

On the Statistical Treatment of the Cabibbo Angle Anomaly

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Abstract

We point out that testing the equality of the Cabibbo angle as extracted from $\Gamma(K \rightarrow \pi l \nu)$, the ratio $\Gamma(K \rightarrow l \nu)/\Gamma(\pi \rightarrow l \nu)$ and nuclear β decays is not identical to a test of first row unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The reason is that a CKM unitarity test involves only two parameters, while beyond the Standard Model all different processes could in principle give different Cabibbo angles. Consequently, the difference between the two tests becomes relevant starting from three observables giving results for the Cabibbo angle that are in tension with each other. With current data, depending on the treatment of the nuclear β decays, we find that the SM is rejected at 5.1σ or 3.6σ while CKM unitarity is rejected at 4.8σ or 3.0σ , respectively. We argue that the best method to test the SM is to test the equality of the Cabibbo angle, because CKM unitarity is only one aspect of the SM.

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I. INTRODUCTION

Among several methods to determine the magnitude of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements V_{ud} and V_{us} , the most precise ones today are the extraction from:

$$K_{l3} : \Gamma(K \rightarrow \pi l \nu), \text{ where } l = \mu, e \text{ [1–9]}, \quad (1)$$

$$K_{\mu 2} : \frac{\Gamma(K \rightarrow \mu \nu)}{\Gamma(\pi \rightarrow \mu \nu)} \text{ [5–8, 10–13]}, \quad (2)$$

$$\beta : \text{Nuclear } 0^+ \rightarrow 0^+ \beta \text{ decays [7, 8, 14–19]}. \quad (3)$$

For brevity, we use V_{ij} to denote the magnitude of a CKM matrix element. Employing the Wolfenstein expansion, in the SM, we can parametrize V_{ud} and V_{us} up to corrections of order $\mathcal{O}(\lambda^6) \simeq 0.0001$ by using the Cabibbo angle describing the mixing of the first two generations

$$V_{ud} = \cos \theta_C, \quad V_{us} = \sin \theta_C, \quad (4)$$

i.e. we practically have a two-generational model. A high-order Wolfenstein expansion can be found e.g. in Ref. [20]. Because of the smallness of $|V_{ub}|^2 \simeq 1.6 \cdot 10^{-5}$ [7] the first row CKM unitarity relation reads

$$1 = V_{ud}^2 + V_{us}^2 + V_{ub}^2 \approx V_{ud}^2 + V_{us}^2. \quad (5)$$

Eq. (5) has been extensively employed in order to probe for, or constrain, new physics (NP) [5, 6, 21–31].

In order to denote the origin of an extraction of the respective CKM matrix element (or their ratio) from experimental data, we use the notation $V_{us}^{K_{l3}}$, $(V_{us}/V_{ud})^{K_{l2}}$ and V_{ud}^β , respectively. As of now, there are two anomalies: Firstly, there is a significant tension of $V_{us}^{K_{l3}}$, $(V_{us}/V_{ud})^{K_{l2}}$ and V_{ud}^β with CKM unitarity [32]. Second, there is an even higher tension between $V_{us}^{K_{l3}}$ and V_{ud}^β [8, 32, 33].

Motivated by these developments, in this paper we discuss the statistical methodology and the differences between testing the SM versus testing CKM unitarity, Eq. (5). In a general model beyond the SM (BSM) n different processes that we use as measurements of the Cabibbo angle could result in n different values. Thus, a test of the SM is a comparison of a one-parameter fit to a trivial fit containing as many parameters as different extractions for the Cabibbo angle that are available. However, tests of CKM unitarity involve only two

parameters, namely V_{us} and the violation of unitarity Δ (see Eq. (11) below). In that case we compare a one-parameter fit to a two-parameter fit only. No matter how many measurements are available, the degree of freedom of the CKM unitarity test is always fixed. In the past, when only two out of three measurements in Eqs. (1)–(3) showed a tension between each other, this was not significant. However, when tensions between all three measurements are present, as is the current situation, one gets sensitive to the fact that in general the significances for the rejection of the SM and CKM unitarity are different.

The point that there is more to test in the measurements of V_{us} and V_{ud} than CKM unitarity was made in specific cases before [5, 23, 34]. Our aim here is to generalize this observation and give a universal methodology for SM tests with an arbitrary number of measurements of θ_C .

We emphasize that the point in this paper is only about the methodology of testing the SM with data on V_{us} and V_{ud} . We do not advocate any of the extractions of V_{ud} , which we use as examples, and are agnostic about the validity of the used models. Especially, we do not claim that the SM is excluded at or beyond 5σ .

In Sec. II we analyze the difference between testing the SM and CKM unitarity. Subsequently, in Sec. III we present our likelihood ratio tests of the SM and CKM unitarity with current data. In Sec. IV we discuss a specific NP model. In Sec. V we conclude.

II. GENERAL FORMALISM FOR TESTING THE SM AND CKM UNITARITY

A. SM test

Beyond the SM, the analysis of n different observables could in principle result in n different mixing angles of the first two generations

$$\theta_1, \theta_2, \dots, \theta_n. \quad (6)$$

We assume here for simplicity that measurements of the same observable by different experiments are already averaged.

In the SM, all of the angles of Eq. (6) are equal to the Cabibbo angle

$$\theta_C = \theta_1 = \theta_2 = \dots = \theta_n. \quad (7)$$

Eq. (7) is the one-parameter SM null hypothesis. It has a constrained nested parameter space relative to the general BSM parameter space Eq. (6). By “nested” we mean as usual that one parameter space can be obtained by fixing some of the parameters of the other. The difference in dimensionality between the two theory spaces, i.e. the number of relatively fixed parameters $\nu_{\text{SM test}}$ of the two hypotheses, is always one less than the total number of observables. The trivial χ^2 fit of the BSM parameter space gives by design always the result

$$\chi_{\text{min, non-SM}}^2 = 0. \quad (8)$$

We perform a likelihood ratio test of SM vs. BSM by calculating the p -value and the significance z of the rejection of the SM as (see e.g. Ref. [35])

$$z = \sqrt{2} \text{Erf}^{-1}(1 - p), \quad p = 1 - P_{\nu/2}(\Delta\chi^2/2), \quad (9)$$

Here, $P_{\nu/2}(\Delta\chi^2/2)$ is the regularized lower incomplete gamma function and we use $\nu = \nu_{\text{SM test}}$ and $\Delta\chi^2 = \Delta\chi_{\text{SM test}}^2$ with

$$\Delta\chi_{\text{SM test}}^2 \equiv \chi_{\text{min, SM}}^2 - \chi_{\text{min, non-SM}}^2 = \chi_{\text{min, SM}}^2. \quad (10)$$

B. CKM unitarity test

In order to test CKM unitarity with n observables one uses the two parameters V_{us} and Δ , where

$$V_{ud} = \sqrt{1 - V_{us}^2} + \Delta, \quad (11)$$

and Δ is the parameter measuring the deviation from unitarity. We test the null hypothesis $\Delta = 0$ against the general case including $\Delta \neq 0$, which is effectively the same as varying V_{us} and V_{ud} freely. We use the Δ notation in order to make completely clear that the two models that we compare are nested. We denote the corresponding minimal χ^2 values as χ_{unitary}^2 and $\chi_{\text{non-unitary}}^2$, respectively, and define for the CKM unitarity test

$$\Delta\chi_{\text{unitarity test}}^2 \equiv \chi_{\text{min, unitary}}^2 - \chi_{\text{min, non-unitary}}^2. \quad (12)$$

n	1	2	≥ 3
$\Delta\chi_{\text{SM test}}^2$	0	χ^2	$\chi^2 > \Delta\chi_{\text{unitarity test}}^2$
χ_{SM}^2	0	χ^2	χ^2
$\chi_{\text{min, non-SM}}^2$	0	0	0
$\nu_{\text{SM test}}$	0	1	$n - 1 \geq 2$
$p_{\text{SM test}}$	1	p	$\neq p_{\text{unitarity test}}$
$z_{\text{SM test}}$	0	z	$\neq z_{\text{unitarity test}}$
$\Delta\chi_{\text{unitarity test}}^2$	0	χ^2	$< \Delta\chi_{\text{SM test}}^2$
χ_{unitary}^2	0	χ^2	χ^2
$\chi_{\text{min, non-unitary}}^2$	0	0	> 0
$\nu_{\text{unitarity test}}$	1	1	1
$p_{\text{unitarity test}}$	1	p	$\neq p_{\text{SM test}}$
$z_{\text{unitarity test}}$	0	z	$\neq z_{\text{SM test}}$

TABLE I. General comparison of SM tests and CKM unitarity tests, showing that the test results are different starting from three observables.

C. Comparison of SM test and CKM unitarity test

The null hypothesis fits of the SM test and the CKM unitarity test are equivalent. They are both one-parameter fits and lead to the same χ^2 :

$$\chi_{\text{min, SM}}^2 = \chi_{\text{min, unitary}}^2. \quad (13)$$

For the two tests however, we compare these fits to different opposite hypotheses. For the CKM unitarity test the difference of dimensionality of the two theory spaces that we compare is always fixed to

$$\nu_{\text{unitarity test}} = 1. \quad (14)$$

For the SM test it is

$$\nu_{\text{SM test}} = n - 1. \quad (15)$$

Furthermore, the non-unitary fit allowing for $\Delta \neq 0$ is nontrivial, resulting in general in $\chi_{\text{non-unitary}}^2 \neq 0$ in contrast to $\chi_{\text{min, non-SM}}^2 = 0$ always.

Whether or not the SM test and CKM unitarity test give the same results depends on the number of observables n that are taken into account, as we show in Table I.

- $n = 1$ is the trivial case where no SM test is needed or possible at all, because SM and BSM hypothesis are degenerate. Also, no violation of unitarity can possibly be detected, so everything is in agreement equally with the SM and unitarity.
- For $n = 2$, the unconstrained fits give the same result (namely a vanishing χ^2), because we can always explain two measurements with two free parameters. The tests have also the same number of degrees of freedom and $z_{\text{SM test}} = z_{\text{unitarity test}}$.
- $n \geq 3$: In this case in general the unconstrained two-parameter CKM unitarity fit cannot explain the data perfectly anymore, i.e. $\chi^2_{\text{non-unitary}} > 0$ and therefore we have in general $\Delta\chi^2_{\text{SM test}} > \chi^2_{\text{unitarity test}}$. The point is that some patterns in the data can not be accounted for by just employing a two parameter fit without unitarity, i.e. this procedure does not account for the generality of possible BSM models.

Note that for $n \geq 3$ one cannot a priori say if $z_{\text{SM test}} > z_{\text{unitarity test}}$ or vice versa, because this does not only depend on $\Delta\chi^2_{\text{SM test}}$ and $\Delta\chi^2_{\text{unitarity test}}$ but also on the specific value of $\nu_{\text{SM test}} \geq 2$ vs. $\nu_{\text{unitarity test}} = 1$, that is the number of observables n . For example, for given values of $\Delta\chi^2_{\text{SM test}} = 20$ and $\Delta\chi^2_{\text{unitarity test}} = 10$, we have $z_{\text{SM test}} > z_{\text{unitarity test}}$ if $n = 5$ or $z_{\text{SM test}} < z_{\text{unitarity test}}$ if $n = 8$, see Fig. 1.

The above discussion makes clear what are the differences between unitarity tests and SM tests in a completely general perspective. In Sec. III we apply the above formalism to the current status of the data.

III. APPLICATION OF FORMALISM TO CURRENT DATA

Current data provides $n = 3$ precision determinations of the Cabibbo angle

$$\sin \theta_{K_{l3}} = V_{us}^{K_{l3}} = V_{us}, \quad (16)$$

$$\cos \theta_{K_{l2}} = V_{ud}^\beta = \left| \sqrt{1 - V_{us}^2} + \Delta \right|, \quad (17)$$

$$\tan \theta_\beta = \left(\frac{V_{us}}{V_{ud}} \right)^{K_{l2}} = \frac{V_{us}}{\left| \sqrt{1 - V_{us}^2} + \Delta \right|}, \quad (18)$$

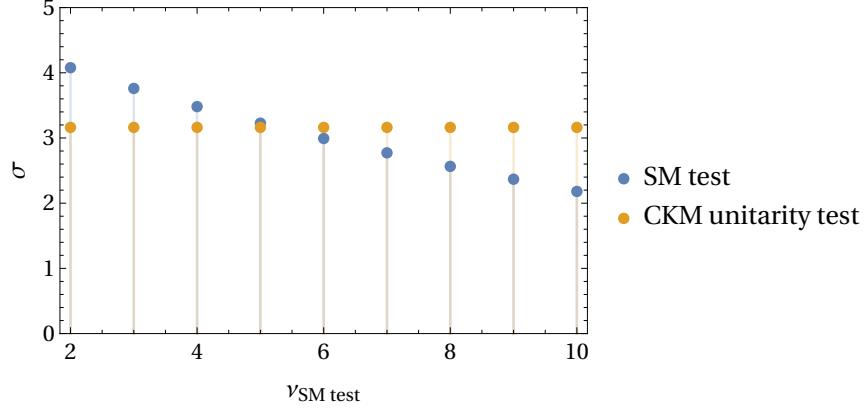


FIG. 1. Toy example for the comparison of significances of the rejection of the SM and CKM unitarity for fixed $\Delta\chi^2_{\text{SM test}} = 20$ and $\Delta\chi^2_{\text{unitarity test}} = 10$ as a function of $\nu_{\text{SM test}} \geq 2$. Note that $\nu_{\text{unitarity test}} = 1$ always and $\nu_{\text{SM test}} = n - 1$ for n observables, see Eq. (15). Of course in reality $\Delta\chi^2_{\text{SM test}}$ and $\Delta\chi^2_{\text{unitarity test}}$ would in general also change when $\nu_{\text{SM test}}$ does. However, we can see from this example that in principle either significance can be larger than the other one.

where $\theta_{K_{l3}}$, $\theta_{K_{l2}}$ and θ_{β} could all be different in BSM models. On the right hand side of Eqs. (16)–(18) we write also the expressions in terms of the parametrization for the CKM unitarity test. In the SM, all of these extractions should be equal up to corrections of order $\mathcal{O}(\lambda^6)$

$$\theta_C = \theta_{K_{l3}} = \theta_{K_{l2}} = \theta_{\beta}. \quad (19)$$

CKM unitarity on the other hand implies

$$\Delta = 0. \quad (20)$$

Eqs. (19) and (20) are the SM and CKM unitarity null hypotheses, respectively.

We summarize the latest determinations of V_{us} and V_{ud} in Table II. The obtained value for V_{ud} depends on the details of the treatment of nuclear β decays. There are extraction available from Seng, Gorchtein, Patel, Ramsey-Musolf (SGPRM) [18] and Czarnecki, Marciano, Sirlin (CMS) [19].

Our fit results are shown in Table III. Therein, also subsets of observables are considered for illustration purposes. As discussed in Sec. II, for $n = 2$ fits SM test and CKM unitarity fit give the same results, and for the current full data set with $n = 3$ they differ. While the

Observable	Measurement	Method	References
$ V_{us} ^{K_{l3}}$	0.22326 ± 0.00058	K_{l3} decays	[5, 9]
$ V_{us}/V_{ud} ^{K_{l2}}$	0.23129 ± 0.00045	K_{l2}/π_{l2} decays	[5, 12]
$ V_{ud} ^\beta$	0.97370 ± 0.00014	Nuclear β decays, SGPRM extraction	[18]
$ V_{ud} ^\beta$	0.97389 ± 0.00018	Nuclear β decays, CMS extraction	[19]

TABLE II. Observables and data used in the fits. In case of the new physics scenario these are interpreted as effective values, see Eqs. (21)–(23). The obtained value for V_{ud} depends on the details of the treatment of nuclear β decays. There are extraction available from Seng, Gorchtein, Patel, Ramsey-Musolf (SGPRM) [18] and Czarnecki, Marciano, Sirlin (CMS) [19].

Fit	n	$\Delta\chi_{\text{SM test}}^2$	$\nu_{\text{SM test}}$	$p_{\text{SM test}}$	$z_{\text{SM test}}$	$\Delta\chi_{\text{unitarity test}}^2$	$p_{\text{unitarity test}}$	$z_{\text{unitarity test}}$
$K_{l3} + K_{l2}$	2	8.5	1	0.0036	2.9σ	8.5	0.0036	2.9σ
$K_{l3} + K_{l2} + \beta$ (SGPRM)	3	30.0	2	$3.1 \cdot 10^{-7}$	5.1σ	22.8	$1.8 \cdot 10^{-6}$	4.8σ
$K_{l2} + \beta$ (SGPRM)	2	11.6	1	0.00065	3.4σ	11.6	0.00065	3.4σ
$K_{l3} + \beta$ (SGPRM)	2	30.0	1	$4.4 \cdot 10^{-8}$	5.5σ	30.0	$4.4 \cdot 10^{-8}$	5.5σ
$K_{l3} + K_{l2} + \beta$ (CMS)	3	16.5	2	0.00027	3.6σ	9.0	0.0027	3.0σ
$K_{l2} + \beta$ (CMS)	2	3.6	1	0.056	1.9σ	3.6	0.056	1.9σ
$K_{l3} + \beta$ (CMS)	2	15.1	1	0.00010	3.9σ	15.1	0.00010	3.9σ

TABLE III. SM and CKM unitarity tests for different data sets. $z_{\text{SM test}}$ is the significance of the SM rejection and $z_{\text{unitarity test}}$ is the significance of the CKM unitarity rejection.

difference of significances of CKM unitarity test and SM test is not dramatic, in case of the SPRGM interpretation the significances of rejection of SM and CKM unitarity are 5.1σ vs. 4.8σ , and for the CMS interpretation 3.6σ vs 3.0σ , respectively.

IV. NEW PHYSICS MODELS

In this section we demonstrate the ability of a concrete BSM model to describe the data with $\chi_{\text{BSM}}^2 = 0$. We emphasize that this serves as a toy example for illustration only. For a real model many more bounds than considered here would have to be discussed.

We employ the model and notation of Ref. [22] and show that BSM couplings of right-handed (RH) quarks [16, 31, 36–44] to the W boson, i.e. RH currents, could remove the tensions presented in Table III.

Following the notation of Ref. [22], we denote the respective coupling of RH strange quarks by ε_s and the one of down quarks by ε_{ns} . Furthermore, the measured values of the CKM matrix elements given in Table II are interpreted as effective ones and are related to the mixing angle and the RH couplings as [22]

$$V_{us}^{K_{l3}} = |\sin \theta_C + \varepsilon_s| , \quad (21)$$

$$\left(\frac{V_{us}}{V_{ud}} \right)^{K_{l2}} = \left| \frac{\sin \theta_C - \varepsilon_s}{\cos \theta_C - \varepsilon_{ns}} \right| , \quad (22)$$

$$V_{ud}^\beta = |\cos \theta_C + \varepsilon_{ns}| . \quad (23)$$

Note that ε_s and ε_{ns} are in general complex. However, to keep things simple for our purposes it is enough to study the real case here. The SM is obtained in the limit

$$\varepsilon_s = \varepsilon_{ns} = 0 . \quad (24)$$

Fitting the general model of right handed currents Eqs. (21)–(23), we obtain a perfect description of the data with $\chi_{\min, \text{RH}}^2 = 0$.

Moving to a different model, in case we switch off the down-quark right handed currents $\varepsilon_{ns} = 0$ we have a more constrained fit. We perform a likelihood ratio test comparing only strange RH currents with the more general case of strange and down RH currents and define

$$\Delta\chi^2 \equiv \chi_{\min, \text{RH strange}}^2 - \chi_{\min, \text{RH}}^2 . \quad (25)$$

We consider only toy NP fits to the SGPRM data set as only for that data set there is a tension with the SM beyond 5σ , and compare the toy model with RH strange quark currents to a more general toy model that includes both strange and down quark RH currents. The relatively fixed number of parameters is always one. For any two observables out of Eqs. (1, 2, 3), we obtain a vanishing $\Delta\chi^2$. However, once we take all observables Eqs. (1)–(3) into account, we get $\Delta\chi^2 = 25.2$ and a significance of rejection of $z = 5.0\sigma$. This example makes it completely obvious that it is very important to include all available data for any test for NP.

While the CKM unitarity test is a smoking gun for the presence of new physics, it is not clear how to relate it to the considered model with RH currents. The above procedure on

the other hand is completely unambiguous. Furthermore, the CKM unitarity test is included in the SM test as outlined in Sec. II A.

V. CONCLUSIONS

Recent precision determinations of V_{us} and V_{ud} enable unprecedented tests of the SM and constraints on possible NP models like right-handed currents. We showed that SM tests go beyond just tests of CKM unitarity and give different test results if more than two observables are taken into account.

In a CKM unitarity test one compares a constrained fit with a fit of free floating V_{us} and V_{ud} . The latter can not necessarily describe the data as well as a BSM model, in case the patterns go beyond just violating unitarity. This matters starting from three independent observables being taken into account. In a SM test one compares a fit of the Cabibbo angle which gives the same absolute χ^2 as the CKM unitarity fit with a completely general BSM scenario where all measured effective angles could be different, as we demonstrated explicitly for a concrete model.

That means the significance of SM tests can in general be different from the one of CKM unitarity fits once more than two observables are considered. In the foreseeable future, τ decays may provide a further precision determination of the Cabibbo angle via the ratio $\Gamma(\tau \rightarrow K\nu_\tau)/\Gamma(\tau \rightarrow \pi\nu_\tau)$, see Ref. [45], and the number of precision observables for the determination of the Cabibbo angle rises to four. With more measurements in the future the differences between SM tests and CKM unitarity tests could become even more significant.

Consequently, we encourage to test the SM by testing for the universality of the Cabibbo angle, rather than testing for CKM unitarity only, with the general methodology laid out above.

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- [1] A. Sirlin, Nucl. Phys. **B196**, 83 (1982).
- [2] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 517 (1985).
- [3] J. Bijnens and P. Talavera, Nucl. Phys. **B669**, 341 (2003), arXiv:hep-ph/0303103 [hep-ph].
- [4] V. Cirigliano, M. Giannotti, and H. Neufeld, JHEP **11**, 006 (2008), arXiv:0807.4507 [hep-ph].
- [5] M. Antonelli *et al.* (FlaviaNet Working Group on Kaon Decays), Eur. Phys. J. **C69**, 399 (2010), arXiv:1005.2323 [hep-ph].
- [6] M. Moulson, *Proceedings, 9th International Workshop on the CKM Unitarity Triangle (CKM2016): Mumbai, India, November 28-December 3, 2016*, PoS **CKM2016**, 033 (2017), arXiv:1704.04104 [hep-ex].
- [7] M. Tanabashi *et al.* (Particle Data Group), Phys. Rev. **D98**, 030001 (2018).
- [8] S. Aoki *et al.* (Flavour Lattice Averaging Group), (2019), arXiv:1902.08191 [hep-lat].
- [9] M. Moulson, “The status of V_{us} ,” Talk given at the workshop "Current and Future Status of First-Row CKM Unitarity", Amherst Center for Fundamental Interactions, UMass Amherst, Amherst, Massachusetts, USA, May 17, 2019.
- [10] V. Cirigliano and H. Neufeld, Phys. Lett. **B700**, 7 (2011), arXiv:1102.0563 [hep-ph].
- [11] D. Giusti, V. Lubicz, G. Martinelli, C. T. Sachrajda, F. Sanfilippo, S. Simula, N. Tantalo, and C. Tarantino, Phys. Rev. Lett. **120**, 072001 (2018), arXiv:1711.06537 [hep-lat].
- [12] E. Passemar, “Extraction of V_{us} from experimental measurements,” Talk given at the International Conference on Kaon Physics 2019, University of Perugia, Perugia, Italy, September 10, 2019.
- [13] M. Di Carlo, D. Giusti, V. Lubicz, G. Martinelli, C. T. Sachrajda, F. Sanfilippo, S. Simula, and N. Tantalo, Phys. Rev. **D100**, 034514 (2019), arXiv:1904.08731 [hep-lat].
- [14] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **96**, 032002 (2006), arXiv:hep-ph/0510099 [hep-ph].
- [15] J. C. Hardy and I. S. Towner, Phys. Rev. **C79**, 055502 (2009), arXiv:0812.1202 [nucl-ex].

- [16] I. S. Towner and J. C. Hardy, Rept. Prog. Phys. **73**, 046301 (2010).
- [17] J. C. Hardy and I. S. Towner, Phys. Rev. **C91**, 025501 (2015), arXiv:1411.5987 [nucl-ex].
- [18] C.-Y. Seng, M. Gorchtein, H. H. Patel, and M. J. Ramsey-Musolf, Phys. Rev. Lett. **121**, 241804 (2018), arXiv:1807.10197 [hep-ph].
- [19] A. Czarnecki, W. J. Marciano, and A. Sirlin, (2019), arXiv:1907.06737 [hep-ph].
- [20] J. Charles, A. Hocker, H. Lacker, S. Laplace, F. R. Le Diberder, J. Malcles, J. Ocariz, M. Pivk, and L. Roos (CKMfitter Group), Eur. Phys. J. **C41**, 1 (2005), arXiv:hep-ph/0406184 [hep-ph].
- [21] W. J. Marciano, Phys. Rev. Lett. **93**, 231803 (2004), arXiv:hep-ph/0402299 [hep-ph].
- [22] V. Bernard, M. Oertel, E. Passemar, and J. Stern, JHEP **01**, 015 (2008), arXiv:0707.4194 [hep-ph].
- [23] M. Antonelli *et al.* (FlaviaNet Working Group on Kaon Decays), in *PHIPSI08, proceedings of the International Workshop on $e+e-$ Collisions from ϕ to ψ , Frascati (Rome) Italy, 7-10 April 2008* (2008) arXiv:0801.1817 [hep-ph].
- [24] V. Cirigliano, J. Jenkins, and M. Gonzalez-Alonso, Nucl. Phys. **B830**, 95 (2010), arXiv:0908.1754 [hep-ph].
- [25] V. Cirigliano, G. Ecker, H. Neufeld, A. Pich, and J. Portoles, Rev. Mod. Phys. **84**, 399 (2012), arXiv:1107.6001 [hep-ph].
- [26] T. Bhattacharya, V. Cirigliano, S. D. Cohen, A. Filipuzzi, M. Gonzalez-Alonso, M. L. Graesser, R. Gupta, and H.-W. Lin, Phys. Rev. **D85**, 054512 (2012), arXiv:1110.6448 [hep-ph].
- [27] V. Cirigliano, M. Gonzalez-Alonso, and M. L. Graesser, JHEP **02**, 046 (2013), arXiv:1210.4553 [hep-ph].
- [28] V. Cirigliano, S. Gardner, and B. Holstein, Prog. Part. Nucl. Phys. **71**, 93 (2013), arXiv:1303.6953 [hep-ph].
- [29] O. Naviliat-Cuncic and M. Gonzalez-Alonso, Annalen Phys. **525**, 600 (2013), arXiv:1304.1759 [hep-ph].
- [30] J. L. Rosner, S. Stone, and R. S. Van de Water, Submitted to: Particle Data Book (2015), arXiv:1509.02220 [hep-ph].
- [31] S. Alioli, V. Cirigliano, W. Dekens, J. de Vries, and E. Mereghetti, JHEP **05**, 086 (2017), arXiv:1703.04751 [hep-ph].
- [32] B. Belfatto, R. Beradze, and Z. Berezhiani, (2019), arXiv:1906.02714 [hep-ph].
- [33] W. Tan, (2019), arXiv:1906.10262 [hep-ph].

- [34] M. Gonzalez-Alonso and J. Martin Camalich, JHEP **12**, 052 (2016), arXiv:1605.07114 [hep-ph].
- [35] M. Wiebusch, Comput. Phys. Commun. **184**, 2438 (2013), arXiv:1207.1446 [hep-ph].
- [36] J. C. Pati and A. Salam, Phys. Rev. Lett. **32**, 1083 (1974).
- [37] R. N. Mohapatra and J. C. Pati, Phys. Rev. **D11**, 2558 (1975).
- [38] R. N. Mohapatra and J. C. Pati, Phys. Rev. **D11**, 566 (1975).
- [39] F. A. Wilczek, A. Zee, R. L. Kingsley, and S. B. Treiman, Phys. Rev. **D12**, 2768 (1975).
- [40] G. Senjanovic and R. N. Mohapatra, Phys. Rev. **D12**, 1502 (1975).
- [41] G. Senjanovic, Nucl. Phys. **B153**, 334 (1979).
- [42] V. Bernard, M. Oertel, E. Passemar, and J. Stern, Phys. Lett. **B638**, 480 (2006), arXiv:hep-ph/0603202 [hep-ph].
- [43] G. Senjanovic and V. Tello, Phys. Rev. Lett. **114**, 071801 (2015), arXiv:1408.3835 [hep-ph].
- [44] V. Cirigliano, W. Dekens, J. de Vries, and E. Mereghetti, Phys. Lett. **B767**, 1 (2017), arXiv:1612.03914 [hep-ph].
- [45] M. Antonelli, V. Cirigliano, A. Lusiani, and E. Passemar, JHEP **10**, 070 (2013), arXiv:1304.8134 [hep-ph].