,\rho) = \frac{\alpha_s C_F}{2\pi}\begin{cases} -\frac{3\rho-2x^2-2}{2(1-\rho)^{3/2}(x-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} & \\ +\frac{4x^2}{(2x-1)(\rho+4x^2-4x)} + \frac{1-2x^2}{(\rho-1)(x-1)(2x-1)} - \frac{8\left(x^3-x^2\right)}{(\rho+4x^2-4x)^2} - \frac{3}{2(x-1)}, & x>1 \\ -\frac{3\rho-2x^2-2}{2(1-\rho)^{3/2}(x-1)}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}} + \frac{1-2x^2}{(\rho-1)(x-1)} + \frac{3}{2(x-1)}, & 0<x<1 \\ -\frac{-3\rho+2x^2+2}{2(1-\rho)^{3/2}(x-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}} & \\ -\frac{4x^2}{(2x-1)(\rho+4x^2-4x)} + \frac{2x^2-1}{(\rho-1)(x-1)(2x-1)} + \frac{8\left(x^3-x^2\right)}{(\rho+4x^2-4x)^2} + \frac{3}{2(x-1)}, & x<0. \end{cases} \tag{115}$$

In the limit $\rho\to 0$, it reduces to:

$$\Delta\tilde{f}^{(1)}_{q/q,z}(x,\rho)|_{\rho\to 0} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\left(x^2+1\right)\ln\frac{x-1}{x}+x-1}{x-1}, & x>1 \\ \frac{2\left(x^2+1\right)\ln\frac{\rho}{4}+4x^2+1}{2(x-1)}, & 0<x<1 \\ -\frac{\left(x^2+1\right)\ln\frac{x-1}{x}+x-1}{x-1}, & x<0 \end{cases} \tag{116}$$

The matching coefficient can then be extracted as

$$\Delta C^{(1)}_{qq}\left(x,r,\frac{p^z}{\mu},\frac{pz}{p_z^R}\right) = \left[\Delta\tilde{f}^{(1)}_{q/q,z}(x,\rho\to 0) - f^{(1)}_{q/q}\left(x,\frac{\mu^2}{-p^2}\right) - (\tilde{f}^{(1)}_{q/q})_{C.T.}\right]_+, \tag{117}$$

where the counterterm in the RI/MOM scheme is determined as:

$$(\tilde{f}^{(1)}_{q/q})_{C.T.} = \left|\frac{p_z}{p_z^R}\right|\Delta\tilde{f}^{(1)}_{q/q,t}\left(\frac{p_z}{p_z^R}(x-1)+1,r\right). \tag{118}$$

C. Gluon in Quark

The matrix element of the local gluon operator between the polarized quark state reads:

$$\langle q|F^{\mu\alpha}(0)F^{\nu\beta}(0)|q\rangle = -\frac{i\alpha_s C_F}{24\pi\epsilon}\left(p^\alpha\epsilon^{\beta\mu\nu\delta} + p^\beta\epsilon^{\alpha\mu\nu\delta} + p^\mu\epsilon^{\alpha\beta\nu\delta} + p^\nu\epsilon^{\alpha\beta\mu\delta}\right)\bar{u}(p)\gamma_\delta\gamma^5 u(p), \tag{119}$$

where we have used the following identity

$$\gamma^\mu\gamma^\nu\gamma^\alpha = \gamma^\mu g^{\alpha\nu} - \gamma^\nu g^{\alpha\mu} + \gamma^\alpha g^{\mu\nu} + i\epsilon^{\mu\nu\alpha\delta}\gamma_\delta\gamma^5. \tag{120}$$

Projecting onto $\Delta O_{g,R}^{(3)}(0,0)$ gives:

$$\langle q|\Delta O_{g,R}^{(3)}(0,0)|q\rangle = \frac{\alpha_s C_F}{6\pi\epsilon}[(p^t)^2+(p^z)^2] + \mathcal{O}(\epsilon^0). \tag{121}$$

If one requests the same momentum dependence with the matrix element of quark operator, one could make the following projection:

$$\bar{u}\gamma^z\gamma^5 u \to p^t, \quad \bar{u}\gamma^t\gamma^5 u \to p^z. \tag{122}$$

The light-cone results are given as

$$x\Delta f_{g/q}^{(1)}(x,\mu) = \frac{\alpha_s C_F}{2\pi}\left(x(x-2)\ln\frac{-p^2(1-x)x}{\mu^2} + x^2 - 5x\right). \tag{123}$$

The quasi-PDF has the one-loop results

$$x\Delta\tilde{f}_{g/q}^{(3,1)}(x,\rho) = \frac{\alpha_s C_F}{2\pi}\begin{cases} -\frac{\left(\rho\left(\rho^2-3\rho+8\right)+4\left(\rho^2+3\rho+2\right)x^2-8\left(\rho^2+2\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(\rho-1)^3} \\ -\frac{(2x-1)\left(\rho\left(\rho^2-5\rho-2\right)-4\left(\rho^2+3\rho+2\right)x^2+8(2\rho+1)x\right)}{4(1-\rho)^{5/2}(\rho+4x^2-4x)}, & x>1 \\ -\frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho\left(\rho^2-3\rho+8\right)+4\left(\rho^2+3\rho+2\right)x^2-8\left(\rho^2+2\right)x\right)}{8(\rho-1)^3} \\ -\frac{\rho^2-5\rho-12(\rho+1)x^2+8(\rho+2)x-2}{4(1-\rho)^{5/2}}, & 0<x<1 \\ \frac{\left(\rho\left(\rho^2-3\rho+8\right)+4\left(\rho^2+3\rho+2\right)x^2-8\left(\rho^2+2\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(\rho-1)^3} \\ +\frac{(2x-1)\left(\rho\left(\rho^2-5\rho-2\right)-4\left(\rho^2+3\rho+2\right)x^2+8(2\rho+1)x\right)}{4(1-\rho)^{5/2}(\rho+4x^2-4x)}, & x<0 \end{cases} \tag{124}$$

In the limit $\rho \to 0$, we have

$$x\Delta\tilde{f}_{g/q}^{(3,1)}(x,\rho) = \tfrac{\alpha_s C_F}{2\pi}\begin{cases} \frac{1}{2}\left(2x+2(x-2)x\ln\frac{x-1}{x}-1\right), & x>1 \\ \frac{1}{2}\left(2(x-2)x\ln\frac{\rho}{4}+6x^2-8x+1\right), & 0<x<1 \\ \frac{1}{2}\left(-2x-2(x-2)x\ln\frac{x-1}{x}+1\right), & x<0. \end{cases} \tag{125}$$

The matching coefficient can be extracted as

$$x\Delta C_{g/q}^{(3,1)}(x,r,\frac{p^z}{\mu},\frac{p^z}{p_z^R}) = \left[x\Delta\tilde{f}_{g/q}^{(3,1)}(x,\rho\to 0) - x\Delta f_{g/q}^{(1)}\left(x,\frac{\mu^2}{-p^2}\right) - (x\Delta\tilde{f}_{g/q}^{(3,1)})_{C.T.}\right], \tag{126}$$

with the counterterm in the RI/MOM scheme determined as:

$$(x\Delta\tilde{f}_{g/q}^{(3,1)})_{C.T.} = \left|\frac{p_z}{p_z^R}\right| x\Delta\tilde{f}_{g/q}^{(3,1)}\left(\frac{p_z}{p_z^R}(x-1)+1, r\right). \tag{127}$$

D. Quark in Gluon

In this case, the light-cone results are:

$$\Delta f_{q/g}^{(1)}(x,\mu) = \frac{\alpha_s T_f}{2\pi}\left((1-2x)\ln\frac{-p^2(1-x)x}{\mu^2} - 4x + 1\right). \tag{128}$$

The results for the quasi PDF are:

$$\Delta\tilde{f}_{q/g}^{(1)} = \frac{\alpha_s T_f}{2\pi}\begin{cases} -\frac{\rho+8x^2+2(\rho-4)x}{\sqrt{1-\rho}(\rho+4x^2-4x)} + \frac{\rho+4x-2}{2(\rho-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}, & x>1 \\ \frac{1-4x}{\sqrt{1-\rho}} + \frac{\rho+4x-2}{2(\rho-1)}\ln\frac{1-\sqrt{1-\rho}}{1+\sqrt{1+\rho}}, & 0<x<1 \\ \frac{\rho+8x^2+2(\rho-4)x}{\sqrt{1-\rho}(\rho+4x^2-4x)} - \frac{\rho+4x-2}{2(\rho-1)}\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}, & x<0 \end{cases}$$

In the limit $\rho \to 0$, we have

$$\Delta \tilde{f}^{(1)}_{q/g} = \frac{\alpha_s T_f}{2\pi} \begin{cases} (1-2x)\ln\frac{x-1}{x} - 2, & x > 1 \\ (1-2x)\ln\frac{\rho}{4} - 4x + 1, & 0 < x < 1 \\ (2x-1)\ln\frac{x-1}{x} + 1, & x < 0 \end{cases} \tag{129}$$

The matching coefficient is then given by

$$\Delta C^{(1)}_{qg}(x, r, \frac{p^z}{\mu}, \frac{p^z}{p^R_z}) = \left[\Delta \tilde{f}^{(1)}_{q/g}(x, \rho \to 0) - \Delta f^{(1)}_{q/g}\left(x, \frac{\mu^2}{-p^2}\right) - (\Delta \tilde{f}^{(1)}_{q/g})_{C.T.}\right], \tag{130}$$

with

$$(\Delta \tilde{f}^{(1)}_{q/g})_{C.T.} = \left|\frac{p_z}{p^R_z}\right| \Delta \tilde{f}^{(1)}_{q/g}\left(\frac{p_z}{p^R_z}(x-1)+1, r\right). \tag{131}$$

V. CONCLUSION

In this paper, we have studied how to extract the flavor-singlet quark PDF and gluon PDF from LaMET, both in the unpolarized and polarized case. After briefly reviewing the auxiliary "heavy quark" formalism used in our earlier work to prove the multiplicative renormalizability of quark and gluon quasi-PDF operators, we explained how a nonperturbative RI/MOM renormalization can be carried out for the quark and gluon quasi-PDFs on the lattice in the presence of mixing. Using OPE, we also derived the factorization formulas that connect them to the usual quark and gluon PDFs in $\overline{MS}$ scheme. In the one-loop calculation of the hard matching kernel, we found that certain gluon quasi-PDF operators are more favorable than others in the sense that the mixing with gauge variant operators can be avoided. We then focused on these operators and presented the corresponding one-loop matching kernel. Our results can thus be used to extract flavor-singlet quark PDFs as well as the gluon PDFs from lattice simulations of the corresponding quasi-PDFs. We therefore completed the procedure of extracting quark and gluon PDFs from LaMET at leading power accuracy in the hadron momentum.

It is interesting to note that the matrix elements of those non-favorable gluon quasi-PDF operators have nontrivial momentum dependence in their asymptotic behavior at large x, which is also exhibited in the UV divergences of their local limit. This is a sign of the potential mixing with gauge variant operators. For these operators, it shall also be possible to work out an appropriate RI/MOM renormalization and matching, but one needs to take into account the gauge variant operators that are allowed to mix with the original operators. This makes the situation much more complicated and is beyond the scope of the present paper. We leave it to future work.

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Appendix A: RI/MOM Counterterm

The renormalization of quark and gluon quasi operators is given by

$$\widetilde{h}_{i/k,R} = Z_{ij}\widetilde{h}_{j/k}, \tag{A1}$$

with i, j denoting gluon/quark. Here $\widetilde{h}_{i/k}$ are the coordinate space matrix elements of quasi operators, which have the Fourier transformation:

$$\widetilde{h}_{i/k}(z) = \int dx e^{-ixzp_z} \widetilde{f}_{i/k}(x, \rho). \tag{A2}$$

The RI/MOM renomalization condition

$$\widetilde{h}^{(0)}_{i/k} = Z_{ij}\widetilde{h}_{j/k}(z, p_z, \rho)\Big|_{p^2=-\mu_R^2, p_z=p_z^R}, \tag{A3}$$

has the perturbative expansion:

$$\widetilde{h}^{(0)}_{i/k} = (Z^{(0)}_{ij} + Z^{(1)}_{ij})(\widetilde{h}^{(0)}_{j/k} + h^{(1)}_{j/k})\Big|_{p^2=-\mu_R^2, p_z=p_z^R} = Z^{(0)}_{ij}\widetilde{h}^{(0)}_{j/k} + Z^{(0)}_{ij}\widetilde{h}^{(1)}_{j/k} + Z^{(1)}_{ij}\widetilde{h}^{(0)}_{j/k}. \tag{A4}$$

It implies

$$Z^{(0)}_{ij} = \delta_{ij},\ Z^{(0)}_{ij}\widetilde{h}^{(1)}_{j/k} + Z^{(1)}_{ij}\widetilde{h}^{(0)}_{j/k} = 0. \tag{A5}$$

Therefore, the renormalization matrix reads

$$Z_{ij} = \delta_{ij} - \frac{\widetilde{h}^{(1)}_{i/j}}{\widetilde{h}^{(0)}_{j/j}} = |\frac{p_z}{p_z^R}| \int dx e^{-ixzp_z}\left[\delta_{ij}\delta\left(\frac{p_z}{p_z^R}x\right)\left(1 + \frac{\widetilde{f}^{(1)}_{j/j,v}}{\widetilde{f}^{(0)}_{j/j}}\right) - \frac{\widetilde{f}^{(1)}_{i/j,r}(\frac{p_z}{p_z^R}x + 1, r)}{\widetilde{f}^{(0)}_{j/j}}\right]. \tag{A6}$$

Thus, one can get the renormalized quasi PDF:

$$\widetilde{f}_{i/j}(x, r, \rho, p_z, p_z^R) = \delta_{ij}\left(\widetilde{f}^{(0)}_{j/j} - \widetilde{f}^{(1)}_{j/j,v}(\rho) + \widetilde{f}^{(1)}_{j/j,v}(r)\right) + \widetilde{f}^{(1)}_{i/j,r}(x, \rho) - |\frac{p_z}{p_z^R}|\widetilde{f}^{(1)}_{i/j,r}\left(\frac{p_z}{p_z^R}(x-1) + 1, r\right), \tag{A7}$$

where the last term is the counter-term.

Appendix B: One-Loop Results in the R_ξ gauge

In this Appendix, we present the one-loop results for all gluon quasi-PDF operators in the general R_ξ gauge. For the gluon to gluon case, we only give the real contribution:

$$\begin{aligned}
x\tilde{f}^{(1,1)}_{g/g}\Big|_{real} &= \frac{\alpha_s C_A}{2\pi}\begin{cases} -\frac{4(x-1)x^2}{(2x-1)^2(\rho+4x^2-4x)} + \frac{2\left(2x^4-8x^3+6x^2-x\right)}{(\rho-1)(x-1)(2x-1)^2} + \frac{12x+1}{6(\rho-1)^2} - \frac{(2x-1)^2}{2(\rho-1)^3} - \frac{1}{x-1}, & x>1 \\ -\frac{2\left(2x^2-x\right)}{(\rho-1)(x-1)} + \frac{8x^3+12x^2-6x-1}{6(\rho-1)^2} + \frac{-8x^3+12x^2-6x+1}{2(\rho-1)^3} + \frac{1}{x-1}, & 0<x<1 \\ \frac{4(x-1)x^2}{(2x-1)^2(\rho+4x^2-4x)} - \frac{2\left(2x^4-8x^3+6x^2-x\right)}{(\rho-1)(x-1)(2x-1)^2} + \frac{-12x-1}{6(\rho-1)^2} + \frac{(2x-1)^2}{2(\rho-1)^3} + \frac{1}{x-1}, & x<0 \end{cases} \\
&+ \frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{\left(-6\rho^3+19\rho^2-20\rho+8x^4-4\left(\rho^2+4\right)x^3+6\left(3\rho^2-4\rho+4\right)x^2+\left(2\rho^3-17\rho^2+24\rho-16\right)x+8\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{7/2}(x-1)}, & x>1 \\ \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(-6\rho^3+19\rho^2-20\rho+8x^4-4\left(\rho^2+4\right)x^3+6\left(3\rho^2-4\rho+4\right)x^2+\left(2\rho^3-17\rho^2+24\rho-16\right)x+8\right)}{4(1-\rho)^{7/2}(x-1)}, & 0<x<1 \\ -\frac{\left(-6\rho^3+19\rho^2-20\rho+8x^4-4\left(\rho^2+4\right)x^3+6\left(3\rho^2-4\rho+4\right)x^2+\left(2\rho^3-17\rho^2+24\rho-16\right)x+8\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{7/2}(x-1)}, & x<0 \end{cases} \\
&+ \frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{(\xi-1)\rho^2\left(\rho^2+8x^4-20x^3+2(2\rho+7)x^2-(6\rho+1)x\right)}{2(\rho-1)^2(x-1)(\rho+4x^2-4x)^2} + \frac{(\xi-1)\rho^2\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}}, & x>1 \\ -\frac{(\xi-1)\left(\rho^2-2\rho x+x\right)}{2(\rho-1)^2(x-1)} + \frac{(\xi-1)\rho^2\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)}{4(1-\rho)^{5/2}}, & 0<x<1 \\ -\frac{(\xi-1)\rho^2\left(\rho^2+8x^4-20x^3+2(2\rho+7)x^2-(6\rho+1)x\right)}{2(\rho-1)^2(x-1)(\rho+4x^2-4x)^2} - \frac{(\xi-1)\rho^2\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}}, & x<0. \end{cases}
\end{aligned} \tag{B1}$$

$$\begin{aligned}
x\tilde{f}^{(2,1)}_{g/g}\Big|_{real} &= \frac{\alpha_s C_A}{2\pi}\begin{cases} -\frac{4(x-1)x^2}{\rho+4x^2-4x} + \frac{-12x^2-x+10}{6(\rho-1)(x-1)} + \frac{(2x-1)^2}{2(\rho-1)^2} - \frac{1}{2(x-1)} + \frac{(\xi-1)\rho^2x(2x-1)}{2(x-1)(\rho+4x^2-4x)^2}, & x>1 \\ \frac{4x^2-2x+1}{2(x-1)} + \frac{8x^3-12x^2+6x-1}{2(\rho-1)^2} + \frac{-8x^4-4x^3-6x^2+25x-10}{6(\rho-1)(x-1)} - \frac{(\xi-1)x}{2(x-1)}, & 0<x<1 \\ \frac{4(x-1)x^2}{\rho+4x^2-4x} + \frac{12x^2+x-10}{6(\rho-1)(x-1)} - \frac{(2x-1)^2}{2(\rho-1)^2} + \frac{1}{2(x-1)} - \frac{(\xi-1)\rho^2x(2x-1)}{2(x-1)(\rho+4x^2-4x)^2}, & x<0 \end{cases} \\
&+ \frac{\alpha_s C_A}{2\pi}\begin{cases} -\frac{\left((\rho-2)^3-8x^4+4\left(\rho^2+4\right)x^3-2\left(5\rho^2-8\rho+12\right)x^2+\left(3\rho^2-12\rho+16\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}(x-1)}, & x>1 \\ -\frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left((\rho-2)^3-8x^4+4\left(\rho^2+4\right)x^3-2\left(5\rho^2-8\rho+12\right)x^2+\left(3\rho^2-12\rho+16\right)x\right)}{4(1-\rho)^{5/2}(x-1)}, & 0<x<1 \\ \frac{\left((\rho-2)^3-8x^4+4\left(\rho^2+4\right)x^3-2\left(5\rho^2-8\rho+12\right)x^2+\left(3\rho^2-12\rho+16\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}(x-1)}, & x<0 \end{cases}
\end{aligned} \tag{B2}$$

$$x\tilde{f}^{(3,1)}_{g/g}\Big|_{real} = \frac{\alpha_s C_A}{2\pi}\begin{cases} -\frac{4\left(x^3-x^2\right)}{(2x-1)(\rho+4x^2-4x)} + \frac{8x^4-16x^3-22x^2+34x-9}{4(\rho-1)(x-1)(2x-1)} + \frac{3x(2x-1)}{2(\rho-1)^2} - \frac{2x+1}{4(x-1)} + \frac{(\xi-1)\rho^2 x}{2(x-1)(\rho+4x^2-4x)^2}, & x>1 \\ \frac{-30x^2+34x-9}{4(\rho-1)(x-1)} + \frac{3\left(4x^3-4x^2+x\right)}{2(\rho-1)^2} + \frac{4x+1}{4(x-1)} - \frac{(\xi-1)x}{2(x-1)}, & 0<x<1 \\ \frac{4\left(x^3-x^2\right)}{(2x-1)(\rho+4x^2-4x)} + \frac{-8x^4+16x^3+22x^2-34x+9}{4(\rho-1)(x-1)(2x-1)} - \frac{3x(2x-1)}{2(\rho-1)^2} + \frac{2x+1}{4(x-1)} - \frac{(\xi-1)\rho^2 x}{2(x-1)(\rho+4x^2-4x)^2}, & x<0 \end{cases}$$
$$+\frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{\left(-(\rho-4)^2(\rho-1)+8(\rho+2)x^4-16(\rho+2)x^3-2\left(\rho^2+8\rho-24\right)x^2+\left(6\rho^2+20\rho-32\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(1-\rho)^{5/2}(x-1)}, & x>1 \\ \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(-(\rho-4)^2(\rho-1)+8(\rho+2)x^4-16(\rho+2)x^3-2\left(\rho^2+8\rho-24\right)x^2+\left(6\rho^2+20\rho-32\right)x\right)}{8(1-\rho)^{5/2}(x-1)}, & 0<x<1 \\ \frac{\left((\rho-4)^2(\rho-1)-8(\rho+2)x^4+16(\rho+2)x^3+2\left(\rho^2+8\rho-24\right)x^2+\left(-6\rho^2-20\rho+32\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(1-\rho)^{5/2}(x-1)}, & x<0 \end{cases} \tag{B3}$$

$$x\tilde{f}^{(4,1)}_{g/g}\Big|_{real} = \frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{4x^2-4x-1}{2(x-1)} + \frac{-4x^3+4x^2-2x+1}{2(\rho-1)(x-1)} - \frac{4\left(3x^3-5x^2+2x\right)}{\rho+4x^2-4x} + \frac{(\xi-1)\rho^2 x(2x-1)}{2(x-1)(\rho+4x^2-4x)^2}, & x>1 \\ \frac{8x^2-6x+1}{2(x-1)} + \frac{-8x^4+12x^3-8x^2+4x-1}{2(\rho-1)(x-1)} - \frac{(\xi-1)x}{2(x-1)}, & 0<x<1 \\ -\frac{4x^2-4x-1}{2(x-1)} + \frac{4x^3-4x^2+2x-1}{2(\rho-1)(x-1)} + \frac{4\left(3x^3-5x^2+2x\right)}{\rho+4x^2-4x} - \frac{(\xi-1)\rho^2 x(2x-1)}{2(x-1)(\rho+4x^2-4x)^2}, & x<0 \end{cases}$$
$$+\frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{\left(\rho^2-8\rho+8x^4+4(\rho-4)x^3-8(2\rho-3)x^2+4(3\rho-4)x+8\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}(x-1)}, & x>1 \\ \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho^2-8\rho+8x^4+4(\rho-4)x^3-8(2\rho-3)x^2+4(3\rho-4)x+8\right)}{4(1-\rho)^{3/2}(x-1)}, & 0<x<1 \\ -\frac{\left(\rho^2-8\rho+8x^4+4(\rho-4)x^3-8(2\rho-3)x^2+4(3\rho-4)x+8\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}(x-1)}, & x<0 \end{cases} \tag{B4}$$

$$x\Delta\tilde{f}^{(1,1)}_{g/g}\Big|_{real} = \frac{\alpha_s C_A}{2\pi}\begin{cases} -\frac{4\left(x^3-x^2\right)}{(2x-1)^2(\rho+4x^2-4x)} + \frac{20x^4-52x^3+31x^2-2x-1}{2(\rho-1)(x-1)(2x-1)^2} + \frac{10x-3}{2(\rho-1)^2} - \frac{1}{x-1}, & x>1 \\ \frac{-3x^2-2x+1}{2(\rho-1)(x-1)} + \frac{20x^2-16x+3}{2(\rho-1)^2} + \frac{1}{x-1}, & 0<x<1 \\ \frac{4\left(x^3-x^2\right)}{(2x-1)^2(\rho+4x^2-4x)} + \frac{-20x^4+52x^3-31x^2+2x+1}{2(\rho-1)(x-1)(2x-1)^2} + \frac{3-10x}{2(\rho-1)^2} + \frac{1}{x-1}, & x<0 \end{cases}$$
$$+\frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{\left(\rho(5\rho-8)+4(\rho+4)x^3-12(\rho+2)x^2+\left(-\rho^2+4\rho+16\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}(x-1)}, & x>1 \\ \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho(5\rho-8)+4(\rho+4)x^3-12(\rho+2)x^2+\left(-\rho^2+4\rho+16\right)x\right)}{4(1-\rho)^{5/2}(x-1)}, & 0<x<1 \\ \frac{\left((8-5\rho)\rho-4(\rho+4)x^3+12(\rho+2)x^2+\left(\rho^2-4\rho-16\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}(x-1)}, & x<0 \end{cases}$$
$$+\frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{(\xi-1)\rho^2\left(\rho^2+8x^4-20x^3+2(2\rho+7)x^2-(6\rho+1)x\right)}{2(\rho-1)^2(x-1)(\rho+4x^2-4x)^2} + \frac{(\xi-1)\rho^2\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}}, & x>1 \\ -\frac{(\xi-1)\left(\rho^2-2\rho x+x\right)}{2(\rho-1)^2(x-1)} + \frac{(\xi-1)\rho^2\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)}{4(1-\rho)^{5/2}}, & 0<x<1 \\ -\frac{(\xi-1)\rho^2\left(\rho^2+8x^4-20x^3+2(2\rho+7)x^2-(6\rho+1)x\right)}{2(\rho-1)^2(x-1)(\rho+4x^2-4x)^2} - \frac{(\xi-1)\rho^2\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}}, & x<0 \end{cases} \tag{B5}$$

$$x\Delta\tilde{f}^{(2,1)}_{g/g}\Big|_{real} = \frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{9x-10x^2}{2(\rho-1)(x-1)} - \frac{4\left(x^3-x^2\right)}{\rho+4x^2-4x} + \frac{(x-2)x}{2(x-1)} + \frac{x\left((\rho-4)^2+4(\rho+4)x^2-4(\rho+6)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}(x-1)}, & x>1 \\ \frac{3x^2}{2(x-1)} + \frac{-20x^3+28x^2-9x}{2(\rho-1)(x-1)} + \frac{x\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left((\rho-4)^2+4(\rho+4)x^2-4(\rho+6)x\right)}{4(1-\rho)^{3/2}(x-1)}, & 0<x<1 \\ \frac{10x^2-9x}{2(\rho-1)(x-1)} + \frac{4\left(x^3-x^2\right)}{\rho+4x^2-4x} - \frac{(x-2)x}{2(x-1)} - \frac{x\left((\rho-4)^2+4(\rho+4)x^2-4(\rho+6)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}(x-1)}, & x<0 \end{cases}$$
$$+\frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{(\xi-1)\rho^2 x(2x-1)}{2(x-1)(\rho+4x^2-4x)^2}, & x>1 \\ -\frac{(\xi-1)x}{2(x-1)}, & 0<x<1 \\ -\frac{(\xi-1)\rho^2 x(2x-1)}{2(x-1)(\rho+4x^2-4x)^2}, & x<0 \end{cases} \tag{B6}$$

$$x\Delta\tilde{f}^{(3,1)}_{g/g}\bigg|_{real} = \frac{\alpha_s C_A}{2\pi}\begin{cases} \frac{-8x^3-8x^2+14x-3}{4(\rho-1)(x-1)(2x-1)} - \frac{4\left(x^3-x^2\right)}{(2x-1)(\rho+4x^2-4x)} + \frac{3(2x-1)}{2(\rho-1)^2} - \frac{2x+1}{4(x-1)} + \frac{(\xi-1)\rho^2 x}{2(x-1)(\rho+4x^2-4x)^2}, & x>1 \\ \frac{3\left(4x^2-4x+1\right)}{2(\rho-1)^2} + \frac{-16x^3+8x^2+6x-3}{4(\rho-1)(x-1)} + \frac{4x+1}{4(x-1)} - \frac{(\xi-1)x}{2(x-1)}, & 0<x<1 \\ \frac{8x^3+8x^2-14x+3}{4(\rho-1)(x-1)(2x-1)} + \frac{4\left(x^3-x^2\right)}{(2x-1)(\rho+4x^2-4x)} - \frac{3(2x-1)}{2(\rho-1)^2} + \frac{2x+1}{4(x-1)} - \frac{(\xi-1)\rho^2 x}{2(x-1)(\rho+4x^2-4x)^2}, & x<0 \end{cases}$$
$$+ \frac{\alpha_s C_A}{2\pi}\begin{cases} -\frac{\left(\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(1-\rho)^{5/2}(x-1)}, & x>1 \\ -\frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x\right)}{8(1-\rho)^{5/2}(x-1)}, & 0<x<1 \\ \frac{\left(\rho\left(\rho^2-3\rho+8\right)+8(\rho-4)x^3+8\left(\rho^2-\rho+6\right)x^2-2\left(9\rho^2-10\rho+16\right)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(1-\rho)^{5/2}(x-1)}, & x<0 \end{cases} \tag{B7}$$

The quark to gluon case is given as:

$$x\tilde{f}^{(1,1)}_{g/q,t} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\left(-5\rho^2+16\rho+4(\rho+2)x^2-12\rho x-8\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(1-\rho)^{5/2}} + \frac{-5\rho+2(\rho+2)x+2}{4(\rho-1)^2}, & x>1 \\ \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(-5\rho^2+16\rho+4(\rho+2)x^2-12\rho x-8\right)}{8(1-\rho)^{5/2}} + \frac{5\rho+12x^2-4(2\rho+1)x-2}{4(\rho-1)^2}, & 0<x<1 \\ \frac{\left(5\rho^2-16\rho-4(\rho+2)x^2+12\rho x+8\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(1-\rho)^{5/2}} + \frac{5\rho-2(\rho+2)x-2}{4(\rho-1)^2}, & x<0 \end{cases} \tag{B8}$$

$$x\tilde{f}^{(1,1)}_{g/q,z} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{-3\rho^2+8(\rho+2)x^3-4(\rho+2)^2x^2+2\rho(2\rho+7)x}{2(\rho-1)^3(\rho+4x^2-4x)} - \frac{\left(5\rho^2-6\rho+4(\rho+2)x^2+\left(-6\rho^2+2\rho-8\right)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{7/2}}, & x>1 \\ \frac{3(2x-1)(2x-\rho)}{2(\rho-1)^3} - \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(5\rho^2-6\rho+4(\rho+2)x^2+\left(-6\rho^2+2\rho-8\right)x+4\right)}{4(1-\rho)^{7/2}}, & 0<x<1 \\ \frac{\left(5\rho^2-6\rho+4(\rho+2)x^2+\left(-6\rho^2+2\rho-8\right)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{7/2}} + \frac{3\rho^2-8(\rho+2)x^3+4(\rho+2)^2x^2-2\rho(2\rho+7)x}{2(\rho-1)^3(\rho+4x^2-4x)}, & x<0 \end{cases} \tag{B9}$$

$$x\tilde{f}^{(1,1)}_{g/q,p} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\rho(\rho-2x)\left(\rho+12x^2+2(\rho-7)x+2\right)}{2(\rho-1)^3(\rho+4x^2-4x)} - \frac{\rho(\rho-2x)(\rho+6x-4)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{7/2}}, & x>1 \\ \frac{(\rho-2x)(-\rho+(4\rho+2)x-2)}{2(\rho-1)^3} - \frac{\rho\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)(\rho-2x)(\rho+6x-4)}{4(1-\rho)^{7/2}}, & 0<x<1 \\ \frac{\rho(\rho-2x)(\rho+6x-4)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{7/2}} - \frac{\rho(\rho-2x)\left(\rho+12x^2+2(\rho-7)x+2\right)}{2(\rho-1)^3(\rho+4x^2-4x)}, & x<0 \end{cases} \tag{B10}$$

$$x\tilde{f}^{(2,1)}_{g/q,t} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\rho(\rho+2)-8(\rho+2)x^3-4(\rho-10)x^2-2\left(5\rho^2-8\rho+12\right)x}{4(\rho-1)(\rho+4x^2-4x)} - \frac{\left((\rho-4)\rho-4(\rho+2)x^2-4(\rho-4)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(1-\rho)^{3/2}}, & x>1 \\ \frac{\rho+12x^2+8\rho x-20x+2}{4-4\rho} - \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left((\rho-4)\rho-4(\rho+2)x^2-4(\rho-4)x\right)}{8(1-\rho)^{3/2}}, & 0<x<1 \\ \frac{-\rho(\rho+2)+8(\rho+2)x^3+4(\rho-10)x^2+2\left(5\rho^2-8\rho+12\right)x}{4(\rho-1)(\rho+4x^2-4x)} - \frac{\left(-(\rho-4)\rho+4(\rho+2)x^2+4(\rho-4)x\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(1-\rho)^{3/2}}, & x<0 \end{cases} \tag{B11}$$

$$x\tilde{f}^{(2,1)}_{g/q,z} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{-3(\rho-2)\rho-8(\rho+2)x^3+4\left(\rho^2-4\rho+12\right)x^2+\left(-8\rho^2+22\rho-32\right)x}{2(\rho-1)^2(\rho+4x^2-4x)} \\ \quad -\frac{\left(3\rho^2-8\rho+4(\rho+2)x^2+2\left(\rho^2+\rho-8\right)x+8\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}}, & x>1 \\ \frac{3\rho-12x^2-10\rho x+22x-6}{2(\rho-1)^2} - \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(3\rho^2-8\rho+4(\rho+2)x^2+2\left(\rho^2+\rho-8\right)x+8\right)}{4(1-\rho)^{5/2}}, & 0<x<1 \\ \frac{\left(3\rho^2-8\rho+4(\rho+2)x^2+2\left(\rho^2+\rho-8\right)x+8\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}} \\ \quad +\frac{3(\rho-2)\rho+8(\rho+2)x^3-4\left(\rho^2-4\rho+12\right)x^2+\left(8\rho^2-22\rho+32\right)x}{2(\rho-1)^2(\rho+4x^2-4x)}, & x<0 \end{cases} \tag{B12}$$

$$x\tilde{f}^{(2,1)}_{g/q,p} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\rho\left(\rho^2-2\rho+12x^2+4(\rho-4)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}} + \frac{\rho\left((\rho-4)\rho+24x^3+(8\rho-44)x^2+2\left(\rho^2-2\rho+10\right)x\right)}{2(\rho-1)^2(\rho+4x^2-4x)}, & x>1 \\ \frac{-(\rho-4)\rho+(8\rho+4)x^2+4\left(\rho^2-3\rho-1\right)x}{2(\rho-1)^2} + \frac{\rho\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho^2-2\rho+12x^2+4(\rho-4)x+4\right)}{4(1-\rho)^{5/2}}, & 0<x<1 \\ -\frac{\rho\left(\rho^2-2\rho+12x^2+4(\rho-4)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}} - \frac{\rho\left((\rho-4)\rho+24x^3+(8\rho-44)x^2+2\left(\rho^2-2\rho+10\right)x\right)}{2(\rho-1)^2(\rho+4x^2-4x)}, & x<0 \end{cases} \tag{B13}$$

$$x\tilde{f}^{(3,1)}_{g/q,t} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\left(3\rho+4x^2-4x-2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}} + \frac{\rho-8x^3+4(\rho+2)x^2-6\rho x}{2(\rho-1)(\rho+4x^2-4x)}, & x>1 \\ \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(3\rho+4x^2-4x-2\right)}{4(1-\rho)^{3/2}} - \frac{(1-2x)^2}{2(\rho-1)}, & 0<x<1 \\ -\frac{\left(3\rho+4x^2-4x-2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{3/2}} - \frac{\rho-8x^3+4(\rho+2)x^2-6\rho x}{2(\rho-1)(\rho+4x^2-4x)}, & x<0 \end{cases} \tag{B14}$$

$$x\tilde{f}^{(3,1)}_{g/q,z} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{(5-2\rho)\rho-8(\rho+2)x^3+4(\rho+8)x^2-2(\rho+8)x}{2(\rho-1)^2(\rho+4x^2-4x)} - \frac{\left(2\rho^2-5\rho+4(\rho+2)x^2-12x+6\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}}, & x>1 \\ -\frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(2\rho^2-5\rho+4(\rho+2)x^2-12x+6\right)}{4(1-\rho)^{5/2}} - \frac{(2x-1)(2\rho+6x-5)}{2(\rho-1)^2}, & 0<x<1 \\ \frac{\left(2\rho^2-5\rho+4(\rho+2)x^2-12x+6\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}} + \frac{\rho(2\rho-5)+8(\rho+2)x^3-4(\rho+8)x^2+2(\rho+8)x}{2(\rho-1)^2(\rho+4x^2-4x)}, & x<0 \end{cases} \tag{B15}$$

$$x\tilde{f}^{(3,1)}_{g/q,p} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\rho\left(\rho+12x^2-12x+2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}} + \frac{3\rho(2x-1)}{2(\rho-1)^2}, & x>1 \\ \frac{3\rho+(8\rho+4)x^2-4(2\rho+1)x}{2(\rho-1)^2} + \frac{\rho\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho+12x^2-12x+2\right)}{4(1-\rho)^{5/2}}, & 0<x<1 \\ \frac{\rho(3-6x)}{2(\rho-1)^2} - \frac{\rho\left(\rho+12x^2-12x+2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(1-\rho)^{5/2}}, & x<0 \end{cases} \tag{B16}$$

$$x\tilde{f}^{(4,1)}_{g/q,t} = 0 \tag{B17}$$

$$x\tilde{f}^{(4,1)}_{g/q,z} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{-\rho+4x^3+2(3\rho-8)x^2-6(\rho-2)x}{(\rho-1)(\rho+4x^2-4x)} - \frac{\left(-3\rho+2x^2-4x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(1-\rho)^{3/2}}, & x>1 \\ \frac{2x^2-4x+1}{\rho-1} - \frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(-3\rho+2x^2-4x+4\right)}{2(1-\rho)^{3/2}}, & 0<x<1 \\ \frac{\rho-4x^3+(16-6\rho)x^2+6(\rho-2)x}{(\rho-1)(\rho+4x^2-4x)} - \frac{\left(3\rho-2x^2+4x-4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(1-\rho)^{3/2}}, & x<0 \end{cases} \tag{B18}$$

$$x\tilde{f}^{(4,1)}_{g/q,p} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\rho\left(\rho-4x^3+10x^2-6x\right)}{(\rho-1)(\rho+4x^2-4x)} - \frac{\rho\left(\rho-2x^2+4x-2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(1-\rho)^{3/2}}, & x>1 \\ -\frac{\rho+2x^2-2(\rho+1)x}{\rho-1} - \frac{\rho\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho-2x^2+4x-2\right)}{2(1-\rho)^{3/2}}, & 0<x<1 \\ \frac{\rho\left(\rho-2x^2+4x-2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(1-\rho)^{3/2}} - \frac{\rho\left(\rho-4x^3+10x^2-6x\right)}{(\rho-1)(\rho+4x^2-4x)}, & x<0 \end{cases} \tag{B19}$$

$$x\Delta\tilde{f}^{(1,1)}_{g/q,t} = \frac{\alpha_s C_F}{2\pi}\begin{cases} -\frac{x-1}{(1-\rho)^{3/2}} - \frac{(x-1)(\rho+2x-2)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^2}, & x>1 \\ -\frac{(x-1)(2x-1)}{(1-\rho)^{3/2}} - \frac{(x-1)\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)(\rho+2x-2)}{2(\rho-1)^2}, & 0<x<1 \\ \frac{x-1}{(1-\rho)^{3/2}} + \frac{(x-1)(\rho+2x-2)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^2}, & x<0 \end{cases} \tag{B20}$$

$$x\Delta\tilde{f}^{(1,1)}_{g/q,z}=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{-3\rho+4(\rho+2)x^3-4(\rho+5)x^2+\left(2\rho^2+\rho+12\right)x}{(1-\rho)^{5/2}(\rho+4x^2-4x)}-\frac{\left(\rho+2(\rho+2)x^2-(\rho+8)x+2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^3}, & x>1\\ \frac{6x^2+(2\rho-11)x+3}{(1-\rho)^{5/2}}-\frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho+2(\rho+2)x^2-(\rho+8)x+2\right)}{2(\rho-1)^3}, & 0<x<1\\ \frac{\left(\rho+2(\rho+2)x^2-(\rho+8)x+2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^3}+\frac{3\rho-4(\rho+2)x^3+4(\rho+5)x^2-\left(2\rho^2+\rho+12\right)x}{(1-\rho)^{5/2}(\rho+4x^2-4x)}, & x<0\end{cases}\tag{B21}$$

$$x\Delta\tilde{f}^{(1,1)}_{g/q,p}=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{\rho(x-1)\left(-3\rho-12x^2+2(\rho+5)x\right)}{(1-\rho)^{5/2}(\rho+4x^2-4x)}+\frac{\rho(x-1)(-\rho+6x-2)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^3}, & x>1\\ \frac{\rho(x-1)\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)(-\rho+6x-2)}{2(\rho-1)^3}-\frac{(x-1)((4\rho+2)x-3\rho)}{(1-\rho)^{5/2}}, & 0<x<1\\ \frac{\rho(x-1)\left(3\rho+12x^2-2(\rho+5)x\right)}{(1-\rho)^{5/2}(\rho+4x^2-4x)}-\frac{\rho(x-1)(-\rho+6x-2)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^3}, & x<0\end{cases}\tag{B22}$$

$$x\Delta\tilde{f}^{(2,1)}_{g/q,t}=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{x(2x-\rho)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)}-\frac{x\left(-3\rho+4x^2+4(\rho-2)x+4\right)}{\sqrt{1-\rho}(\rho+4x^2-4x)}, & x>1\\ \frac{x-2x^2}{\sqrt{1-\rho}}+\frac{x\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)(2x-\rho)}{2(\rho-1)}, & 0<x<1\\ \frac{x\left(-3\rho+4x^2+4(\rho-2)x+4\right)}{\sqrt{1-\rho}(\rho+4x^2-4x)}+\frac{x(\rho-2x)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)}, & x<0\end{cases}\tag{B23}$$

$$x\Delta\tilde{f}^{(2,1)}_{g/q,z}=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{x\left(2\rho^2-3\rho+4(\rho+2)x^2-12x+4\right)}{(1-\rho)^{3/2}(\rho+4x^2-4x)}+\frac{x(\rho+2(\rho+2)x-4)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^2}, & x>1\\ \frac{x(2\rho+6x-5)}{(1-\rho)^{3/2}}+\frac{x\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)(\rho+2(\rho+2)x-4)}{2(\rho-1)^2}, & 0<x<1\\ -\frac{x\left(2\rho^2-3\rho+4(\rho+2)x^2-12x+4\right)}{(1-\rho)^{3/2}(\rho+4x^2-4x)}-\frac{x(\rho+2(\rho+2)x-4)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^2}, & x<0\end{cases}\tag{B24}$$

$$x\Delta\tilde{f}^{(2,1)}_{g/q,p}=\frac{\alpha_s C_F}{2\pi}\begin{cases}-\frac{\rho x\left(\rho+12x^2+2(\rho-7)x+2\right)}{(1-\rho)^{3/2}(\rho+4x^2-4x)}-\frac{\rho x(\rho+6x-4)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^2}, & x>1\\ -\frac{x(-\rho+(4\rho+2)x-2)}{(1-\rho)^{3/2}}-\frac{\rho x\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)(\rho+6x-4)}{2(\rho-1)^2}, & 0<x<1\\ \frac{\rho x\left(\rho+12x^2+2(\rho-7)x+2\right)}{(1-\rho)^{3/2}(\rho+4x^2-4x)}+\frac{\rho x(\rho+6x-4)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{2(\rho-1)^2}, & x<0\end{cases}\tag{B25}$$

$$x\Delta\tilde{f}^{(3,1)}_{g/q,t}=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{(2x-1)\left((\rho-4)\rho-4(\rho+2)x^2+4(\rho+2)x\right)}{4(1-\rho)^{3/2}(\rho+4x^2-4x)}-\frac{\left(\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(\rho-1)^2}, & x>1\\ \frac{\rho-12x^2+12x-4}{4(1-\rho)^{3/2}}-\frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4\right)}{8(\rho-1)^2}, & 0<x<1\\ \frac{\left(\rho^2-2\rho+4(\rho+2)x^2-4(\rho+2)x+4\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{8(\rho-1)^2}-\frac{(2x-1)\left((\rho-4)\rho-4(\rho+2)x^2+4(\rho+2)x\right)}{4(1-\rho)^{3/2}(\rho+4x^2-4x)}, & x<0\end{cases}\tag{B26}$$

$$x\Delta\tilde{f}^{(3,1)}_{g/q,z}=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{(2x-1)\left(3\rho+4(\rho+2)x^2+2\left(\rho^2-3\rho-4\right)x\right)}{2(1-\rho)^{5/2}(\rho+4x^2-4x)}-\frac{\left(\rho+4(\rho+2)x^2-2\left(\rho^2-\rho+6\right)x+2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(\rho-1)^3}, & x>1\\ \frac{12x^2+2(\rho-7)x+3}{2(1-\rho)^{5/2}}-\frac{\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho+4(\rho+2)x^2-2\left(\rho^2-\rho+6\right)x+2\right)}{4(\rho-1)^3}, & 0<x<1\\ \frac{\left(\rho+4(\rho+2)x^2-2\left(\rho^2-\rho+6\right)x+2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(\rho-1)^3}-\frac{(2x-1)\left(3\rho+4(\rho+2)x^2+2\left(\rho^2-3\rho-4\right)x\right)}{2(1-\rho)^{5/2}(\rho+4x^2-4x)}, & x<0\end{cases}\tag{B27}$$

$$x\Delta\tilde{f}^{(3,1)}_{g/q,p} = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{\rho\left(\rho+12x^2-12x+2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(\rho-1)^3} + \frac{\rho(3-6x)}{2(1-\rho)^{5/2}}, & x>1 \\ \frac{\rho\ln\left(\frac{1-\sqrt{1-\rho}}{\sqrt{\rho+1}+1}\right)\left(\rho+12x^2-12x+2\right)}{4(\rho-1)^3} - \frac{3\rho+(8\rho+4)x^2-4(2\rho+1)x}{2(1-\rho)^{5/2}}, & 0<x<1 \\ \frac{3\rho(2x-1)}{2(1-\rho)^{5/2}} - \frac{\rho\left(\rho+12x^2-12x+2\right)\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}}{4(\rho-1)^3}, & x<0 \end{cases} \tag{B28}$$

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