Unitary Bethe-Salpeter Methods in Two- and Three-Body Systems

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Abstract of Dissertation

Unitary Bethe-Salpeter Methods in Two- and Three-Body Systems

The study of resonances, unstable particles formed in particle scattering, is motivated by the questions about the strong interaction and the composition of hadrons. In order to advance this study in the intermediate energy region, a variety of techniques are used. In this thesis, we present work done on two different analyses that use the principle of unitarity and the Bethe-Salpeter equation to study the resonances $\Lambda(1405)$, $\Sigma(1385)$, and $a_1(1260)$. Firstly, we perform a simultaneous analysis of S - and P -waves of the Strangeness $= -1$ two-body meson-baryon scattering amplitude using all low-energy data. For the first time, differential cross section data are included for chiral unitary coupled-channel models. From this model, S - and P -wave amplitudes are extracted and we observe both well-known $I(J^P) = 0(1/2^-)$ S -wave states as well as a new $I(J^P) = 1(1/2^+)$ state absent in quark models and lattice QCD results. Multiple statistical and phenomenological tests suggest that, while the data clearly require an $I = 1$ P -wave resonance, the new state just accounts for the absence of the decuplet $\Sigma(1385)3/2^+$ in the model which is subsequently included in the parameterization. Secondly, we formulate the final state interaction of the $a_1(1260)$ resonance decay in a manifestly three-body-unitary parameterization and fit it to the $a_1(1260)$ lineshape measured by the ALEPH experiment. Dalitz plots calculated from this fit are presented. The work demonstrates the feasibility to numerically solve a previously derived amplitude and its generalization to isobars with spin and coupled channels. The model is a good test case because it can also be applied to other meson decays including exotic states and modified for the finite-volume problem as it arises in lattice QCD due to its manifest unitarity. The understanding of the resonances being analyzed in both the two-body work and the three-body work gives a better understanding of intermediate energy QCD.

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1 Introduction

1.1 Background

All of the material world is composed of particles and forces. The theory that describes the behavior of fundamental particles is known as the standard model. In this theory, all particles are grouped as either fermions or bosons. The fermions can be subdivided into leptons and quarks which compose hadrons. These quarks can take one of six flavors: up, down, top, bottom, charm, and strange. In a simplified view, two quarks can compose a hadron state known as a meson and three can compose a hadron called a baryon. The properties of mesons and baryons are of great interest to physics because they provide most of the mass of the matter we are made of and interact with. Several questions which dominate the study of hadrons are: What generates their mass? How do they interact with other particles? What can we learn about them from unstable particles? In this thesis, we present work that contributes to humanity's knowledge of the answers to these questions. There are four fundamental interactions that describe the behavior of particles: the strong interaction, the weak interaction, the electromagnetic interaction, and the gravitational interaction. The first three are studied in the standard model with Quantum Field Theory, QFT [\[31\]](#).

The work done in this thesis is focused on the strong interaction at lower energies.

This research is of interest because the strong force is primarily responsible for binding quarks together within a hadron and for binding nucleons together within a nucleus. Knowledge of it can be used to understand a number of phenomena related to the composition and properties of hadrons. The term confinement describes the phenomenon that one quark cannot be observed by itself. Another interesting phenomenon concerns the mass coming from the binding of quarks within a hadron, which is a consequence of non-linear, highly relativistic gluon forces. This process is

Figure 1: Some examples of Feynmann diagrams for different types of photon production with time running from top to bottom. The lines marked q and $\bar{q}$ denote quarks and anti-quarks; the lines marked g represent gluons; and the lines marked γ represent photons. Figure taken from Ref. [1].

responsible for generating the majority of the mass of hadrons and, therefore, the majority of the mass of all objects which we interact with. These phenomena, among others, motivate the study of the strong interaction and its underlying theoretical formulation, Quantum Chromodynamics (QCD).

Furthermore, one is motivated to study the strong interaction to explain data from the many experiments being done on this subject, a few of which are described in Refs. [32–35]. Billions of dollars are being spent on these experiments and quite a bit of effort has been devoted to producing results. It is fitting that some amount of effort be spent analysing the results.

The strong interaction in the intermediate energy region cannot be approached in the usual way. Typically, one performs a perturbative calculation based on a Lagrangian density composed of particle fields. The QCD interaction is associated with a gauge boson that carries the force and a coupling that describes the strength of vertices involving quarks and gluons. One can calculate the amplitude of particle interactions by writing it as a sum of infinitely many contributions. Each one of these contributions can be represented by a Feynman diagram, some examples of which are

shown in Fig. 1. The diagrams contain vertices depicted with black dots that describe gluon interactions and propagators depicted with lines that describe the propagation of particles. This sum can then be written in orders of the power of the coupling.

This coupling will have an energy dependence due to a procedure known as renormalization that allows for rigorous calculations of particle interaction amplitudes despite divergent integrals. This procedure involves introducing an energy scale at which calculations are being made, (regularization scale). Renormalization gives the coupling constant, which would otherwise be scale-independent, a dependence on the regularization scale. For the strong interaction, shown in Fig. 2, it decreases at higher energies, but at lower energies it increases to values that can exceed 1. Thus, QCD can be handled perturbatively at high energies, but at low energies, a different approach must be taken. Several methods that are used in the lower energy region at various levels of rigor are, for example, Lattice QCD, Chiral Perturbation Theory, approaches inspired by unitarity, and Quark Models.

Lattice QCD is an ab-initio approach that makes calculations for particles confined to a finite volume on a lattice. The lattice consists of a number of sites on a grid, as depicted in Fig. 3. Fields which represent quarks are defined at these sites, and fields representing gluons are defined on the links between each site and its neighbors. The aim of lattice QCD is to calculate the so-called path integral which is used to compute contributions to, e.g., amplitudes for all possible trajectories of the particles of interest. This path integral cannot be solved analytically which necessitates calculations in a finite-volume in grid points on a lattice for numerical evaluation. The strength of this method is that it works from first principles. Some properties such as symmetries are different on the lattice than in reality. However, there are methods with which calculations on the lattice can be related to those in the infinite volume. For a description of lattice QCD, we direct the interested reader to Ref. [36].

For a mapping of finite-volume lattice QCD results to the infinite volume, see,

Figure 2: The running coupling constant of the strong interaction measured in a number of different experiments. Q is the momentum transfer and α_s is the coupling constant for QCD, a dimensionless parameter that gives the strength of a vertex. It "runs", meaning its value changes as a function of Q . Figure taken from Ref. [2]

Figure 3: A depiction of a lattice. The colored dots represent sites on which quark fields are defined. These are connected by the black lines on which gluon fields are defined. (Figure taken from Ref. [3]).

e.g. [37–42]. We do not make use of lattice QCD in this thesis but we present results that could be related to equivalent lattice results.

Chiral Perturbation Theory (ChPT) exploits the chiral symmetry which indicates that rotating the left-handed and right-handed components of the quark field has no effect on the QCD Lagrangian if the quarks are massless. Through these symmetries, an effective Lagrangian can be constructed that makes a perturbative approach with hadrons instead of quarks as degrees of freedom possible. This method works very well in the low-energy region at which the systematic expansion of amplitudes in meson masses and momenta converges rapidly. In this thesis, we make use of ChPT in an ansatz we employ. For that reason, we describe it in more detail in Sec. 2.6.

The principle of unitarity, (described in Sec. 2.3), follows from conservation of probability. It can be used to extend the energy range of ChPT at the cost of losing systematic power counting. This method, which is the focus of the first part of this thesis, is reviewed in Ref. [43]. Another method, which is the focus of the second part of this thesis, uses unitarity in conjunction with the Bethe-Salpeter Equation,

(described in Sec. 2.2), to construct a model for three-particle interactions.

Quark models are phenomenological approaches to the study of hadrons that attempt to account for confinement in a rather ad-hoc way. We wish to distinguish these models — which merely attempt to explain the data without the rigor of full systematic — from effective field theories (EFTs) which are more rigorous. These quark models group quarks together. The simplest is the naive bag model which treats quarks within a hadron as if they are confined within the radius of a spherical “bag.” This model introduces a number of phenomenological parameters that cannot be calculated theoretically, including an external pressure that needs to be present to cancel the internal pressure from the quarks in the bag. Chodos and Thorn [44] made modifications to account for pions coupling to the quarks on the surface of the bag. The effect introduces an external pressure on the bag that can affect the size of the bag. Another quark model involves joining quarks via a potential. The simplest of these was given by Isgur and Carl [45] but other potentials have also been used [46–48]. The quarks in these models have constituent masses which are not calculated through theoretical or experimental methods but rather adjusted to give the most accurate observable results. In the non-relativistic case these constituent masses are roughly one third of the baryon mass or half of the meson mass. It should therefore be understood that these quarks are not the actual quarks of the QCD Lagrangian. The constituent mass, unlike the renormalized running quark mass, has no physical meaning outside the quark model and can be several orders of magnitude larger than the physical quark mass.

All of these methods, (and every other method of theoretical physics), must be related to experimental measurements. In the study of hadrons, a large amount of data is collected from scattering particles. The main observable is the cross section, σ , which can be measured by firing a beam of particles at a target and measuring the fraction of scattered particles. One can further compute the differential cross section, $\frac{d\sigma}{d\Omega}$.

In non-relativistic quantum mechanics one can impose the Sommerfeld radiation condition that, far away from the source of the scattering, the particle should be a sum of a plane wave in the direction of the incoming wave and a spherical wave with some angular dependence for the angle of the outgoing particle, θ relative to the incoming direction. In the limit, $r \rightarrow \infty$ the amplitude can be written

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{e^{ikr}}{r} f(\theta), \quad (1)$$

where bold symbols denote three-dimensional vectors and standard font denotes either the equivalent four-vector or the magnitude of the three-vector. We either use context or give an explanation to distinguish between the two possibilities for standard font. This scattering is depicted diagrammatically in Fig. 4. The differential cross section for this wave function will be $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$. In relativistic quantum mechanics, there is a scattering operator, $\mathbb{S}$ which is analogous to the right side of Eq. (1) and a reaction operator, $\mathbb{T}$, that is analogous to the second term of the right side. This $\mathbb{T}$ can be related to the cross section. The plot of the intensity at different energies is called a line shape.

For $2 \rightarrow 2$, spinless scattering, there are two kinematically independent degrees of freedom. Typically these are chosen to be the Mandelstam s and θ , the scattering angle. The former can be calculated for incoming four-momentum p_1 and p_2 and outgoing four-momentum p_3 and p_4 with $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$. This gives the total energy of the system. Note that the term $\sqrt{s}$ is also denoted in the literature as

E or W . Throughout this thesis these three terms are equivalent. Fig. 5 schematically depicts a two particle collision in the center-of-mass frame. Note that if the particles being scattered are not spinless, more degrees of freedom are possible.

The case of three-body scattering of spinless particles can be described by 8 kinematically independent degrees of freedom. Fig. 6 shows the incoming and

Figure 4: A quantum mechanical depiction of scattering. An incoming plane wave in the z -direction interacts with a target and emits an outgoing plane wave and spherical wave with some angular distribution. The Sommerfeld radiation condition requires that at large distances, the plane wave and the spherical wave with some angular dependence are the only contributions to the wave function in the spinless case. Figure taken from Ref. [4].

outgoing momenta in the three-body center-of-mass for momenta p_1, p_2, p_3 , and q_1, q_2, q_3 . The line shape plots for the $3 \rightarrow 3$ case are similar to those of the $2 \rightarrow 2$ case

with the difference that they depend on the total three-body energy, $\sqrt{s}$, where $s = (p_1 + p_2 + p_3)^2 = (q_1 + q_2 + q_3)^2$, rather than the two-body energy. There is no unique way to parameterize the 8 degrees of freedom, but a common method is to have several of them account for the energy of two-body subsystems. For outgoing momenta, q_1 and q_2 , the total energy would be σ_{12} , where $\sigma_{12} = (q_1 + q_2)^2$. Dalitz plots, which we describe in Sec. 2.4, show the distribution of likelihood of an event as a function of the energies of two-body subsystems. We describe these subsystems,

called isobars, in Sec. 2.1.

Peaks in the cross section can be caused by resonances which are unstable particles that are formed and then decay on the time-scale of the strong interaction of the order of 10^{-22} seconds. This time is much shorter than the travel time to the detector, such that only the resonance decay products, but not states themselves, can

Figure 5: A schematic depiction of $2 \rightarrow 2$ collision in the center-of-mass rest frame.

Figure 6: The three-body center-of-mass rest frame depiction of the incoming (left) and outgoing (right) four-momenta p_1, p_2, p_3 , and q_1, q_2, q_3 .

be detected. The position of the peak is associated with the resonance mass M , while the broadness of the peak is associated with the width Γ . A narrower width implies a more stable particle due to the energy-time uncertainty principle. Fig. 7 shows an example of a peak in the line-shape data associated with a resonance.

While this provides an intuitive understanding of resonance parameters, the definition of a resonance can be made more rigorous by considering pole positions in the complex energy plane. If one continues the amplitude describing a peak on the real $\sqrt{s}$ axis into the complex plane, one may find a singularity. The real and half of the imaginary energy values of the pole position give the respective mass and width of the resonances. Finding resonances in the analysis of scattering data, and specifying the masses and width is useful because many theoretical approaches like the discussed quark models can only predict these parameters. In Sec. 2.5, we briefly justify this method, known as analytic continuation, as a model independent way to study resonances. There, we will also provide a more detailed discussion including partial waves, in which resonances exist, branch points, and other non-analyticities that the S-matrix can in general possess.

These resonances, just like all particles, come with conserved quantum numbers which we briefly describe. The isospin, I , of a particle is closely related to its charge states and gives information about the difference of up and down quark content that

Figure 7: An example of a resonance. The peak in these line-shape data from The ALEPH Experiment [5] is caused by the $a_1(1260)$. The mass, M , is given by the energy value at which the peak occurs and the width, Γ , can be determined in the absence of other resonances and background by taking the width of the peak at half the maximum height. This naive parameterization can be improved in a model independent way as described in Sec. 2.5

constitutes the particle. It is defined in analogy to quantum spin. In the same way

that the total spin, S has a component, S_3 , that can take values

$S_3 = -S, -S + 1, \dots, S$, the isospin has a component, I_3 , that can take values in integer steps. The isospin component, I_3 , gives half the difference of the number of up quarks and the number of down quarks: $I_3 = \frac{1}{2}((n_u - n_{\bar{u}}) - (n_d - n_{\bar{d}}))$, where n_u is the number of up quarks, n_d is the number of down quarks and $\bar{u}$ and $\bar{d}$ represent the

respective anti-quarks. Thus, for a given isospin there are a number of isospin components possible; these components are ordered in a multiplets. For example, a proton, ($I_3 = \frac{1}{2}$), and a neutron, ($I_3 = -\frac{1}{2}$), form a doublet for $I = \frac{1}{2}$ and there is a

triplet of pions with $I = 1$, $I_3 = -1, 0, 1$.

Each of the other flavors of quarks has its own quantum number that works in a similar way. For example, strangeness, S , has a projection $S_3 = (n_s - n_{\bar{s}})$, where n_s and $n_{\bar{s}}$ are the numbers of strange and anti-strange quarks respectively. There are

also quantum numbers charm, top, and bottom, defined in similar ways.

The total angular momentum quantum number is denoted J . The decay products of a resonance of spin J couple together such that their total spin S and relative orbital

angular momentum L gives J , $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

All of the quantum numbers described so far are additive, meaning that the sum of the quantum numbers in an interaction (that preserves this quantum number) is the

same before and after. There are also multiplicative quantum numbers whose product, rather than sum is conserved. One example is parity, P , which gives the sign

of how a state transforms under the parity operation, $\hat{P}$, which replaces every coordinate with its negative: $\hat{P}\mathbf{r} = -\mathbf{r}$. Thus, $P = +1$ for state ψ if $\hat{P}|\psi\rangle = |\psi\rangle$ and $P = -1$ if $\hat{P}|\psi\rangle = -|\psi\rangle$. Two related multiplicative quantum numbers are C -parity and G -parity. C -parity specifies how a state transforms under charge conjugation, a

transformation where each particle becomes its anti-particle. The G -parity tells how

a state transforms under the operation in which a rotation in isospin space is performed followed by charge conjugation. It can generalize the concept of C -parity

to systems in a multiplet with particles that are not neutral. If we consider the pion

triplet, $\{\pi^+, \pi^0, \pi^-\}$, for example, the π^0 has well-defined C -parity but the π^+ and π^- do not. The rotation involved in G -parity allows this generalization to the other

states in the multiplet. Five of these, isospin, G -parity, spin, parity, and C -parity, are

commonly grouped together with the notation $I^G(J^{PC})$. For example, the ρ meson

has quantum numbers $I^G(J^{PC}) = 1^+(1^{--})$. Under the strong interaction these

quantum numbers are conserved.

1.2 Motivation

In this thesis we study three resonances, the $\Lambda(1405)$, the $\Sigma(1385)$ and the $a_1(1260)$.

We discuss these resonances below.

$\Lambda(1405)$: The $\Lambda(1405)$, which has quantum numbers $J^P = \frac{1}{2}^-$, was first observed over 60 years ago [49]. Its exact structure is still up for debate because the real part of its mass is below the $\bar{K}N$ threshold and therefore its peak cannot be seen in $\bar{K}N$ scattering. The hypothesis exists that it consists of two states but this is difficult to verify due to the scarcity of data at the energy value of the pole [50]. Our research on the structure of the $\Lambda(1405)$, discussed in Sec. 4, can help resolve a decades long controversy.

$\Sigma(1385)$: The $\Sigma(1385)$ is similar to the $\Lambda(1405)$ in that it is also a $S = -1$ baryon that can be studied through meson-baryon interaction. Unlike the $\Lambda(1405)$, it has quantum number $J^P = \frac{3}{2}^+$. These quantum numbers mean that it has a significant effect on the differential cross section data, $\frac{d\sigma}{d\Omega}$. This will be described in more detail in Sec. 3.1. The introduction of such data to models previously used to study the $\Lambda(1405)$ allows one to simultaneously analyze both resonances giving a more realistic scenario of each.

$a_1(1260)$: The $a_1(1260)$ almost exclusively decays into three pions, thus it makes a good case study for three body analysis. Three-body amplitudes constitute a frontier in our fundamental understanding of meson dynamics and can be used to analyze so-called exotic states with quantum numbers forbidden by the simple constituent quark-antiquark model.

Such states are forbidden because, according to the quark model, the quantum numbers are determined as follows: $J = L \oplus S$, $P = (-1)^{L+1}$, $C = (-1)^{L+S}$. According to these formulae, the J^{PC} states 0^{+-} , 1^{-+} , 2^{+-} , 3^{-+} , ... are forbidden [51]. A discovery of this type of state would provide direct evidence for explicit gluon dynamics at low energies. Such exotic states are long sought-for. The $a_1(1260)$ is not an exotic state but, as a state that decays into three pions, it is a valuable case for the testing of methods that can be used for the analysis of upcoming data. The most

Figure 8: A plot showing the intensity for the exotic $J^{PC} = 1^{-+}$ quantum numbers from the COMPASS experiment. the best fit, (red), which can be composed of a Breit-Wigner term, (blue, dashed), and a non-resonant background, (purple, dotted). (Figure from Ref. [6]).

prominent modern experimental efforts in this regard are:

COMPASS Experiment: COMPASS at CERN claimed evidence [6] of an exotic state. This experiment studied diffractive dissociation of negative pions into a $\pi^- \pi^- \pi^+$ final state, using a 190 GeV π^- beam on a 3 mm lead target. In addition to their findings of the $a_1(1260)$, $a_2(1320)$, and $\pi_2(1670)$, they report finding a state with $J^{PC} = 1^{-+}$. They fit this state yielding a mass of 1660 MeV and a width of 269 MeV. Their fit is shown with the intensity distribution for the $J^{PC} = 1^{-+}$ quantum numbers in Fig. 8.

For the fit shown, they use a Breit-Wigner, which we will describe in Sec. 2.5, with a non-resonant background. The signal shown in Fig. 8 may be an artifact originating from an over-simplification in their analysis. Several other models have also been used to analyse this data [52–55], generally referred to as *isobar models* (not to be

Figure 9: A cartoon of the setup of the GlueX experiment which has an ideal design for the search for exotic mesons. (Figure from Ref. [7]).

confused with the isobar formulation used in this thesis). Such amplitudes are quite basic and usually lack two-body unitarity for subsystems, let alone three-body unitarity, which is the main topic of this thesis. The absence of these S- matrix principles may lead to artifacts that could even resemble resonant enhancements as shown in the figure. The work of this thesis, suitably extended to more quantum numbers, can thus help provide theoretically more sound analysis tools for the analysis of the COMPASS and other experiments.

GlueX Experiment: The search for exotic states is continued by the GlueX experiment at Jefferson Lab which has a unique experimental setup in which a high-energy photon beam is used for the production of two, three, and multi-meson final states [56]. This experiment is depicted schematically in Fig. 9.

There are several reasons why this experiment is ideal for the study of exotics: Firstly, a number of exotic states are predicted in photon hadron collisions. Secondly, the beam energy is high enough to allow for full coverage of the expected range of light quark exotics. Finally, the photon beam is linearly polarized which allows the beam

to be set up in a way that measures parity.

The GlueX experiment has already given results [7, 57], and there are more to come.

The need for analysis of these results motivates the need for an understanding of three-body methods which in turn motivates a better understanding of the $a_1(1260)$.

The studies of the states $a_1(1260)$, $\Lambda(1405)$, and $\Sigma(1385)$ are structured as follows.

Sec. 2 describes in more detail subjects that were briefly referenced, namely, the isobar formulation, unitarity, the Bethe-Salpeter equation, Dalitz plots, analytic continuation and Chiral Perturbation Theory. Sec. 3 describes previous work done on the subjects discussed. Sec. 4 describes an analysis of differential cross section data, antikaon-nucleon scattering, and differentiation of universal parameters of the $\Lambda(1405)$ and the $\Sigma(1385)$. Sec. 5 describes the model we develop of the Final State Interaction (FSI) of the $\tau^- \rightarrow \pi^- \pi^- \pi^+ \nu_\tau$ decay. This model is used to study the $a_1(1260)$ resonance. We give a conclusion in Sec. 6. In order to present readable results, we have placed some of the more technical work in Appendices.

2 Technical Toolbox

2.1 The Isobar Formulation

Three-body dynamics is more complicated than two-body dynamics for several reasons, including the fact that there is an increase in the number of subsystem-interactions that must be accounted for. A full calculation will sum over all possible cases in which none of the particles interact, two of the particles interact with each other but one does not, and all three of the particles interact. This challenge can be approached with the isobar formulation. An isobar, which is denoted in Feynmann-like diagrams (Fig. 10) with a double solid line, represents a two-body subsystem. It does not need to be resonant but simply represents the on-shell two-body scattering amplitude, suitably decomposed for integration into the three-body system. When two of the three particles form an isobar, the remaining particle is called a spectator. One possible interaction between the isobar and the spectator occurs when one of the particles in the isobar is emitted and then forms a different isobar with the spectator. This is called one particle exchange. This is not ad-hoc modeling, but a consequence of three-body unitarity as discussed in Sec. 3.4. Later in this thesis, when we apply the isobar formulation to $\pi\pi\pi$ interaction, it is called One Pion Exchange (OPE).

It is also possible for all three of the particles to interact at the same time. We call this interaction the three-body contact term, and it includes all those interactions which do not contain any intermediate on-shell particles like OPE. We denote this term with an incoming and outgoing isobar/spectator pair joined in the center with a diamond. In our formalism, which we describe in Sec. 3.4, the existence of this term is a consequence of unitarity, and does not come from modeling or perturbative expansion of a Lagrangian.

If there is no interaction between the isobar and the spectator, either through one

Figure 10: Left: An isobar and a spectator that are disconnected, meaning they do not interact. Center left: An isobar and a spectator with one particle exchange. Center right: A three-body contact term where all three particles interact. Right: The decay vertex of a source decaying into an isobar and a spectator. This process is followed by strong three-body final-state interaction (FSI) as shown in Fig. 38 in more detail in a later chapter. Note that these processes are calculated through dispersion relations and not a-priori expansions of a given Lagrangian, although effective Lagrangians may be used to model these processes.

particle exchange or through three-body contact, we say the process is disconnected.

Note that even then, there is a two-body interaction in the subsystem, and the process cannot be discarded. Only the fully disconnected process in which none of the particles interact can be omitted. If there is interaction of all three particles, we say

that the diagram is connected. The first three images in Fig. 10 illustrates a disconnected isobar and specator diagram, a (connected) one particle exchange diagram, and a (connected) three-body contact diagram respectively. These are the three ingredients of three-body interaction in the isobar formulation. A proof exists that every subinteraction in the three body system can be recast as an isobar spectator interaction [58, 59].

In theory, there is no problem making calculations for interactions with three incoming and outgoing on-shell particles. But experimentally it is prohibitively difficult to prepare an experiment in which 3 particles from multiple beams simultaneously interact. In order to get data for three-body interaction, we look at situations where a resonance decays into three particles, or, more generally, where three particles undergo strong final state interaction. This decay is depicted diagrammatically by a circle with an outgoing isobar and spectator in the right diagram in Fig. 10.

2.2 The Bethe-Salpeter Equation

This section reviews standard material on the Bethe-Salpeter equation which was first introduced by Ref. [60]. For a more detailed explanation, see for example Ref. [61] or

Ref. [62]. The Bethe-Salpeter Equation is the relativistic version of the Lippman-Schwinger Equation which is used in non-relativistic quantum mechanics to describe scattering processes. The latter equation is derived from Schrödinger's equation for a sufficiently short-ranged potential, V , given by

$$\left(-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r})\right) \psi(\mathbf{r}) = E\psi(\mathbf{r}). \quad (2)$$

From this equation and the Sommerfeld radiation condition, it follows that, far enough from the scattering source,

$$\psi(\mathbf{r}) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(2\pi\hbar)^{\frac{3}{2}}} - \frac{2m}{4\pi\hbar^2} \frac{e^{ikr}}{r} \int d^3\mathbf{r}' e^{-i\mathbf{k}'\cdot\mathbf{r}'} V(\mathbf{r}') \psi(\mathbf{r}'), \quad (3)$$

as given in Ref. [63], where $\mathbf{k}$ is the incident wave vector and the potential V falls off sufficiently rapidly with r . This equation can be written in several other ways. In

momentum space an equivalent formulation is

$$T(\mathbf{p}', \mathbf{p}) = V(\mathbf{p}', \mathbf{p}) + \int d^3\mathbf{q} V(\mathbf{p}', \mathbf{q}) \frac{1}{E + i\epsilon - \frac{q^2}{2m}} T(\mathbf{q}, \mathbf{p}) \quad (4)$$

where V is the Fourier transform of the potential given by

$$V(\mathbf{p}', \mathbf{p}) = \langle \mathbf{p}' | V | \mathbf{p} \rangle = \frac{1}{(2\pi\hbar)^3} \int d^3\mathbf{r} e^{i(\mathbf{k}' - \mathbf{k})\cdot\mathbf{r}} V(\mathbf{r}), \quad (5)$$

and the T matrix is proportional to $f(\theta)$ from Eq. (1) and can thus be related to the cross section with, for example, Eq. (7.50) in Ref. [63].

The Bethe-Salpeter Equation is a relativistic version of the Lippmann-Schwinger

equation given by

$$\langle q_n | T(s) | p_m \rangle = \langle q_n | B(s) | p_m \rangle + \int \frac{d^4 k}{(2\pi)^4} \langle q_n | B(s) | k \rangle \tau(\sigma(k)) \langle k | T(s) | p_m \rangle . \quad (6)$$

Here, $\tau(\sigma(k))$ denotes the free Green's function which describes propagation and $\langle q_n | B(s) | p_m \rangle$ is the interaction kernel, respectively which describes particle interaction.

They are not determined by the BSE but in Sec. 3.4 we discuss how unitarity constrains them. The incoming and outgoing four-momenta on which T depends are denoted p_m and q_n . This T is the amplitude that can be related to the cross-section.

It can represent either 2-particle interaction or isobar spectator interaction.

There is an obvious similarity between Eq. (6) and Eq. (4) where $\langle q_n | B(s) | k \rangle$ is

$$\text{analogous to } V(\mathbf{p}', \mathbf{p}) \text{ and } \tau(\sigma(k)) \text{ is equivalent to } \frac{1}{E + i\epsilon - \frac{q^2}{2m}}.$$

A big difference is that the relativistic Bethe-Salpeter Equation includes an integral over four-momentum rather than three-momentum. The Bethe-Salpeter equation,

thus, accounts for anti-particle propagation, as well. As described in Sec. 3.4, unitarity can be matched to the BSE, reducing the four-dimensional integration to a three-dimensional one. The result may be regarded as a Lippmann-Schwinger type of equation, without anti-particle contributions, but with relativistic kinematics.

If the product $\langle q_n | B(s) | k \rangle \tau(\sigma(k))$ is small enough and analytic everywhere, the

equation can be approached iteratively, in a way analogous to the Born

Approximation. The equation and its iterative solution for this circumstance are depicted diagrammatically in Fig. 11. Because one cannot generally assume that the iterative solution is valid, especially if the interaction term has singularities, it is not the approach in this thesis. However, the iterative expansion can be used as test for the solution in certain circumstances, e.g., at low energies where one expects

Figure 11: A diagrammatic depiction of the Bethe-Salpeter Equation and its iterative solution, (convergence issues put aside).

convergence of the series.

2.3 Unitarity

In this section, we rely heavily on Ref. [63]. Unitarity applies to the scattering operator, $\mathbb{S}$, which is defined in terms of the incoming and outgoing free states, ϕ_{in} and ϕ_{out} , by $\mathbb{S}|\phi_{in}\rangle = |\phi_{out}\rangle$. If there is no interaction between the particles, the (iso)spin and momenta of the incoming states are identical to the outgoing ones. In that case, $\mathbb{S} = \mathbb{1}$. The transition operator, $\mathbb{T}$ is introduced to account for the component of $\mathbb{S}$ that describes particle interaction. In the convention of Ref. [63], it can be written as

$$\mathbb{S} = \mathbb{1} - 2\pi i \mathbb{T}, \quad (7)$$

where $\mathbb{T}$ is the reaction amplitude operator. Unitarity, which follows from conservation of probability, imposes the constraint that

$$\mathbb{S}^\dagger \mathbb{S} = \mathbb{1}. \quad (8)$$

This imposes a constraint on $\mathbb{T}$ called the optical theorem,

$$\mathbb{T} - \mathbb{T}^\dagger = 2\pi i \mathbb{T}^\dagger \mathbb{T}. \quad (9)$$

We have written these constraints in operator notation but it is also common to represent the operators in momentum space as (continuous) matrices. One can calculate the matrix elements by putting the operators between wave functions of in/outgoing asymptotic states. For the $2 \rightarrow 2$ case with incoming and outgoing momentum $\mathbf{p}$ and $\mathbf{p}'$, the $\mathcal{S}$ -matrix will be

$$\langle \mathbf{p}' | \mathbb{S} | \mathbf{p} \rangle = \delta(\mathbf{p} - \mathbf{p}') - 2\pi i \delta(E_{\mathbf{p}'} - E_{\mathbf{p}}) T(\mathbf{p}', \mathbf{p}). \quad (10)$$

where $E_{\mathbf{p}'}$ and $E_{\mathbf{p}}$ are the energies of $\mathbf{p}'$ and $\mathbf{p}$ respectively, and the T matrix is defined by

$$\langle \mathbf{p}' | \mathbb{T} | \mathbf{p} \rangle = \delta(E_{\mathbf{p}'} - E_{\mathbf{p}}) T(\mathbf{p}', \mathbf{p}). \quad (11)$$

Note that we once again follow the convention of Ref. [63]. The coefficient which is multiplied by $\mathbb{T}$ can be changed if it is done consistently. In matrix notation, the unitarity constraint of Eq. (9) on the T can be formulated

$$T(\mathbf{p}', \mathbf{p}) - T^*(\mathbf{p}', \mathbf{p}) = -4\pi m i \int d^3 q T^*(\mathbf{p}', \mathbf{q}) \delta(2mE - q^2) T(\mathbf{q}, \mathbf{p}), \quad (12)$$

where m is the reduced mass of a two-body system with masses m_1 and m_1 given by $m = \frac{m_1 m_1}{m_1 + m_2}$. This equation corresponds to Eq.(7.311) in Ref. [63]. Notably, the right-hand side contains an integral over all possible intermediate states and a delta function that puts these states on-shell. For three-body unitarity, we will use a very similar equation (see Eq. (29) below). Indeed, in order to make a model in low-energy

QCD that accurately captures behavior that follows from conservation of probability, one must impose the unitarity constraint. In Sec. 3, we describe how unitarity, in conjunction with the Bethe-Salpeter equation, can be used to construct models for two- and three-body scattering.

2.4 Dalitz Plots

The $2 \rightarrow 2$ spinless scattering process has only 2 independent kinematic degrees of freedom, usually parameterized as the Mandelstam- s and the scattering angle θ . Thus the data can be completely represented with the differential cross section, $\frac{d\sigma}{d\Omega}$, at different energies.

For scattering involving more particles, there will be more variables. The full $3 \rightarrow 3$ case has 8 degrees of freedom: The six four-momenta have 24 components from which one subtracts six for the on-shell condition, four for conservation of momentum, and 6 for the Euler angles. If the interaction has fewer incoming particles there will be fewer degrees of freedom, but in any case, two independent variables are not enough. Those

extra degrees of freedom are captured in the form of Dalitz plots which are two-dimensional plots given for a specific value or a range of Mandelstam- s . Each of the two axes of the plot corresponds to one of the three invariant masses. If the outgoing particles have four-momenta, q_1 , q_2 , and q_3 , then the invariant mass of the first two particles, (1 and 2), will be $\sigma_{12} = (q_1 + q_2)^2 = (\sqrt{s} - \sqrt{\mathbf{q}_3^2 + m^2})^2 - \mathbf{q}_3^2$, where m is the mass of the third particle. .

The Dalitz plots show the likelihood of an event occurring with given values of the invariant masses on the axes. This can be done with data by choosing a bin size and plotting the number of events in each bin. These methods can be compared to theoretical Dalitz plots which calculate the likelihood of a given event from the amplitude.

Only two of the three possible invariant masses are used in the plot, but any

two-body invariant mass can be calculated from s and the other two invariant masses. Thus, the Dalitz plot might look different depending on which two invariant masses are chosen for the axes but it will contain the same information. A sample Dalitz plot for $D_s^+ \rightarrow K^+ K^- \pi^+$ decay is shown in Fig. 12. The likelihood distribution gives the

Figure 12: A sample Dalitz plot for the process $D_s^+ \rightarrow K^+ K^- \pi^+$ from the BABAR detector. Each point represents a decay event. In this case, $m^2 = \sigma$. The horizontal axis gives the invariant mass for the $K^+ K^-$ and the vertical axis gives the invariant mass for $K^- \pi^+$. The dense bands show the positions where the two-body subsystem on the axis can form a resonance. The vertical band at around 1.1 GeV^2 is caused by the $\phi(1020)$ and the horizontal band at around 0.800 GeV^2 is caused by the $\bar{K}^*(892)^0$ resonance. Figure taken from Ref. [8].

information about the structure of the process being measured. If two of the particles can form a resonance it might dominate the interaction. In that case, one will see enhanced bands at values of the invariant mass equal to the mass of the resonance.

2.5 Analytic Continuation

In this section, we give a brief overview of analytic continuation. For more details see [2, 64, 65]. This method takes a physical amplitude calculated as a function of real

energy and analytically continues it to the complex energy plane. This is done to extract the real and imaginary position and residue of a pole associated with peaks in the data. These parameters are useful for comparison with other models like quark models which can predict resonance parameters, but not observables. Then, usually the real part of a pole position is associated with a quark model state, but this method is not rigorous and particularly ignores the fact that quark model spectra are usually real. In contrast, from the perspective of scattering theory, the association of a resonance with a complex pole is well defined and reaction-independent. In the simplest case, when there is no external background or nearby poles, one can model a resonance with the Breit-Wigner distribution,

$$T(E) \sim \frac{1}{E^2 - M^2 + iM\Gamma}, \quad (13)$$

where E is the energy and M and Γ are free parameters. In the case where a Breit-Wigner distribution describes the cross section data for a resonance, one can obtain the real pole positions simply by finding the height of the peak and the imaginary part simply by taking the full width at half maximum. These quantities correspond to M and $\Gamma/2$ respectively. The pole parameters can be extracted directly from the data without calculating the function for imaginary energy values. Very few resonances, however, can be approximated with a Breit-Wigner distribution. Close-by thresholds distort the Breit-Wigner shape, and the generalized version of Eq. (13) depends on the orbital angular momenta of the decay products which induces momentum-dependent width. Also, if there is a multi-channel decay, the widths in the denominator add. More sophisticated resonance parametrizations like Flatté distributions are discussed, e.g., in Ref. [66]. Only a few resonances, for example the $\rho(770)$, can be approximated well with a Breit-Wigner. Most other resonances have a completely different shape as illustrated in Fig. 13 which shows the

Figure 13: A plot of the real (solid blue) and imaginary (dashed red) $\bar{K}N$ amplitude for $I(J^P) = 0(\frac{1}{2}^-)$. The shaded regions show the 1σ uncertainty. The shape shown, caused by the two poles of the $\Lambda(1405)$ is distinctly different from the Breit-Wigner shape. This difference illustrates the benefit of calculating pole parameters which are independent of background.

$\bar{K}N$ amplitude influenced by the poles of the $\Lambda(1405)$ and by different thresholds (see also Fig. 13 below).

In addition to the complex position of the pole, one can extract the residue of the pole. It can be shown that the amplitude as a function of complex energy can be written as a Laurent series around the pole, or, more generally if there is more than one pole, as

a sum of the residue terms at the different pole positions and a smooth function (Mittag-Leffler theorem). The background is then absorbed in the higher terms in the expansion making the residue another background independent resonance parameter that can be extracted from analytic continuation. If a resonance is in a multi-channel environment, it can also be shown that the resonance position is channel independent, but the residues are not, playing the role of branching fractions (or, more precisely, products of branching fractions according to the in- and out-states of the amplitude).

In order to continue the amplitude to the complex plane, the analytic structure of the function must be taken into account. The amplitude as a function of complex energy

will be connected to the real axis by several Riemann sheets [65]. The energy threshold of the particles being scattered determines a branch point on the real axis.

There will be a sheet that is connected above the real axis which is usually referred to as first, or physical, sheet. If one follows this sheet to values below the real axis it exhibits a discontinuity and its part below the real axis is no longer analytically connected to the physical axis. Similarly, the second sheet is connected below the real axis and will not be connected above the real axis. At the branch points, both sheets are connected. The branch points are endpoints of branch cuts at which the function is discontinuous. For example, if one chooses to run the cut along the real axis, it is responsible for the mentioned discontinuity. Unstable particles have branch points in their two-body sub-amplitude, at the respective two-body threshold which induce additional branch points in the three-body amplitude, at $\sqrt{s} = 3M$ if all particles have the same mass M . If the sub-amplitude has a resonance pole itself (i.e., in the unstable-particle case of a resonant isobar), these poles likewise induce branch points in the complex three-body energy, $\sqrt{s}$. All these additional branch points are simple and induce one additional cut [64]. In a coupled channel approach, there will be multiple sheets, each with different branch points, but the resonance poles on each sheet will have the same location for each amplitude associated with channel transition $i \rightarrow j$. When these poles are found, they can be used to compare results to other models. The branch points and branch cuts define the structure of the complex plane. Two examples of the analytic structure of the complex plane are shown in

Fig. 14.

The analytic structure of the complex plane is further complicated when one uses the common method of Partial Wave Analysis, (PWA). This involves taking an amplitude for particle scattering and writing it as a sum of contributions corresponding to different relative angular momenta of the particles being scattered. For more details and more rigorous definitions, see Sec. 5.1.2 in a later chapter and Ref. [67]. Each partial wave, a contribution for a given relative angular momentum, can be handled separately, allowing one to study only the part of the amplitude corresponding to the

Figure 14: Depictions of the analytic structure of the complex $\sqrt{s}$ plane for $\pi\pi$ scattering on the left and $S = -1$ meson-baryon scattering on the right. The quantity $\sqrt{s}$ stands for the total three-body and two-body energy, respectively. The gray regions contain partial wave specific cuts and the left-hand cuts. The colored dots depict the threshold branch points for the particle collisions, including unstable particles with complex branch points. The black lines give the branch cuts. The black stars show the position of resonances, namely the $a_1(1260)$ on the left and the two poles of the $\Lambda(1405)$ and the $\Sigma(1385)$ on the right.

resonance of interest. This method introduces new cuts for a given partial wave.

When all partial waves are summed over, these wave-dependent cuts will vanish, but they must be accounted for when one is performing analytic continuation for a given partial wave. These sub-threshold cuts are indicated with gray areas in Fig. 14. Note that there is one more cut, the so-called left-hand cut, which is sub-threshold as well, but not only present in partial wave amplitudes but also the plane-wave amplitude.

Poles on the first Riemann sheet — which is connected above the real axis — are forbidden by causality. This can be shown with a proof that, if the Green's function obeys the causality commutation relations, it must be analytic for all energies with a positive imaginary parts [65]. The second Riemann sheet, connected below the axis, does not satisfy these requirements, meaning that it does not have to be analytic.

Thus, it is where one looks for poles when doing analytic continuation.

2.6 Chiral Perturbation Theory

This thesis does not explicitly use chiral perturbation theory, but it does make heavy use of Refs. [9, 11, 68] which take ChPT as an inspiration for the Bethe-Salpeter kernel for 2-body scattering. Thus, we give a brief description of the theory here, but we aim only to give the most rudimentary understanding necessary to motivate the ways in which it is applied in this thesis. For a more detailed description, see Ref. [69]. Throughout this description, we rely heavily on Refs. [9, 69, 70]. ChPT relies on approximate chiral symmetries of the QCD Lagrangian density. These QCD symmetries are implemented in a tower of effective Lagrangians that can be used to perturbatively calculate low-energy processes. The QCD Lagrangian is

$$\mathcal{L}_{QCD} = \sum_{f=u,d,s,c,b,t} \bar{q}_f(i\not{D} - m_f)q_f - \frac{1}{4}\mathcal{G}_{a\mu\nu}\mathcal{G}_a^{\mu\nu}, \quad (14)$$

where we have omitted the CP -violating θ term [71]. The index f sums over the flavor for quark fields, q_f and quark masses m_f . The symbol $\not{D}$ is the Feynman slash on the gauge covariant derivative, $D_\mu = \partial_\mu + ig\mathcal{A}_\mu$, where $\mathcal{A}_\mu \equiv \mathcal{A}_{a\mu}\lambda_a/2$ is the gluon field, λ_a are the Gell-Mann matrices, a is the color index, and g is the strong coupling. The symbol $\mathcal{G}$ is the gauge invariant gluon field strength tensor which can be written

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{G}_{a\mu\nu} \frac{\lambda_a}{2} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + ig[\mathcal{A}_\mu, \mathcal{A}_\nu].$$

The different flavors of quarks have very different masses. In the $\overline{MS}$ scheme, (which is not the same as the constituent quark models), they are: $m_u = [2.16^{+0.49}_{-0.26}]$ MeV,

$$m_d = [4.67^{+0.48}_{-0.17}] \text{ MeV}, m_s = [93^{+11}_{-5}] \text{ MeV},$$

$$m_c = [1270 \pm 20] \text{ MeV}, m_b = [4180 \pm 30] \text{ MeV}, \text{ and } m_t = [172900] \pm 400 \text{ MeV} \quad [2].$$

Chiral Perturbation Theory deals with the low energy limit, well below the masses of the top, bottom and charm quarks.

The chiral operators are $P_L = \frac{1}{2}(\mathbb{1} - \gamma_5)$ and $P_R = \frac{1}{2}(\mathbb{1} + \gamma_5)$ which project the quark fields to the left-handed and right-handed component fields via $q_R = P_R q$ and

$q_L = P_L q$. These operators satisfy $P_R + P_L = \mathbb{1}$.

In the limit of small momenta and masses, using the decomposition of quark fields into left-handed and right-handed fields and setting $m_{u,d,s} \rightarrow 0$ one can write the

Lagrangian as

$$\mathcal{L}_{QCD} = \sum_{f=u,d,s} (\bar{q}_{R,f} i \not{D} q_{R,f} + \bar{q}_{L,f} i \not{D} q_{L,f}) - \frac{1}{4} \mathcal{G}_{\alpha\mu\nu} \mathcal{G}_\alpha^{\mu\nu}. \quad (15)$$

The charm, top, and bottom quarks are neglected in this Lagrangian because their masses are much larger than the energy range at which ChPT is valid. This Lagrangian is $U_L(3) \times U_R(3)$ invariant for quark field transformations $q_R \rightarrow U_R q_R$ and

$$q_L \rightarrow U_L q_L \text{ [69].}$$

This symmetry can be decomposed according to

$$U_L(3) \times U_R(3) = SU_V(3) \times SU_A(3) \times U_V(1) \times U_A(1), \quad (16)$$

where the subscripts V and A for vector and axial can be related to the right-handed and left-handed subscripts with $V = (L + R)$ and $A = (L - R)$.

From Noether's theorem it follows that symmetries lead to conservation laws. The

$U_V(1)$ symmetry induces Baryon number conservation. From each of these symmetries, there exists a conserved current and invariant charge. The invariant charges for the vector and axial currents can be written Q_{V_a} and Q_{A_a} . When one applies the charge operators to the vacuum state, one gets $Q_{V_a}|V_a\rangle = 0$ and $Q_{A_a}|A_a\rangle \neq 0$ [69]. This means that the vector symmetry remains but the axial symmetry is broken. This corresponds to symmetry breaking observed in nature.

This symmetry breaking is called spontaneous symmetry breaking as opposed to explicit symmetry breaking because the Lagrangian preserves the symmetry but it is not present in the ground state. Furthermore, the $U_A(1)$ current is not conserved due to quantum effects.

Thus the $U_L(3) \times U_R(3)$ spontaneously break to a $SU_V(3)$ symmetry. The 8 generators of the broken/hidden $SU(3)_A$ are called Goldstone Bosons, ϕ_a , which are identified with π^+ , π^- , π^0 , K^+ , K^- , K^0 , $\bar{K}^0$, and η . They are grouped in the matrix

$$\phi = \sum_{a=1}^8 \phi_a \lambda_a \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{\sqrt{1}}{3}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \quad (17)$$

The effective ChPT Lagrangian can be written in terms of the Goldstone Bosons as an expansion of increasing powers of derivatives and quark-mass terms [72]:

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots \quad (18)$$

Note that odd power terms do not contribute because the Lagrangian is a scalar, therefore the fields must be contracted in pairs. On top of different powers of masses and momenta in the Lagrangian, one needs to take the structure of a given perturbative expansion into account (number of vertices, loops, propagators). This method to arrive at a systematic expansion scheme is referred to as power counting [72].

In this thesis, we are interested in ChPT extended to meson-baryon interaction [73].

In this case, the odd powers will contribute to the expansion of the Lagrangian because the nucleon (and baryon) in general have a mass which does not disappear in the limit $m_{u,d,s} \rightarrow 0$. The corresponding Lagrangian reads

$$\mathcal{L}_{\phi B}^{eff} = \mathcal{L}_{\phi B}^{(1)} + \mathcal{L}_{\phi B}^{(2)} + \mathcal{L}_{\phi B}^{(3)} + \dots, \quad (19)$$

where the low-lying baryons are given in the matrix

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}. \quad (20)$$

The Goldstone bosons and the baryons are used to construct the chiral Lagrangian from which Feynman rules can be derived following the usual methods of QFT. We describe the method that is used to construct this Lagrangian in Sec. [3.2](#).

3 Literature Review

In this section, we summarize the literature that gives the state-of-the-art prior to work done in this thesis. We begin, in Sec. 3.1 by describing the available experimental data for $S = -1$ meson-baryon scattering. This consists of total cross section measurements, differential cross section measurements, and threshold measurements. In Sec. 3.2 we describe the model developed by M. Mai and U. Meißner which is the basis for the $\bar{K}N$ analysis performed in this work. This model had previously been fit to the total cross section data and threshold data and made predictions for the pole position of the $\Lambda(1405)$.

Sec. 3.3 gives some very general background, without going into any specifics, about three body approaches in low-energy QCD. In Sec. 3.4, we summarize Ref. [17], which derives a three-body model based on the Bethe-Salpeter Equation and the principle of unitarity. This model is the basis for the three-body analysis presented in this thesis. It is the first one to implement the proof of manifest unitarity. We summarize some experimental data in Sec. 3.5 and the theoretical approaches in Sec. 3.6.

3.1 Experimental Data for $S = -1$ Meson-Baryon Scattering

The available data that can be used for the study of the $\Lambda(1405)$ and the $\Sigma(1385)$ has been collected over several decades and takes a number of different forms. Because we show fits to the data in Sec. 3.2 and Sec. 4 it would be redundant to show it here.

Instead, we describe the data in this section and show plots in the next one.

There are 10 possible meson-baryon pairs with the right quantum numbers

$I(J^P) = 0(\frac{1}{2}^-)$, $S = -1$ which can form the $\Lambda(1405)$. They are K^-p , $\bar{K}^0n$, $\pi^0\Lambda$, $\pi^0\Sigma^0$, $\pi^+\Sigma^-$, $\pi^-\Sigma^+$, $\eta\Lambda$, $\eta\Sigma^0$, $K^+\Xi^-$, and $K^0\Xi^0$. Out of these, only the first 6 have thresholds close enough to sizeably contribute to the $\bar{K}N$ amplitude around its threshold. In principle there are 6 possible incoming states and 6 possible outgoing states for cross section data that could be used to study the $\Lambda(1405)$. However, data

is available for only one incoming state, K^-p . This is not surprising because of the instability of other baryons. Thus, there are 6 channels for the $\Lambda(1405)$. These data are given in Refs. [25–27, 74] and shown in Fig. 15.

The K^-p incoming channel has a threshold higher than 1405 MeV so that the peak caused by the $\Lambda(1405)$ cannot be seen directly. But the width of $\Lambda(1405)$ is large enough that it can affect cross sectional data with real energy values above the threshold. Thus, data sets for processes with thresholds above the mass of the $\Lambda(1405)$ can be used to study it.

Differential cross section data are also available [28]. The bulk of these data corresponds to energies where the ChPT kernel ceases to be a good approximation, but some are available at lower energies for the $K^-p \rightarrow K^-p$ and $K^-p \rightarrow K^0n$ channels. The differential cross section makes it possible to simultaneously study the effects of the $\Sigma(1385)$ and the $\Lambda(1405)$ with a coupled channel analysis. Prior to this thesis, no such analysis had been performed.

Measurements of the energy shift and width of kaonic hydrogen can also constrain theoretical predictions. This was done by the SIDDHARTA collaboration, see Ref. [75]. Ref. [76] relates these results to the complex $\bar{K}N$ scattering lengths at the threshold including isospin breaking. There are plans to continue this experimental research to study kaonic deuterium at JPARC and Frascati. Such results can be related to $S = -1$ interactions but they would require a three-body analysis to account for the three components of the kaonic deuterium system [77]. This additionally highlights the importance of the development of three-body analysis models which is also a focus of this work.

The threshold amplitudes can be further constrained by decay ratios $\gamma = (K^-p \rightarrow \Sigma^- \pi^+) / (K^-p \rightarrow \Sigma^+ \pi^-)$, $R_n = (K^-p \rightarrow \Lambda \pi^0) / (K^-p \rightarrow \Lambda \pi^0, \Sigma^0 \pi^0)$, $R_c = (K^-p \rightarrow \text{charged particles}) / (K^-p \rightarrow \text{all final states})$ from Refs. [24, 78]. The strength of having SIDDHARTA data and ratios calculated at the K^-p threshold is

that the threshold is the point closest to the pole of the $\Lambda(1405)$ in the complex plane, thus, it is of great importance to constrain the amplitude with observables at the threshold.

Further comparisons to data can be made by utilizing measurements from experiments for which the process of interest is a subprocess. One such scenario is $\gamma p \rightarrow K^+(\pi\Sigma)$ which is measured by the CLAS collaboration [10] giving data shown in Fig. 16. Ref. [11] introduces generic couplings for $\gamma p \rightarrow K^+MB$ to the 10 channels mentioned above, where M and B represent a meson and a baryon respectively. These couplings, in addition to free parameters from the interaction of interest, can be fit to data. Another such reaction is from Hemingway [30] with data shown in Fig. 35, is $K^-p \rightarrow \Sigma^+(1660)\pi^-$, $\Sigma^+(1660) \rightarrow \Lambda(1405)\pi^+$, $\Lambda(1405) \rightarrow \Sigma\pi$. The generic couplings for $K^-p \rightarrow \Sigma(1660)\pi^-$ and $\Sigma(1660) \rightarrow (\pi^-\Sigma^+)\pi^+$ can be fit to the Hemingway data, giving a process that can be used to study the $\Lambda(1405)$. Note that the $\gamma p \rightarrow K^+(\pi\Sigma)$ data require additional fit parameters unrelated to the $\Lambda(1405)$ which inhibits its ability to constrain the values of pole positions as the K^-p cross section data. Nevertheless, it is still useful in this study.

In summary, the data of interest take the form of total cross section data, differential cross section data, kaonic hydrogen measurements, threshold ratios, and more complicated processes from the CLAS collaboration and Hemingway which can be fit with additional couplings. All of these data are utilized in this thesis in Sec. 4.

3.2 A Model for $S = -1$ Meson Baryon Scattering

We here refer to a model derived over a series of works [9, 11, 68], developed by Maxim Mai and Ulf Meißner which we extend in this thesis. In this section we summarize its state prior to work done. It makes predictions for and is constrained by the data previously described. The model is the only approach which has the correct low-energy behavior, fulfills two-body unitarity and describes S - and P -waves

simultaneously without introducing explicit states. Ref. [68] gives the method of calculating the Bethe-Salpeter Kernel from the Lagrangian and analytic solution of the BSE building on previous work [79–81], Ref. [9] describes the development of the model, and Ref. [11] gives the most recent iteration of the model, (prior to work in this thesis).

The model starts from the chiral Lagrangian for meson-baryon interaction which is expanded in Eq. (19) with a first order term,

$$\mathcal{L}_{\phi B}^{(1)} = \langle \bar{B}(i\gamma_\mu D^\mu - m_0)B \rangle + \frac{D/F}{2} \langle \bar{B}\gamma_\mu\gamma_5[u^\mu, B]_\pm \rangle , \quad (21)$$

where $\langle \dots \rangle$ denotes the trace in flavor space, $D_\mu B := \partial_\mu B + \frac{1}{2}[[u^\dagger, \partial_\mu u], B]$, m_0 is the baryon mass, D , F are the axial coupling constants. The “/” between them should not be understood as a ratio but as an alternative between the two with D corresponding to “+” and F corresponding to “-”. They parametrize the Born terms which are free parameters. However these terms are neglected in this model, because they induce overlapping ultraviolet divergences in the rescattering series that are difficult to renormalize. The mesons and baryons are grouped in matrices B and ϕ given in Eq. (17) and Eq. (20). The meson matrix is related through U ,

$$U = \exp\left(i \frac{\phi}{F_0}\right) \quad (22)$$

$$\begin{aligned} u^2 &:= U , \quad u^\mu := iu^\dagger \partial^\mu u - iu \partial^\mu u^\dagger , \\ \chi_\pm &:= u^\dagger \chi u^\dagger \pm u \chi^\dagger u , \quad \chi := 2B_0 \mathbf{u} , \end{aligned} \quad (23)$$

with ϕ given in Eq. (17). $F_0 = 92.3$ MeV is the meson decay constant in the chiral limit, B_0 is related to quark condensates in the chiral limit, and $\mathbf{u}$ is the quark mass matrix, $\mathbf{u} = \text{diag}(m_u, m_d, m_s)$. This Lagrangian leads to the Weinberg-Tomaza term in the amplitude which is the first order approximation for the Bethe-Salpeter kernel,

but the strength of the model we employ is that it considers up to next to leading order chiral expansion. The second order Lagrangian term in the power counting scheme is given by

$$\begin{aligned}
\mathcal{L}_{\phi B}^{(2)} = & b_{D/F} \langle \bar{B} [\chi_+, B]_{\pm} \rangle + b_0 \langle \bar{B} B \rangle \langle \chi_+ \rangle \\
& + b_{1/2} \langle \bar{B} [u_\mu, [u^\mu, B]_{\mp}] \rangle + b_3 \langle \bar{B} \{u_\mu, \{u^\mu, B\}\} \rangle + b_4 \langle \bar{B} B \rangle \langle u_\mu u^\mu \rangle \\
& + ib_{5/6} \langle \bar{B} \sigma^{\mu\nu} [u_\mu, u_\nu], B]_{\mp} \rangle + ib_7 \langle \bar{B} \sigma^{\mu\nu} u_\mu \rangle \langle u_\nu B \rangle \\
& + \frac{i b_{8/9}}{2m_0} \left(\langle \bar{B} \gamma^\mu [u_\mu, [u_\nu, [D^\nu, B]_{\mp}]]_{\mp} \rangle + \langle \bar{B} \gamma^\mu [D_\nu, [u^\nu, [u_\mu, B]_{\mp}]]_{\mp} \rangle \right) \\
& + \frac{i b_{10}}{2m_0} \left(\langle \bar{B} \gamma^\mu \{u_\mu, \{u_\nu, [D^\nu, B]\}\} \rangle + \langle \bar{B} \gamma^\mu [D_\nu, \{u^\nu, \{u_\mu, B\}\}] \rangle \right) \\
& + \frac{i b_{11}}{2m_0} \left(2 \langle \bar{B} \gamma^\mu [D_\nu, B] \rangle \langle u_\mu u^\nu \rangle + \langle \bar{B} \gamma^\mu B \rangle \langle [D_\nu, u_\mu] u^\nu + u_\mu [D_\nu, u^\nu] \rangle \right), \quad (24)
\end{aligned}$$

where b_i are the Low-Energy Constants (LEC)s. From these Lagrangians, one can evaluate the full interaction kernel, V , up to Next to Leading Order (NLO). If the Lagrangian at NNLO were considered, it would require 78 independent operators. [82]. The perturbative amplitude calculated from both Lagrangians can then be used in the Bethe-Salpeter Equation to calculate the unitary amplitude of the interactions.

$$T(q_2, q_1; p) = V(q_2, q_1; p) + i \int \frac{d^d \ell}{(2\pi)^d} \frac{V(q_2, \ell; p)}{\ell^2 - M^2 + i\epsilon} \frac{1}{\not{p} - \not{\ell} - m + i\epsilon} T(\ell, q_1; p), \quad (25)$$

where p is the total 4-momentum of the system and q_1 and q_2 are the incoming and outgoing 4-momentum of one of the scattered particles, respectively. This equation can be solved for the amplitude, T , which is then related to the total cross section data, threshold data, and SIDDHARTA data described in the previous section.

The Lagrangian in Eq. (21) also induces baryon s and u -channel exchange, but especially the u -channel exchange cannot be solved for in the full Bethe-Salpeter equation due to its topology. Instead, this contribution is neglected and absorbed in the values of the low-energy constants of Eq. (25).

Figure 15: The best fit solutions collected in Tab. 3.2 and error bands in comparison with the total cross section data. These data do the majority of the constraining of the free-parameters. Figure taken from Ref. [9].

The free parameters of the model are the LECs, (not including D/F which are neglected), that parameterize the NLO Lagrangian as well as the regularization scale for each channel, which make the integral solvable. There are 14 LECs and a regularization scales for each channel, thus there are 20 free parameters. Because there are so many degrees of freedom, and because the mathematical expression for the amplitude is quite intricate, fitting presents a challenge. The size of the sample space means that there are many local minima in the χ^2 function. Therefore, there is no guarantee that there is only one set of free parameters that describe the data well; in fact, there are quite a few such solutions.

In order to address this challenge, the C++ library Minuit2 was used to perform a rapid minimization starting from thousands of randomly generated starting parameters. This allowed the collection of a large number of solutions. These solutions must meet a further condition. Causality requires that the amplitude must not have any poles when it is analytically continued to the first Riemann sheet,

see Sec. 2.5. This condition is not built into the Bethe-Salpeter equation, thus many sets of best-fit parameters do have unphysical poles on the first Riemann sheet.

Fit #	1	2	3	4	5	6	7	8
$\chi^2_{\text{d.o.f.}}$ (hadronic data)	1.35	1.14	0.99	0.96	1.06	1.02	1.15	0.90
$\chi^2_{\text{p.p.}}$ (CLAS data)	3.18	1.94	2.56	1.77	1.90	6.11	2.93	3.14

Table 1: Quality of the various fits in the description of the hadronic and the photoproduction data from CLAS. $\chi^2_{\text{p.p.}}$ is the χ^2 per data point and $\chi^2_{\text{d.o.f.}}$ is the χ^2 per degree of freedom. Table from Ref. [11]

The large set of best fit parameters is tested and only the ones that preserve causality are kept. This yields 8 solutions. The χ^2 for these solutions from the hadronic data, given in the upper row of Tab. 1, range between 0.9 and 1.35. Thus, all solutions are reasonable fits of these data sets.

Fig. 15 shows the fit of the eight solutions to the cross section data. These sets constitute the majority of the data and impose the strongest constraints on the free parameters. The fits come with error bands generated using the bootstrap method [83].

The eight fits are also compared to the CLAS photoproduction data [10] and the Hemingway lineshape data [30]. Because these data sets require the fit of additional free parameters, they are not used for the fit of the LECs or scale factors. Instead, the 8 fits are taken as is and the additional couplings are fit. The χ^2 for these fits is shown in the lower row of Tab. 1. For some sets of LECs and scale factors, there was no fit of the couplings that returned a good χ^2 , thus the CLAS data can rule out solutions, even though it requires the fit of additional free parameters to be compared to results in this sector. The fits are shown for $W = 2.5$ GeV in Fig. 16. The plot shows that solutions 6 is particularly bad. Even though it accurately described the data to which it was fit, it could not describe the CLAS data. On the other hand, solution 4 describes the CLAS data well.

When these fits are analytically continued into the complex plane, they each give the

Figure 16: The 8 best fit solutions of the hadronic data compared with the CLAS photoproduction data [10]. Because these data require additional free-parameters to compare to the processes of interest, the LECs and scale constants are not fit to these data. However, even with the additional degrees of freedom, some fits of the LECs and scale constants to the cross section and threshold data cannot describe this photoproduction data. Thus some fits, like #6 are ruled out. Figure taken from Ref. [11]

broad and narrow poles of the $\Lambda(1405)$. The confidence region for these poles given by the variances of the real and imaginary pole positions are shown in Fig. 17 along with the confidence region for the poles from other models. Some of these models include an NLO term in the Lagrangian that is not complete [13, 14], a model that parametrized the Bethe-Salpeter kernel with free parameters [16], and the fit of this model prior to the iteration being shown [15]

The model we employ in its form immediately prior to work in this thesis solves the Bethe-Salpeter Equation with a kernel from a chiral Lagrangian up to NLO. It is compared to all available data except the differential cross section data. The S -wave component of the amplitude that dominates the total cross section has no angular dependent effect on $\frac{d\sigma}{d\Omega}$. This means that the P -wave components which provide only a small contribution to the total cross section can be isolated and determined by differential cross section data assuming that the D -wave is negligible. Note that there are two different possible contributions to the P -wave corresponding to different

Figure 17: The analytic continuation confidence regions of the poles of the $\Lambda(1405)$ for the model we employ (colored rectangles) and other models (Black crosses) [12–16]. Figure from [9].

parities and the differential cross-section data does not resolve *which* P -wave contribution a structure lies on, as will be discussed later. The neglected D -wave contribution would depend on the $\cos \theta$ to a higher power than the P -wave and, at higher energies than we consider, would be characterized by the $\Lambda(1520)$.

Because the parameters for the LECs and the scale constants are simultaneously fit for the coupled channel model that could describe both S -wave and P -wave, the positions of the two resonances are correlated. An analysis of the differential cross section data in order to determine the effects of the $\Sigma(1385)$ resonance (a P -wave state) constitutes the two-body work in this thesis. It will be discussed in Sec. 4.

3.3 First Approaches to Three-Body Interactions

The relativistic quantum three-body problem is considerably more complicated than the two-body version. There have been several frameworks developed to solve this problem, each with its own strengths and weaknesses.

The Faddeev equations are the most general framework for three-body interactions [84, 85]. A considerable simplification of the Faddeev equations was achieved in Ref. [86]. They were initially derived for the non-relativistic case but they have a relativistic version. In principle, the Faddeev equations give a complete solution to the three-body problem. But numerical implementation for their solutions presents a challenge. One strategy is to use them for potentials that can be separated into incoming and outgoing components, but for potentials that cannot be decomposed into a product of terms from each component, the calculations become very difficult. This motivates the search for a different set of equations for the three-body problem.

The Aaron Amado and Young (AAY) model [87–93] provides a framework for the three-body problem which uses the isobar formulation to apply the BSE to the three-body case. They use the BSE and three-body unitarity to construct a three-body amplitude. However, their derivation for the kernel is only valid for bound-state particle scattering. Ref. [17], which generalizes this proof to hold for higher energies, is the basis of this work.

A third framework can be found in the Khuri-Treiman equations [94–102]. These equations take calculations for $2 \rightarrow 2$ particle scattering and rotate the interaction to a $1 \rightarrow 3$ decay. The limitation of this method is that it does not give results for the $3 \rightarrow 3$ case. However, since data for three-body interactions tend to come from $1 \rightarrow 3$ decays, such equations are still very useful. The comparison between the modified Aaron, Amado, and Young model, and the Khuri-Treiman method is currently being investigated in collaboration with Bonn University.

Figure 18: The total scattering amplitude as composed by the connected part, T_c and the disconnected part, T_D . Single lines depict stable particles and double lines depict isobars. Figure taken from Ref. [17].

The Faddeev equations, Khuri-Treiman equations, and Aaron Amado and Young model give the broad background for the relativistic, quantum three-body problem

Next, we will go more into specifics in our description of more recent work.

3.4 Three-body Unitarity with Isobars Revisited

In the following we will summarize the work of Ref. [17] which derives an expression for a three-body amplitude. The first step is to expand the scattering matrix, $\mathcal{S}$.

$\mathcal{S} := \mathbb{1} + i(2\pi)^4 \delta^4(P_i - P_f) \hat{T}$, where P_i and P_f are the initial and final total four-momentum of the system and the total T -matrix is written $\hat{T}$ which is distinguished from the operator that indicates internal three-body rescattering denoted by T and depicted in Fig. 18.

The scattering amplitude for the $3 \rightarrow 3$ process, $\hat{T}$, has a connected piece, $\hat{T}_c$, and a disconnected piece, $\hat{T}_d$ as shown in Fig. 18. The full amplitude is given by

$$\langle q_1, q_2, q_3 | \hat{T}(s) | p_1, p_2, p_3 \rangle = \langle q_1, q_2, q_3 | \hat{T}_c(s) | p_1, p_2, p_3 \rangle + \langle q_1, q_2, q_3 | \hat{T}_d(s) | p_1, p_2, p_3 \rangle, \quad (26)$$

where q_i are the outgoing momenta and p_i are the incoming momenta. The connected and disconnected pieces can be written:

$$\langle q_1, q_2, q_3 | \hat{T}_c(s) | p_1, p_2, p_3 \rangle = \frac{1}{3!} \sum_{n=1}^3 \sum_{m=1}^3 v(q_{\bar{n}}, q_{\bar{n}}) S(\sigma(q_n)) \langle q_n | T(s) | p_m \rangle S(\sigma(p_m)) v(p_{\bar{m}}, p_{\bar{m}}), \quad (27)$$

and

$$\langle q_1, q_2, q_3 | \hat{T}_d(s) | p_1, p_2, p_3 \rangle = \frac{1}{3!} \sum_{n=1}^3 \sum_{m=1}^3 2E_{q_n} (2\pi)^3 \delta^3(\mathbf{q}_n - \mathbf{p}_m) v(q_{\bar{n}}, q_{\bar{n}}) S(\sigma(q_n)) v(p_{\bar{m}}, p_{\bar{m}}) . \quad (28)$$

Here, S is the isobar propagator and v is the $\rho \rightarrow \pi\pi$ decay vertex which depends on the four-momentum of the outgoing pions.

From the unitarity of the $\mathcal{S}$ matrix it follows that

$$\begin{aligned} \langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle &= i \int \prod_{\ell=1}^3 \left[\frac{d^4 k_\ell}{(2\pi)^4} (2\pi) \delta^+(k_\ell^2 - m^2) \right] \\ &\times (2\pi)^4 \delta^4 \left(P - \sum_{\ell=1}^3 k_\ell \right) \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle . \end{aligned} \quad (29)$$

Here the k_i are the intermediate four-momenta. The right side of Eq. (29) can be expanded in terms of the disconnected and connected amplitudes, T_d and T_c , which can be written as the sums of permutations of states. These terms from this expansion are depicted in the topologies on the left side of Fig. 19. Topologies (1a), (1b), (3a), and (3b) are for disconnected-connected combinations, $\hat{T}_d^\dagger \hat{T}_c$ and $\hat{T}_c^\dagger \hat{T}_d$. Topology (2) is for the disconnected-disconnected diagram $\hat{T}_d^\dagger \hat{T}_d$. Topologies (4a) and (4b) are for the fully connected diagram $\hat{T}_c^\dagger \hat{T}_c$.

These terms will be matched to equivalent terms from a similar expansion from the

Bethe-Salpeter equation, given as:

$$\langle q_n | T(s) | p_m \rangle = \langle q_n | B(s) | p_m \rangle + \int \frac{d^4 k}{(2\pi)^4} \langle q_n | B(s) | k \rangle \tau(\sigma(k)) \langle k | T(s) | p_m \rangle , \quad (30)$$

where $\tau(\sigma(k))$ represents the Green's function, k is the spectator momentum, and $\langle q_n | B(s) | p_m \rangle$ represents the interaction kernel. Without imposing unitarity, neither quantity is fixed.

Figure 19: The left hand side of this figure shows the topological representation of the expansion of unitarity relation (29). The right hand side shows the expansion of the Bethe Salpeter amplitude 32. The figure is displayed in a suggestive way to present matching conditions. The double lines indicate isobars. The black dashed lines indicate intermediate momenta k_i . The red boxes contain the terms considered by the AAY model however the terms in the blue dashed box also exist and only vanish for bound state-particle scattering and at energies below threshold. Figure taken from Ref. [17].

In operator notation, the ansatz for the Bethe-Salpeter Equation, (Eq. (6)), is

$$T = B + B\tau T = B + T\tau B \quad (31)$$

This leads to

$$\begin{aligned} \hat{T} - \hat{T}^\dagger &= v(S - S^\dagger)v + vSTSv - vS^\dagger T^\dagger S^\dagger v \\ &= v(S - S^\dagger)v + \underbrace{(vS - vS^\dagger)TSv}_{\rightarrow(1a)} + \underbrace{vS^\dagger T^\dagger (Sv - S^\dagger v)}_{\rightarrow(1b)} + vS^\dagger(T - T^\dagger)Sv. \\ T - T^\dagger &= \underbrace{(B - B^\dagger)}_{\rightarrow(2)} + \underbrace{(B - B^\dagger)\tau T}_{\rightarrow(3a)} + \underbrace{T^\dagger\tau^\dagger(B - B^\dagger)}_{\rightarrow(3b)} + \underbrace{T^\dagger(\tau - \tau^\dagger)T}_{\rightarrow(4a)} + \underbrace{T^\dagger\tau^\dagger(B - B^\dagger)\tau T}_{\rightarrow(4b)}, \end{aligned} \quad (32)$$

The underbraces in Eq. (32) indicate how their topology is depicted in the right side of Fig. 19. That figure is displayed suggestively to allow a comparison between the right side and the left. From the matches in the figure, the most general form of the matching conditions for S can be derived:

$$\begin{aligned} S(\sigma(k)) - S^\dagger(\sigma(k)) &= i \frac{S(\sigma(k))S^\dagger(\sigma(k))}{2(2\pi)^2} \int d^4\bar{K} \delta^+ \left(\left(\frac{P-k}{2} + \bar{K} \right)^2 - m^2 \right) \\ &\times \delta^+ \left(\left(\frac{P-k}{2} - \bar{K} \right)^2 - m^2 \right) \left(v \left(\frac{P-k}{2} + \bar{K}, \frac{P-k}{2} - \bar{K} \right) \right)^2. \end{aligned} \quad (33)$$

Here τ will take the form $\tau(\sigma(k)) := (2\pi)\delta^+(k^2 - m^2)S(\sigma(k))$, where

$\delta^+(x) = \delta(x)\Theta(x)$. The most general form of the discontinuity for B is

$$\langle q|B|p\rangle - \langle q|B^\dagger|p\rangle = iv(P-p-q, q)(2\pi)\delta^+((P-q-p)^2 - m^2)v(P-p-q, p) \quad (34)$$

Figure 20: One-meson exchange contribution from Eq. (35). The left side shows the forward time-ordered piece and the right side shows the backwards time-ordered part that can be added without violating unitarity because it is strictly real.

The expression for the interaction term, B , is derived from these discontinuities

$$\langle q|B(s)|p\rangle = \frac{v(P-p-q, q)v(P-p-q, p)}{m^2 - u - i\epsilon}. \quad (35)$$

This expression corresponds to a sum of a forward and backward time-ordered OPE as depicted in Fig. 20. Note that Eq. (34) only fixes the imaginary part and the backwards time-ordered exchange will be strictly real in the physical region.

Therefore, it can be added without affecting unitarity, thus providing an analytic approximation to the u -channel kinematics. The kernel does not explicitly include a t -channel exchange, however, this approach does not claim to explicitly include a term for each possible contribution. Instead it writes the kernel in the most general way that the constraint of unitarity allows. In particular, t -process exchange implies the exchange of at least two pions. Such a process can only go on-shell for three-body energies $\sqrt{s} > 5m_\pi$ which is the inelastic threshold beyond which 5-body unitarity needs to be taken into account, anyway.

To find the propagator, τ , we write the equations in terms of the inverse,

$$D(\sigma(k)) := -\tau^{-1}(\sigma(k)). \quad (36)$$

The discontinuity of D for one S -wave isobar with the simple case setting $v(p, q) = \lambda$

is $\text{Disc } D(\sigma(k)) = \frac{i}{8\pi} \frac{K_{\text{cm}}}{\sqrt{\sigma(k)}} \lambda^2$ Here $K_{\text{cm}} = \sqrt{\sigma(k)/4 - m^2}$ is the magnitude of 3-momentum of the particles from isobar decay defined in the isobar center of mass.

D can be reconstructed from this equation, e.g., via a twice-subtracted dispersion relation,

$$D(\sigma(k)) = A + B \sigma(k) + \frac{\sigma(k)^2}{\pi} \int_{4m^2}^{\infty} d\sigma' \frac{\text{Im } D(\sigma')}{\sigma'^2(\sigma' - \sigma(k) - i\epsilon)} . \quad (37)$$

Substituting Eq. (35) into the Bethe Salpeter ansatz gives

$$\begin{aligned} \langle q|T(s)|p \rangle &= \frac{v(P-p-q, p)v(P-p-q, q)}{m^2 - (P-p-q)^2 - i\epsilon} \\ &\quad - \int \frac{d^3\ell}{(2\pi)^3} \frac{1}{2E_\ell} \frac{v(P-q-\ell, \ell)v(P-q-\ell, q)}{m^2 - (P-q-\ell)^2 - i\epsilon} \frac{1}{D(\sigma(\ell))} \langle \ell|T(s)|p \rangle . \end{aligned} \quad (38)$$

Here v is the isobar decay vertex and σ is the invariant mass of the isobar squared,

$D(\sigma(\ell))$ is the inverse propagator.

This expression is unitary beyond the bound state-particle scattering case and valid up to the first inelastic threshold. The scattering problem can be written in terms of on-shell $2 \rightarrow 2$ amplitudes that only depend on the invariant mass. These amplitudes are general, i.e., one can go beyond the simplistic picture of a “resonant” isobar, and,

if wished, directly insert scattering phase shifts from experiment, with or without resonances. It should be mentioned that one needs the analytic continuation below

the two-particle threshold as input. This makes the result dependent on the unphysical two-body sub-threshold region. To account for the physical contact force between all three bodies, we add a term, C , which can be done without affecting our

matching condition because it is strictly real:

$$\begin{aligned} \langle q|T(s)|p \rangle &= \langle q|C(s)|p \rangle + \frac{1}{m^2 - (P-p-q)^2 - i\epsilon} - \int \frac{d^3\ell}{(2\pi)^3} \frac{1}{2E_\ell} T_{22} \\ &\quad \times (\sigma(\ell)) \left(\langle \ell|C(s)|p \rangle + \frac{1}{m^2 - (P-p-\ell)^2 - i\epsilon} \right) \langle \ell|T(s)|p \rangle . \end{aligned} \quad (39)$$

This equation allows one to numerically solve for the amplitude. Much of this thesis involves working with an equivalent integral equation that takes into account spin and isospin and it emphasizes the numerical problems involved in solving this equation in practice.

3.5 Experimental Studies of the $a_1(1260)$

The method for calculation of three-body amplitudes outlined in the previous section can be applied to several sets of available data. As mentioned, the $\pi\pi\pi \rightarrow \pi\pi\pi$ process is not directly observable because one cannot prepare three different states in a way that guarantees they interact simultaneously. Instead, we consider data from decay processes that have three pions in the final state, ensuring the $\pi\pi\pi$ interaction.

Here, we summarize the data from three collaborations.

The Argus Experiment: The first is the work of the Argus collaboration which presented experimental results for the reaction $e^+e^- \rightarrow \tau^+\tau^-$ [18]. The τ^- could decay into 6 different channels. Approximately 6% of the τ^- decays were $\pi^-\pi^-\pi^+\nu_\tau$.

Approximately 7500 events were observed.

The advantage of this process for the study of the $a_1(1260)$ is that the ν_τ interacts with the other particles on the scale of the weak interaction, which has a coupling constant on the order of 10^{-6} . This is vanishingly small compared to the strong interaction with a coupling constant on the order of 10^0 . Thus, the contribution of ν_τ is negligible. One can observe the effects of three-pion interaction in this process.

Even more relevant is the fact that one has a well-defined initial state that automatically filters out the quantum numbers of the $a_1(1260)$, $I^G(J^{PC}) = 1^-(1^{++})$. This can be seen in contrast to pion or photon-induced reactions corresponding to the

COMPASS and GlueX experiments, respectively. There, many partial waves/quantum numbers are produced simultaneously and require a partial wave decomposition first (see discussion in Sec. 1.2). The energy that the neutrino takes

from the system will vary for each event which means that a spectrum of energies for the three pion system can be collected.

The projected and unprojected Dalitz plots from this paper, (Ref. [18]), are shown in Fig. 21. The outline of the ρ band with constructive interference is visible. One sees that at energies close to the peak of the line shape there are many events but those farther away are much more sparsely populated. Thus, one sees the qualitative effects one would expect for the Dalitz plots for the $a_1(1260)$.

In addition to the Argus collaboration, the OPAL collaboration at CERN [103] and the DELPHI collaboration [104], both at CERN present data for the $\tau^- \rightarrow \pi^- \pi^- \pi^+$.

These results follow a similar process as that of the ARGUS collaboration in generating the data.

The CLEO Experiment: A second experimental study that is worth discussing in detail was performed by the CLEO collaboration [19]. This experiment collected results for $\tau^- \rightarrow \pi^- \pi^0 \pi^0 \nu_\tau$. This process will be very similar to $\tau^- \rightarrow \pi^- \pi^- \pi^+$ because each case has two pions with the same isospin projection and one that is different. The different one can form a ρ -meson with one of the other two pions, but the two pions with the same isospin projection cannot form a ρ -meson.

The CLEO experiment records 4.3×10^6 $\tau^- \tau^+$ event pairs. This large data set makes it possible to calculate branching ratios for the $a_1(1260)$ decay process using a

$$\text{model-dependent fit. They are: } \Gamma((\rho\pi)_{S\text{-wave}})/\Gamma_{Total} = 0.6019,$$

$$\Gamma((\rho\pi)_{D\text{-wave}})/\Gamma_{Total} = 0.0130 \pm 0.0060 \pm 0.0022,$$

$$\Gamma((\rho(1450)\pi)_{S\text{-wave}})/\Gamma_{Total} = 0.0056 \pm 0.0084 \pm 0.0032,$$

$$\Gamma((\rho(1450)\pi)_{D\text{-wave}})/\Gamma_{Total} = 0.0204 \pm 0.0120 \pm 0.0028, \text{ and}$$

$\Gamma(\sigma\pi)/\Gamma_{Total} = 0.1876 \pm 0.0429 \pm 0.0148$. These branching ratios indicate that the dominant intermediate contribution in the $a_1(1260) \rightarrow \pi\pi\pi$ decay comes from $\rho\pi$ in S -wave but $\pi\sigma$ also has a notable contribution. Note that for these branching ratios,

Figure 21: The Dalitz plots, (left), and projected Dalitz plots, measured by the Argus collaboration [18]. The plots show the events as a function of two-body invariant mass within bins of three-body invariant mass. The bins are $0.8 \text{ GeV} < m_{3\pi} < 1.0 \text{ GeV}$, $1.0 \text{ GeV} < m_{3\pi} < 1.2 \text{ GeV}$, $1.2 \text{ GeV} < m_{3\pi} < 1.4 \text{ GeV}$, and $1.4 \text{ GeV} < m_{3\pi} < 1.6 \text{ GeV}$. The thick black lines in the left column show the expected location of the ρ band. The density of points within those bands confirms the expectation. This can be compared to the results of this thesis shown in Fig. 45. Figure taken from Ref. [18].

a very simple isobar model was used that did not take into account the three-body unitary rescattering effects which are the main subject of this thesis.

In addition, the CLEO collaboration report the line shape, Dalitz plots and projected Dalitz plots, the latter of which are shown in Fig. 22. These plots show the same ρ band structure and kinematic reflection as the equivalent Argus plots, but the larger number of events allows a smaller $\sqrt{s}$ bin size.

The ALEPH Experiment: Further experimental results for $\tau^- \rightarrow \pi^- \pi^- \pi^+ \nu_\tau$ are available from the ALEPH collaboration at CERN Switzerland [105]. This experiment produced over 300,000 τ decay events which were analysed in Ref. [5] and Ref. [20]. They present lineshapes for $\tau^- \rightarrow \pi^0 \pi^- \nu_\tau$, $\tau^- \rightarrow \pi^0 \pi^0 \pi^0 \pi^- \nu_\tau$, $\tau^- \rightarrow \pi^0 \pi^+ \pi^- \pi^- \nu_\tau$, $\tau^- \rightarrow \pi^0 \pi^0 \pi^- \nu_\tau$, and $\tau^- \rightarrow \pi^+ \pi^- \pi^- \nu_\tau$. Out of these, we are only interested in the last one because it has three outgoing pions and their isospins ensure that in the restriction to the $\pi\rho$ channels made in this thesis any two pion cluster has the quantum numbers of a ρ^0 meson. The line shape for the $\pi^+ \pi^- \pi^-$ decay is shown in Fig. 23.

This line shape data is similar to the data from the ARGUS collaboration in that they both show a pronounced peak that dominates the entire spectrum. Note that differences in shape from the plots are in part a result of the fact that the ALEPH plot [20] gives the invariant mass squared on the horizontal axis and the ARGUS plot gives the invariant mass on the horizontal axis.

These results were updated with a new analysis of the data in Ref. [20]. (See Fig. 24 for the relevant comparison.) This paper uses a different method to separate the detector effects from the results and uses more sophisticated statistical methods to calculate the uncertainties of the data. The new points in the line shape have error bars for two dimensions rather than one. This upgrade gives great benefits to a very detailed comparison of theory to the data. But the centers of the points are very

Figure 22: The projected Dalitz plot data (red points), as a function of the $\pi^- \pi^0$ invariant mass (s_1 or s_2). The solid black line shows the fit result and the colored histogram gives the background. Plots (a) to (h) correspond to the bins for $\sqrt{s} = 0.6-0.9, 0.9-1.0, 1.0-1.1, 1.1-1.2, 1.2-1.3, 1.3-1.4, 1.4-1.5, 1.5-1.8$ GeV. Figure taken from Ref. [19].

Figure 23: The Line Shape data from the ALEPH collaboration for the channels $\tau^- \rightarrow \pi^- \pi^0 \pi^0 \nu_\tau$, (Black points), and $\tau^- \rightarrow \pi^- \pi^- \pi^+ \nu_\tau$, (Green shaded band). The latter channel is relevant to this work. Figure from [5].

Figure 24: The Line Shape data from the ALEPH collaboration that was updated with statistical methods compared to the older line shape [5]. Figure from [20]

similar. With the current state-of-the-art, no theoretical models, (including ours) have reached the precision in which the difference between the data sets becomes relevant. However, we look forward to an era in which detailed three-body analysis makes the newer data necessary.

3.6 Theoretical Approaches to the $a_1(1260)$

To our knowledge, the work presented in this thesis is the first to study the $a_1(1260)$ that both accounts for rescattering and utilizes a manifest proof for unitarity, but it is certainly not the only work to study the $a_1(1260)$. In order to present our results in their proper context, we briefly summarize three other theoretical approaches.

Janssen Thesis: The first of these is the M.S. thesis of Georg Janssen [106] which gave a detailed description of work published in Ref. [21]. This thesis used the Bethe-Salpeter equation to solve for three-body amplitudes, but it did not make use

of unitarity. We outline a few points of interest about the method used in Janssen's thesis.

Janssen's thesis uses OPE for the kernel of the Bethe-Salpeter equation which is similar to the kernel derived in Sec. 3.4, however, Janssen did not invoke unitarity. A further difference between the thesis work and Refs. [17] is that there, the $\pi\rho$ interaction is modeled from a hadronic Lagrangian, whereas in this thesis, the interaction is not modeled but simply absorbed in an energy and momentum dependent counter term, except, of course, for the OPE required by unitarity. The OPE interaction in Janssen has the denominator

$$\sqrt{s} - E_p - E_q - \sqrt{m^2 + p^2 + q^2 + 2pq \cos \theta}, \quad (40)$$

which can be compared to the denominator of Eq. (35), in a comparable form

$$(\sqrt{s} - E_p - E_q)^2 - (m^2 + p^2 + q^2 + 2pq \cos \theta). \quad (41)$$

For positive, real energies and momenta in the physical region, the two denominators will yield the same poles. Janssen's version of OPE has the structure of the left diagram in Fig. 20 whereas the denominator we use in this work includes both diagrams meaning that it accounts for both the forward and backward time-ordered pion exchange, ensuring the correct analytic behavior in the u -channel. Because the backward time-ordered pion exchange denominator cannot equal 0 for real momenta, an integral over it will not yield an imaginary part. This means that the inclusion or exclusion of this term will not affect the unitarity constraint on the OPE term. Thus, an understanding of unitarity not only allows the derivation of the Bethe-Salpeter kernel, but it also allows for an informed decision about which model-dependent choices are allowed.

Furthermore, Janssen's thesis gives detailed calculations for the propagator, τ , for the

Figure 25: The line shape calculated with (solid line) and without (dashed line) the inclusion of a dressed $a_1(1260) \rightarrow \rho\pi$ vertex function. Figure from Ref. [21].

Bethe-Salpeter equation, (Eq. (6)), but in its final results, it uses a stable ρ . In our thesis, an understanding of the constraints of unitarity allows for a calculation of the amplitude. Without this understanding, the approximations that Janssen makes are a reasonable solution to a difficult problem. With these approximations, the line shape is shown in Fig. 25.

Kamano et al. Another theoretical study, Ref. [22] by Kamano et al, makes calculations for the process $a_1(1260) \rightarrow \pi^0\pi^+\pi^-$. Like the three-body work done in this thesis for $1 \rightarrow 3$ decay, they calculated the amplitude from a sum of a connected part that includes rescattering terms and a disconnected part that does not. From this amplitude, they calculated observables including the Dalitz plot which is shown on the left side of Fig. 26. The structure of the plot is dominated by the ρ bands which exhibit constructive interference as one would expect from Bose symmetry. The right side of the figure shows the ratio of the Dalitz plot calculated with the rescattering terms to that calculated without indicating the effect of the

Figure 26: (left) Dalitz plot of $a_1(1260) \rightarrow \pi^+\pi^-\pi^0$; (right) ratio of Dalitz plot distributions with and without the rescattering terms. Figure from Ref. [22].

rescattering. The large ratio particularly in the region in which both invariant masses are small justifies their inclusion of the rescattering terms.

The paper was published in 2011, before a manifest proof of unitarity for the rescattering terms above breakup was released. We have not checked whether their formalism would still fulfill manifest three-body unitarity, as does ours.

Mikhasenko et al. The final paper we discuss here is from 2018 by Mikhasenko et al. [23]. It makes a detailed calculation of the τ decay process that approximates unitarity which simplifies their calculation by neglecting the complicated moving singularities in the OPE process for the Bethe-Salpeter equation and instead giving the formula for the line-shape:

$$\frac{d\Gamma}{ds} = \frac{1}{s} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \frac{c\rho(s)}{|m^2 - s - ig^2C(s)/2|^2}. \quad (42)$$

Here $\rho(s)$ is the propagator for the ρ -meson. This formula has four variations; two of them are called non-dispersive and have $C(s) = \rho(s)$. The other two are in the dispersive (DISP) case where $C(s) = \tilde{\rho}(s)$, with $\tilde{\rho}$ defined by

Figure 27: Fit to ALEPH data with the four models from [23]. The models differ by either including the effect of interference between two $\rho\pi$ decay channels (SYMM) or (QTB), and either using the dispersive integral over the phase space (DISP), or not. The lower panels show the normalized residues. Figure from Ref. [23].

$$i\tilde{\rho}(s) = l_0 + \frac{s}{\pi} \int_{9m_\pi^2}^{\infty} ds' \frac{\rho(s')}{s'(s' - s - i\epsilon)}. \quad (43)$$

Both the dispersive and the non-dispersive amplitude can be related to the data in a Symmetrized (SYMM) way by summing over both possible spectators. They can be written in the Quasi-Two-Body, (QTB) way in which only one π^- case is considered. Thus, there are four cases, QTB, SYMM, DISP-QTB, DISP-SYMM. The fits of these

four models to the data are shown in Fig. 27.

From the figure, it is clear that the dispersive case yields a better fit, but it is unclear whether the SYMM model is better than the QTB model. The dispersive SYMM and

QTB are then used to calculate the pole position with analytic continuation. The

plots of the poles are shown in Fig. 28.

Each case has a pole for the $a_1(1260)$ as well as a spurious pole—an artifact of the parametrization—that is located behind the cut from the complex $\pi\rho$ branch point on

Figure 28: Analytic continuation of the amplitude for different models from [23]: QTBDISP (Left plot), SYMMDISP (Right plot). Lines indicate equipotential levels. The poles of the amplitude are the yellow spots. The red dots indicate branch points corresponding to the opening of decay channels. Figure from [23]. See also Fig. 14.

the second Riemann sheet. In the QTB case, the pole is far enough from the cut that it has a significant effect on the real axis, whereas in the SYMM case, it is well hidden behind the branch cut and the effects of poles on the data come almost entirely from the $a_1(1260)$ pole. This pole has a real and imaginary value of $(1209 - 576i) \pm (4 - 11i)$ MeV. This value is the main result of the paper and is quoted in the PDG [2].

Each of the three theory works summarized here is useful for insight or comparison and presents its own unique set of results.

4 S - and P -wave structure of $S = -1$ meson-baryon scattering in the resonance region

The contents of this chapter coincides to a large part with the work of Ref. [107]. All authors of Ref. [107] have agreed that the article can be used as part of this thesis.

4.1 Introduction

Strangeness plays an important role in various facets of strongly interacting matter. For example, the attractive interaction between antikaons and nucleons leads to the famous $\Lambda(1405)$ -resonance as predicted in Ref. [49]. Nowadays this low-energy,

$S = -1$ region is accessed by chiral unitary coupled-channel models [11, 13, 15, 29, 80, 81, 108–121], amplitude analyses [122–127], lattice QCD [128–132], and quark models [133–135], see, e.g., the reviews in Refs. [2, 136]. A similar mechanism can be responsible for the generation of K^-pp bound states [137, 138] as predicted in Ref. [139]. Furthermore, the equation of state of neutron stars is sensitive to the antikaon condensate [140, 141] and thus to the propagation of antikaons in nuclear medium. In the era of high-precision measurements of neutron star properties with LIGO [142], this ultimately can lead to new interconnection between QCD and astrophysical observations.

At the core of all theoretical studies lies the antikaon-nucleon scattering amplitude, which has to address the non-perturbative regime of QCD valid over a large energy range including the resonance region. Such a twice non-perturbative amplitude can neither rely on perturbative QCD nor its low-energy effective field theory. Thus, some model dependence has to be introduced, with the corresponding parameters being fitted to data. In this thesis we use the model derived in a series of works [9, 11, 11, 68]

which to our knowledge is the only approach, which has the correct low-energy behavior, fulfills two-body unitarity and describes S - and P -waves simultaneously without introducing explicit states. This property is utilized to fit all available scattering and threshold data including (for the first time) differential cross sections,

simultaneously, up to energies well below the D -wave $\Lambda(1520)3/2^-$ resonance.

4.2 Model

The entire database for meson-baryon scattering in the strangeness $S = -1$ sector comes from experiments with K^-p in the initial state. (See Sec. 3.1.) In general, 10 combinations of ground state octet mesons and baryons have the same quantum numbers, meaning that the scattering amplitude T must describe the dynamics of all coupled-channels simultaneously. Two-body unitarity constrains the form of such a scattering amplitude, incorporated exactly via the Bethe-Salpeter equation in d Minkowski dimensions

$$T(\not{q}_2, \not{q}_1; p) = V(\not{q}_2, \not{q}_1; p) + i \int \frac{d^d \ell}{(2\pi)^d} \frac{V(\not{q}_2, \not{\ell}; p)}{\ell^2 - M^2 + i\epsilon} \frac{1}{\not{p} - \not{\ell} - m + i\epsilon} T(\not{\ell}, \not{q}_1; p), \quad (44)$$

where p is the total and $q_{1/2}$ are the incoming/outgoing meson four-momenta, while m/M denotes the mass of the baryon/meson in each channel, respectively. Different channels are related by $SU(3)_f$ symmetry and its breaking in the interaction kernel V and can be further restricted by demanding the correct energy-dependence around the corresponding thresholds. This requirement is addressed by taking the ChPT [143, 144] local potential up to the next-to-leading chiral order

$$\begin{aligned} V(\not{q}_2, \not{q}_1; p) = & A_{WT}(\not{q}_1 + \not{q}_2) + A_{1-4}(\not{q}_1 \cdot \not{q}_2) + A_{5-7}[\not{q}_1, \not{q}_2] \\ & + A_{0DF} + A_{8-11} \left(\not{q}_2(\not{q}_1 \cdot p) + \not{q}_1(\not{q}_2 \cdot p) \right), \end{aligned} \quad (45)$$

where A are matrices over the channel space that depend on the meson decay constants (fixed throughout this work together with meson and baryon masses to the physical values [2]) and low-energy constants (LECs) $\{b_0, b_D, b_F, b_1, \dots, b_{11}\}$. The latter are not known from ChPT (See Sec. 3.2) and must be fitted to the data. The present model corresponds to an infinite set of ChPT Feynman diagrams. Since

this set does not contain all possible diagrams, the regularization scale of the loop integration in Eq. (44) does not cancel out. To express it differently, this scale parameterizes the missing Feynman topologies to some extent and is dealt with as free parameters of the model. Neglecting isospin breaking, there are six such parameters called $\{a_i|i = 1, \dots, 6\}$ corresponding to the 6 channels with available data. Previous analyses [9, 15, 68] have shown that off-shell terms explicitly contained in Eq. (44) have only minor effect and are dropped here as well. With this approximation the scattering amplitude is solved analytically and is projected to physical observables [145]. In particular, for channel indices i and j the partial wave amplitudes read

$$16\pi W f_{L\pm}^{ij} = + \sqrt{E_i + m_i} \sqrt{E_j + m_j} \left(A_L^{ij} + \left(W - \frac{m_i + m_j}{2} \right) B_L^{ij} \right) - \sqrt{E_i - m_i} \sqrt{E_j - m_j} \left(A_{L\pm 1}^{ij} - \left(W + \frac{m_i + m_j}{2} \right) B_{L\pm 1}^{ij} \right). \quad (46)$$

Here $E = \sqrt{m^2 + q_{cms}^2}$ for q_{cms} being the modulus of the three-momentum in the center of mass system, while A_L^{ij} and B_L^{ij} denote the partial wave projected (to angular momentum L) invariant amplitudes. On-shell the latter are related to the scattering amplitude (44) as $T_{ON}^{ij} = A^{ij} + (\not{q} + \not{q}') B^{ij}$, see for more details and further definitions Sec. 2 of Ref [145]. To our knowledge the model extended in this research is the only existing unitary coupled-channel model for antikaon-nucleon interaction which contains explicit S - and P -wave interactions, derived from the low-energy behavior of QCD Green's functions.

4.3 Data and fits

The available data consist of: (a) Total cross sections [25–27, 74] for $\{K^-p \rightarrow \mathcal{S}_i | i = 1, \dots, 6\}$ in the lab-momentum range of $P_{lab} < 300$ MeV; (b) Differential cross sections [28] in the same momentum window for the $K^-p \rightarrow K^-p$

Figure 29: Real (solid, blue) and imaginary (red, dashed) parts with 1σ uncertainty bands of $\bar{K}N$ partial wave amplitudes with $I(J^P)$. The vertical dashed lines show the positions of the $\pi\Lambda$, $\pi^0\Sigma^0$, $\pi^+\Sigma^-$, $\pi^-\Sigma^+$, K^-p , $\bar{K}^0n$ thresholds, and the dots over the plots represent the available data in the respective channel from top to bottom.

and $K^-p \rightarrow \bar{K}^0n$ reactions; (c) Threshold decay ratios [24, 78] $\gamma = 2.38 \pm 0.04$, $R_n = 0.189 \pm 0.015$, $R_c = 0.664 \pm 0.011$, and energy shift $\Delta E = 283 \pm 42$ eV and width $\Gamma = 541 \pm 110$ eV of kaonic hydrogen measured in the SIDDHARTA [75] experiment and related [76] to the K^-p complex-valued scattering length a_{K^-p} . Further data, related indirectly to the $\bar{K}N$ scattering amplitude, require additional model assumptions and are not used in fits. *A-posteriori* checks to these data will be discussed below.

We employ a multi-step fitting routine to explore the parameter space as extensively as possible.

- We set all parameters b to zero, effectively reducing the chiral order of the driving term V of Eq. (44). The remaining parameters are fitted to the data starting from a large set $\mathcal{O}(10^3)$ of randomized starting values of natural size, i.e. $|a_i| \lesssim 5$.
- Using the best fits of the previous step as starting values we fit all parameters of the model simultaneously. The starting values of the so-called dynamical LECs $b_1 - b_{11}$ are chosen randomly and fitted within natural limits $|b_{1-11}| < 10 \text{ GeV}^{-1}$.

The starting values of the symmetry-breaking LECs b_0, b_D, b_F are chosen to be consistent with ground-state octet baryon masses and $\sigma_{\pi N}$ at the next-to-leading chiral order. In the fit, small variations around these values are allowed. We found that this stabilizes the fit considerably.

- We impose an analyticity constraint on the scattering amplitude, disregarding solutions with poles on the first Riemann sheet closer than 150 MeV to the real axis. Note that all poles on the first Riemann sheet are forbidden even those farther than 150 MeV from the real axis. However, we cannot easily verify whether the model with the given parameters has poles at very large energies and such poles will have a very small contribution to the physical amplitude.

χ^2_{dof}	1.10 (62/30/8%)
$ \Sigma \text{ [GeV}^{-14}\text{]}$	0.106

a_{1-6}	$b_{1-11} \text{ [GeV}^{-1}\text{]}$
$+0.44^{+0.28}_{-0.14}$	$-0.34^{+0.11}_{-0.04}$
$+2.04^{+1.37}_{-0.50}$	$+0.29^{+0.21}_{-0.18}$
$+0.14^{+0.15}_{-0.20}$	$-0.62^{+0.04}_{-0.06}$
$-0.95^{+0.30}_{-0.31}$	$-0.02^{+0.16}_{-0.14}$
$-0.75^{+0.16}_{-0.45}$	$+0.24^{+0.06}_{-0.11}$
$-1.99^{+0.30}_{-3.59}$	$-0.82^{+0.31}_{-0.09}$
	$-1.32^{+0.44}_{-0.04}$
$b_{0,D,F} \text{ [GeV}^{-1}\text{]}$	$-0.01^{+0.02}_{-0.10}$
$-0.50^{+0.01}_{-0.01}$	$+0.04^{+0.22}_{-0.14}$
$+0.08^{+0.01}_{-0.01}$	$+0.15^{+0.05}_{-0.06}$
$-0.22^{+0.01}_{-0.01}$	$+0.40^{+0.13}_{-0.05}$

Table 2: Parameters of the best fit. The values in parentheses behind the χ^2_{dof} show the contributions from data sets (a,b,c), respectively. The generalized variance is given by the determinant of the covariance matrix, $|\Sigma|$, and is related to the volume of the error ellipse as described in the main text. Error bars on the model parameters are determined in a re-sampling procedure.

The obtained best fit has an order of magnitude smaller χ^2_{dof} than the next best fits.

Its parameters are collected in Tab. 2. The data is well reproduced by the model as shown in Figs. 30 and 31. In Fig. 30 also the S -wave cross section is shown. The P -wave contribution to the total cross section is very small but P -wave as such still has a major influence on the fit through the differential cross section. The major contribution to the χ^2_{dof} originates from the total cross sections data (a), see values in parenthesis. All model parameters are of natural size with 1σ -uncertainties determined in a re-sampling procedure. This procedure involves generating 200 synthetic data sets in which each data point is calculated by taking a normally distributed random variate for each real data point. The free parameters are then fit to each synthetic data set using the best fit parameters as starting values. This gives 200 sets of fit parameters. The error bands can be calculated by taking the observable values that are 1 standard deviation above and below the mean.

The quantity $|\Sigma|$ shows the generalized variance that is proportional to the volume of the error ellipse of the 20-dimensional parameter space which can serve as a bulk measure of how well the data restrict the parameters. In particular, in the region with confidence $1 - \alpha$ around the best fit parameters (p), the volume of ellipse reads $\frac{(2\pi)^{p/2}}{p\Gamma(\frac{p}{s})}(\chi^2_{p,\alpha})^{p/2}|\Sigma|^{1/2}$. We will set the quoted number (0.106 GeV^{-14}) into relation when discussing the stability of results in the following. Furthermore, exploring the covariance matrix, we have not found any obvious separation between various parameter groups.

The interaction kernel (45) contains explicit (linear) dependence on the (cosine of the) scattering angle, and so does the solution of the dynamical coupled-channel equation (44). The best fit solution is projected to the S - and P -waves, denoted as in Ref. [145] by $f^I_{L\pm}$, indicating total angular momentum $J = L \pm 1/2$ and isospin I . This is depicted in Fig. 29 for the $\bar{K}N$ channel including 1σ error bands up to energies covered by the considered data base (see dots above Fig. 29). Around the $\bar{K}N$ threshold, the dominant contribution is due to the S -wave amplitudes with a clear

Figure 30: Total cross sections [24–27] fitted by the model described in the main part of the manuscript. Error bands represent the 1σ uncertainty determined in a re-sampling procedure. The dashed black line shows the contribution of the S -wave part of the amplitude only. For the channels, $K^-p \rightarrow K^-p$, $K^-p \rightarrow \pi^0\Lambda$, and $K^-p \rightarrow \pi^+\Sigma^-$, cusps can be seen at the $\bar{K}^0n$ threshold.

resonant behavior in the sub-threshold region in the isoscalar channel.

To identify the resonance parameters, the amplitudes $f_{L\pm}^I$ are analytically continued to the complex energy-plane on the second Riemann sheet (II.RS). Indeed, on the II.RS connected to the physical one between the $\pi\Sigma$ and $\bar{K}N$ thresholds we find two poles in the $0(1/2^-)$ sector, associated with the $\Lambda(1405)$, and one in the $1(1/2^+)$ channel, referred in the following to as $\Sigma(1380)1/2^+$. In Sec. 4.4, we describe several tests that give evidence that this state exists in the fit to account for the physical effects of the $\Sigma(1385)3/2^+$ on the data.

The resonance parameters are collected in the right panel of Tab. 3, including the

Figure 31: Differential cross sections [28] fitted by the model described in the main part of the manuscript. Error bands represent 1σ uncertainty determined in a re-sampling procedure.

magnitudes of the couplings to the meson-baryon channels

$$g^2 := \left| \lim_{W \rightarrow W^*} (W - W^*) f_{L\pm}^I(W) \right| \quad (47)$$

for the pole position W^* . As expected, the narrow pole of the $\Lambda(1405)$ couples predominantly to $\bar{K}N$, while the broad one more to the $\pi\Sigma$ channels. The new

$\Sigma(1380)1/2^+$ state couples mostly to the $\pi^0\Lambda$ channel.

The resonance poles in the complex energy-plane are depicted and compared with the outcome of other models in Fig. 32. Note that all these models are purely S -wave approaches, and do not include the differential cross section data (b) that bears information on the partial wave content. This is in line with the observation that,

$I(J^P)$		$0(1/2^-)$	$0(1/2^-)$	$1(1/2^+)$
Pole position [MeV]		$1430 - 15i$	$1360 - 43i$	$1360 - 11i$
Covariance [MeV ²]		$\begin{pmatrix} +20.0 & +3.7 \\ & +14.9 \end{pmatrix}$	$\begin{pmatrix} +169 & -166 \\ & +202 \end{pmatrix}$	$\begin{pmatrix} +354 & -91 \\ & +31 \end{pmatrix}$
Couplings [GeV]	g_{K^-p}	$0.34^{+0.10}_{-0.05}$	$0.35^{+0.06}_{-0.03}$	$0.10^{+0.05}_{-0.03}$
	$g_{\bar{K}^0 n}$	$0.32^{+0.10}_{-0.05}$	$0.35^{+0.06}_{-0.04}$	$0.10^{+0.04}_{-0.03}$
	$g_{\pi^0 \Lambda}$	—	—	$0.19^{+0.03}_{-0.03}$
	$g_{\pi^0 \Sigma^0}$	$0.19^{+0.03}_{-0.02}$	$0.39^{+0.04}_{-0.03}$	—
	$g_{\pi^+ \Sigma^-}$	$0.19^{+0.03}_{-0.02}$	$0.40^{+0.04}_{-0.03}$	$0.11^{+0.06}_{-0.07}$
	$g_{\pi^- \Sigma^+}$	$0.19^{+0.03}_{-0.02}$	$0.38^{+0.04}_{-0.03}$	$0.09^{+0.06}_{-0.06}$
	$g_{\eta \Lambda}$	$0.22^{+0.23}_{-0.09}$	$0.30^{+0.10}_{-0.07}$	—
	$g_{\eta \Sigma^0}$	—	—	$0.13^{+0.25}_{-0.07}$
	$g_{K^+ \Xi^-}$	$0.05^{+0.03}_{-0.03}$	$0.02^{+0.02}_{-0.01}$	$0.08^{+0.07}_{-0.06}$
	$g_{K^0 \Xi^0}$	$0.06^{+0.03}_{-0.03}$	$0.02^{+0.02}_{-0.01}$	$0.08^{+0.07}_{-0.06}$

Table 3: Prediction of resonance parameters found in a channel of given isospin (I) and total angular momentum ($J = L \pm 1/2$) on the second Riemann sheet, including the absolute values of the couplings to meson-baryon channels.

while there is a good agreement on the parameters of both poles in the S -wave with these models, the new P -wave state has not been found in any previous analyses.

4.4 Stability of results

The predictions for both poles of the $\Lambda(1405)1/2^-$ are well in agreement with other modern approaches [11, 13, 29]. The new P -wave $\Sigma(1380)1/2^+$ -resonance has similar mass, width, and branching ratios as the decuplet $\Sigma(1385)3/2^+$, but different total angular momentum and is not listed in the PDG [2]. Before discussing the possible origin of this spurious state we perform several statistical and phenomenological tests to confirm its existence within the present model.

- First: The $\pi\Sigma$ -invariant mass distribution [30] from the $K^-p \rightarrow \Sigma(1660)^+\pi^- \rightarrow ((\pi^-\Sigma^+)\pi^+)\pi^-$ reaction can be used for an *a-posteriori* test of the obtained solu-

Figure 32: Pole positions (black stars at the centers of the ellipses) for the $I(J^P) = 0(1/2^-)$ (left) channel. The error ellipses are from a re-sampling procedure shown explicitly in the corresponding insets. The shaded rectangles show the predictions of Refs. [11, 13, 29] for the narrow (blue) and broad (orange) pole of $\Lambda(1405)$. The black crosses denote the locations of the $\Lambda(1405)$ poles when the $\Sigma(1385)3/2^+$ is included in the model (see “Fifth” in the main text). In that case, the $I(J^P) = 1(1/2^+)$ pole (right, star) disappears and is replaced by the $\Sigma(1385)3/2^+$ (right, cross).

tion. For this and following Ref. [81], we assume energy-independent production and decay vertices of the $\Sigma(1660)1/2^+$. Fitting the associated constants, while not altering the meson-baryon scattering amplitude in the final state, we obtain a good fit with $\chi_{\text{pp}}^2 = 0.89$ (per data point). The result is shown in Fig. 35 in the Appendix.

- Second: A similar test of our scattering amplitude can be performed using the $\pi\Sigma$ -invariant mass distributions [10] of the reactions $\gamma p \rightarrow K^+(\pi\Sigma)$ recently measured with CLAS at Jefferson Lab. Following Refs. [11, 16] we obtain a

Figure 33: Pole position (black star at the centers of the ellipse) $1(1/2^+)$ channel. The error ellipse is from a re-sampling procedure shown explicitly in the corresponding inset.

good fit of the data with $\chi_{\text{pp}}^2 = 1.07$ (see Fig. 34), fitting again only the generic couplings $\gamma N \rightarrow K^+ \mathcal{S}_i$.

- Third: The differential cross section data (b) contributes with 30% to the full χ_{dof}^2 , see Tab. 3 that also contains the total cross section data (a) and threshold decay ratios (c). Additionally, we test the impact of these data by excluding it. Indeed, the generalized variance $|\Sigma|$, that provides a bulk number of how well data restrict parameters, increases from $|\Sigma_{a,b,c}| = 0.106 \text{ GeV}^{-14}$ to $|\Sigma_{a,c}| = 9.74 \text{ GeV}^{-14}$. In other words, the parameters of the model are significantly less well determined when omitting the differential cross section data, demonstrating their relevance. Furthermore, the $1(1/2^+)$ pole in the

Figure 34: Fit ($\chi^2_{\text{pp}} = 1.07$) to the $\pi\Sigma$ invariant mass distribution (M_{inv}) from $\gamma p \rightarrow K^+(\pi\Sigma)$ reaction [10]. The model for the reaction is taken from Ref. [11], where only generic couplings $\gamma p \rightarrow K^+\mathcal{S}$ are fitted to the data at each measured total energy W_{tot} . The meson-baryon scattering amplitude in the final state is taken from the fit to the scattering data. Only the best fits are shown.

complex energy plane disappears – the existence of the $\Sigma(1380)1/2^+$ seems indeed tied to the differential cross section data.

- Fourth: To investigate the stability of the new P -wave resonance further, we use the least absolute shrinkage and selection operator (LASSO) [146] to remove the $\Sigma(1380)1/2^+$. In this we add a penalty term $\lambda \int_{1.29 \text{ GeV}}^{1.46 \text{ GeV}} |\partial^2/\partial W^2 f_{1-}^{I=1}(W)| dW$ to the total χ^2 . Minimizing this function at $\lambda = 0$ and including data (a,b,c) we obtain back our previous result. Subsequently increasing the value of λ and re-fitting, the curvature of the amplitude between the $\pi\Sigma$ and $\bar{K}N$ thresholds decreases forcing the P -wave resonance to disappear into the complex plane as shown in Fig. 36. The figure also reveals that the description of the threshold data (c) deteriorates most visibly as the pole disappears. Of course, this is not *directly* due to the disappearance of the pole, but related to it through parameters b_i and a_i .

The quantity λ_{crit} is the minimum λ , at which $|\partial^2/\partial W^2 f_{1-}^{I=1}(W)|$ is close to zero for $W \approx 1380$ MeV. The difference in quality of the fit of the data when $\lambda = 0$ (our best fit) with the fit of the data at λ_{crit} measures the stability of the resonance. The likelihood ratio of the fit at $\lambda = \lambda_{\text{crit}}$ to the fit at $\lambda = 0$ is 4.3×10^{-28} which shows that the data description indeed deteriorates significantly in the absence of the resonance. In particular, the prediction of kaonic hydrogen data [75] is more than 2 standard deviations away from the experimental value at λ_{crit} .

- Fifth: The P -wave decuplet $\Sigma(1385)$ resonance in the $J^P = 3/2^+$ partial wave is absent in our model because the NLO terms do not fully describe the $J^P = 3/2^+$ sector. On the other hand, the slope of the differential cross section indicates the size of the P -wave contribution (Fig. 31) but is blind to the total angular momentum. It is then obvious to ask if the newly found $\Sigma(1380)1/2^+$ is only

generated to account for the absence of the $\Sigma(1385)3/2^+$. This could indeed be the case because the $\Sigma(1380)1/2^+$ exhibits very similar pole position and branching ratios as the $\Sigma(1385)3/2^+$.

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
-0.34	-0.06	-0.59	0.15	0.14	-0.61	-0.97	0.04	0.37	0.16	0.32
b_0	b_D	b_F	a_1	a_2	a_3	a_4	a_5	a_6	g	m_0
-0.46	0.13	-0.27	0.32	5.48	0.27	-1.07	-0.96	-1.44	1.50	1.52
$I(J^P)$			$0(1/2^-)$			$0(1/2^-)$			$1(1/2^+)$	
Pole Positions [GeV]			1.3644 -0.0431i			1.4302 -0.0151i			1.366-0.009i	

Table 4: Best fit parameters in GeV and pole positions found when an explicit $\Sigma(1385)3/2^+$ state is included in the fit. g and m_0 are the new parameters for this test.

To test this hypothesis, we explicitly include it in order to replace the new $\Sigma(1380)1/2^+$. The two-potential formalism allows the addition of resonances to an amplitude without spoiling unitarity [147]. This is achieved by adding the term $f_{1+}^P = \Gamma^*[W - m_0 - \gamma^T I \Gamma^*]^{-1} \Gamma^T$ to f_{1+} where T indicates the transpose of the “dressed vertex” $\Gamma = \gamma + \gamma I f_{1+}$ which is a vector in channel space. The “bare vertex” vector γ has components $\gamma_i = g g_i$ with one free fit parameter g and the relative decay strengths g_i to channels $\pi\Lambda$, $\pi\Sigma$, and $\bar{K}N$ fixed by the Lagrangian of Ref. [148].

Furthermore, I is the diagonal matrix of meson-baryon loop functions from the integration in Eq. (44). The bare mass m_0 and bare coupling g are new fit parameters. We then fit the previous free parameters as well as the two new ones shown in Tab. 4. These parameters give a $\Sigma(1385)3/2^+$ pole position of $W = 1.366 - 0.009i$ GeV. This value is 16 MeV below the Particle Data Group average value of 1382.8 MeV, but it is within the error ellipse of the spurious structure we observe (see the cross in Fig. 33). The changed pole positions of the $\Lambda(1405)$, also indicated by crosses, lie within the error ellipses of the previous solution. The ratio of partial decay widths for the

Figure 35: Fit of the generic couplings $K^-p \rightarrow \Sigma(1660)\pi^-$ and $\Sigma(1660) \rightarrow (\pi^-\Sigma^+)\pi^+$ to the invariant mass distribution [30] in arbitrary units. The final state interaction is taken from the best fit to the scattering data as described in the main part of the manuscript. Only the best fit is shown.

Figure 36: Dependence of the components of the χ^2_{dof} from each data set on the distance of the W^* pole from the real axis, parameterized with the penalty λ . The sum of all three components equals χ^2_{dof} which increases with the imaginary position of the pole at $W = W^*$. The contribution to the χ^2 from the penalty is subtracted in all cases.

$\Sigma(1385)3/2^+$ is predicted as $\Gamma(\Sigma\pi)/\Gamma(\Lambda\pi) = 0.183$ compared to the PDG value of $\Gamma(\Sigma\pi)/\Gamma(\Lambda\pi) = 0.135$ [2]. This test causes the P -wave $\Sigma(1380)1/2^+$ observed in our previous fit to disappear.

Therefore, the results of this test indicate that the $\Sigma(1380)1/2^+$ is dynamically generated by the free parameters just to account for the missing $\Sigma(1385)3/2^+$.

Indeed, no $\Sigma(1380)1/2^+$ is found in Lattice QCD calculations [130–132].

It should be noted that a more rigorous analysis would require the inclusion of the $\Sigma(1385)3/2^+$ as a spin- $3/2$ field not only in the s but also the u -channel in Eq. (45).

This is beyond the scope of this work. Similar considerations apply to the ground-state Λ and Σ in the s - and u -channels.

4.5 Conclusion

For the first time the available low-energy differential cross section data have been included in a chiral unitary coupled-channel model with fully dynamically generated

S - and P -waves, i.e., all waves being parameterized only by low-energy and regularization constants. Parameters of both $\Lambda(1405)1/2^-$ poles are determined in this approach with improved stability. Additionally, and confirmed by multiple (statistical and phenomenological) tests, the data require the existence of a P -wave pole but are otherwise insensitive to the total angular momentum. Thus, the pole is predicted in the only available P -wave with $J^P = 1/2^+$ but a number of tests show that this dynamically generated pole accounts for the absence of the $\Sigma(1385)3/2^+$ in the model.

5 Dalitz plots and lineshape of $a_1(1260)$ from a relativistic three-body unitary approach

The contents of this chapter coincides to a large part with the work of Ref. [149]. All authors of Ref. [149] have agreed that the article can be used as part of this thesis. Note that Ref. [149] exists only in form of a preprint on arXiv. It has been submitted to and is currently being reviewed by Physical Review D.

Excited states of strongly interacting matter exhibit a complex spectrum at low and intermediate energies. Connecting the underlying non-perturbative realization of Quantum Chromodynamics (QCD) to phenomenology is an ongoing experimental and theoretical challenge. The quest in understanding phenomenology including the decay of excited states but also unraveling the microscopic nature of mesonic and baryonic resonances poses open questions. Many are related to three-body dynamics.

A prime example of such a riddle in the baryon sector (explored by CLAS@JLab [150, 151], ELSA [152], MAMI [153], and other facilities) is the dynamical structure of the enigmatic Roper-state $N(1440)1/2^+$ and its mass pattern compared to the $N(1535)1/2^-$ state. While the microscopic features of the latter state can be parameterized by, e.g., meson-baryon interactions within Chiral Unitary approaches [64, 68, 79, 154, 155], the situation is more involved for the Roper-state due to large branching ratios to the $\pi\pi N$ channels. The $\pi\pi N$ dynamics is included in many analyses of the baryon spectrum with varying degrees of rigor [156–159], with dynamical coupled-channel approaches usually respecting at least aspects of three-body unitarity [64, 160, 161].

In the meson sector, the GlueX [162] and COMPASS experiments [6], and the BESIII accelerator [163] search for the excited states of strongly interacting matter, including exotics, i.e., states with quantum numbers not possible to form from two constituent quarks. Thus, it is expected that the discovery and the understanding of such states will be a direct indicator for gluonic degrees of freedom in QCD at low energies.

Many of such exotic states, but also conventional mesons decay dominantly or exclusively into three particles, which has triggered extended partial wave analysis (PWA) efforts, e.g., by COMPASS [164, 165], BESIII [166, 167], in coupled channels using the PAWIAN framework for $p\bar{p}$ induced meson production [168], or using Khuri-Treiman equations and related frameworks by the Bonn group, JPAC, and others for light meson decays [23, 95–98, 100–102, 169–179].

Lattice QCD is the non-perturbative tool to access the QCD Greens functions at low and intermediate energies from first principles. In addressing three-hadron systems, such calculations need to include a large set of operators including three-hadron operators. This leads to a significantly increased computational effort compared to two-body systems. However, progress has been made over the last years in calculating lattice spectra [180–185], including the $a_1(1260)$ meson [182]. Such discrete and real-valued spectra are inherently different from the infinite volume ones. The quantization condition allows one to map between finite and infinite volume, and is an active field of theoretical research [37, 38, 40, 186–202]. It has been shown in Ref. [37] that the key to the understanding of three-body finite-volume spectra from the lattice lies in the S -matrix principle of unitarity. Both finite-volume and lattice computations in the three-body sector are now capable of addressing simple three pion systems [39, 185, 203, 204] and are about to be extended to more complicated cases like axial mesons and exotics.

The $a_1(1260)$ axial meson has a clean three pion decay if it appears as an intermediate state in τ decays in which other partial waves are suppressed; it is also wide indicating strong and non-trivial three-body effects which makes the $a_1(1260)$ a prime candidate to study three-body dynamics. This is reflected in an increased interest in the structure of the $a_1(1260)$ [21–23, 205–212] which is also the main goal of the present manuscript.

More specifically, in view of the importance of three-body unitarity, our goal is to

Figure 37: The isobar-spectator amplitude leading to a unitary three-pion scattering amplitude. The interaction kernel (quantity in parentheses) is determined from unitarity and comprises a complex valued contribution shown as one-pion exchange, as well as a real valued three-body force C .

extend the manifestly unitary, relativistic three-body scattering amplitude derived in Refs. [17, 213] to the coupled-channel case for isobars with spin, namely the $a_1(1260)$ in which the $\rho\pi$ decay channel is known to provide the dominant contribution [2].

This also means the development of numerical techniques for the solution of the integral equations. Here, we restrict the $a_1(1260)$ dynamics to the $\rho\pi$ S - and D -wave channels; a detailed partial wave analysis of the three-pion system in τ decays measured by CLEO [19] has shown that there are also other channels needed for the detailed description of Dalitz plots and related observables. Demonstrating the feasibility of the approach, we fit the amplitude to the data on the $\tau^- \rightarrow \pi^+\pi^-\pi^-\nu_\tau$

lineshape measured in the ALEPH@CERN [5] experiment. Note also the measurement of the ARGUS collaboration [18] with lower statistics.

This work is organized as follows: In Sec. 5.1 the main definitions of the decay and three-body amplitude are introduced and compared to other approaches in the literature. The strategy for solving these equations is described in Sec. 5.2. Finally, in Sec. 5.3 the result of a fit to ALEPH data and the calculation of Dalitz plots from that fit will be presented and discussed.

5.1 Formalism

The interaction of three pions with $a_1(1260)$ quantum numbers enters the weakly induced decay process $\tau^- \rightarrow \pi^+\pi^-\pi^-\nu_\tau$ via the final state interaction. The presence of the outgoing neutrino leads to the fact that the total energy squared s of the three

Figure 38: The decay of an unpolarized $a_1(1260)$ meson parametrized by a source (left parentheses) into a ρ -meson (double lines) and a pion (single lines), with subsequent decay into $\pi^-\pi^-\pi^+$ and the pertinent symmetrization (not shown). The relativistic, unitary final state interaction (parentheses in the middle) is parametrized in terms of the isobar-spectator amplitude T and isobar propagator τ . The coupling of the ρ to two pions is encoded in the vertex function v .

pions is not fixed. This allows one to “scan” the spectrum of three pions in the final state, thus obtaining the so-called *mass spectrum* or *lineshape* of the $a_1(1260)$ -resonance that was measured in the ALEPH [5] experiment and is the main experimental input of the present work.

To access the mass spectrum theoretically, the decay process is decomposed into the weak and strong parts as $\tau^- \rightarrow W^- \nu_\tau \rightarrow (a_1(1260) \rightarrow \pi^-\pi^-\pi^+) \nu_\tau$ (see, e.g., Ref. [23]). The strong, final state interaction of the three pions is described by the process shown in Fig. 38. As indicated in the first parentheses it contains a part describing the production of the $\rho\pi$ pair in S - and D -wave. This part may be visualized as the direct production of the $\rho\pi$ pair as well as an intermediate propagation of $a_1(1260)$. The $\rho\pi$ system interacts then (second parentheses in the figure) and decays in the final step into three pions, such that three-body unitarity is preserved exactly. Note that while the picture suggests a diagrammatic expansion of the interaction, the approach is not Lagrangian-based as discussed below.

As already mentioned the final state interaction of three pions is taken into account non-perturbatively, ensuring unitarity. This is achieved using the formalism developed in Ref. [17]. We will describe its implementation and numerical applications in more detail below. In a nutshell, the approach is based on a decomposition of the scattering amplitude into a connected and a disconnected part

(cf. “connectedness structure” in Ref. [214]). Each of these pieces is populated by the two-body sub-system (referred to as “*isobar*” in the following) and a spectator.

Utilizing the Bethe-Salpeter-like form of a relativistic integral equation with an unknown kernel B and imposing three-body unitarity fixes the analytic form of B as well as the fully dressed isobar propagator τ up to real functions of energy and momenta [17]. This isobar-spectator amplitude is depicted symbolically in Fig. 37. Note that this approach relies on dispersive techniques making advantage of unitarity and connectedness structure. As such, it does not rely on a Lagrangian formalism for the microscopic interaction but provides a clean separation between on-shell parts and short-range physics encoded in real-valued contributions indicated as C in Fig. 37 and referred to as “*three-body force*” in the following.

The formal decomposition of the entire amplitude into a short-range and a long-range part (“ladders”), if parameterized as in this work or similarly, is discussed in Ref. [179] and sometimes referred to as “two-potential formalism” utilized mostly in the baryonic sector [147, 215]; this decomposition is not unique [216] but can be advantageous for time-consuming fits to large data sets [147]. The matching of different three-body formalisms, including the current one and its mapping to Feynman-diagrammatic expressions is discussed in Ref. [217]. There, global analytic properties of the three-to-three amplitude were discussed. Comparing frameworks like the present one to the analytic properties of the triangle diagram it was shown that in the sub-threshold region, non-analyticities can occur depending, e.g., on specific implementations of the integration over two-body sub-energies in the three-body equation.

Early attempts to solve the three-pion problem in the $a_1(1260)$ channel with a non-perturbative final-state interaction, in a framework similar to the present one, were carried out in Refs. [21, 218]. However, due to problems of how to continue the solution of the T -matrix from complex back to real spectator momenta, the

calculations were restricted to a stable ρ meson in parts of the amplitude, which violates unitarity. The present approach is most closely related to the one by the EBAC collaboration (now ANL/Osaka) [22]. There, Dalitz plots not only for the $a_1(1260)$ meson but also for other three-body decays were predicted. Furthermore, inelasticities in the two-body subsystems were taken into account and the $a_1(1260)$ was allowed to decay also in $\sigma\pi$ and $f_2\pi$, apart from the dominant $\rho\pi$ S -wave channel. On the other hand, we use here an amplitude that is manifestly unitary, with the full proof of unitarity first delivered in Ref. [17], and we also provide data fits and study the energy dependence of the amplitude by comparing to ALEPH data. This is particularly relevant for future extensions to finite-volume calculations for lattice QCD in which manifest unitarity is responsible for a subtle cancellation of unphysical singularities [37].

In the following, we describe our approach to calculate the decay process. In Sec. 5.1.1, we provide the equations needed to describe each term shown in Fig. 38. In Sec. 5.1.2 the partial wave decomposition is discussed. Finally, in Sec. 5.1.3, we describe how the amplitude can be related to observables.

5.1.1 Plane-Wave Amplitudes

The amplitude $\hat{\Gamma}_{\Lambda\lambda}$, describing the decay of the axial $a_1(1260)$ -resonance at rest with helicity Λ measured along the z -axis into a π^- and a $\rho_\lambda^0 \rightarrow \pi^+\pi^-$ with helicity λ , is given by

$$\begin{aligned} \hat{\Gamma}_{\Lambda\lambda}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) &= \frac{1}{\sqrt{2}} \left[\Gamma_{\Lambda\lambda}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) - (\mathbf{q}_1 \leftrightarrow \mathbf{q}_2) \right], \\ \Gamma_{\Lambda\lambda}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) &= \left(D_{\Lambda\lambda}(\mathbf{q}_1) + \sum_{\lambda'} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} D_{\Lambda\lambda'}(\mathbf{p}) \tau(\sigma(|\mathbf{p}|)) T_{\lambda'\lambda}(\mathbf{p}, \mathbf{q}_1) \right) \\ &\quad \times \tau(\sigma(|\mathbf{q}_1|)) v_\lambda^\pm(q_2, q_3), \end{aligned} \tag{48}$$

where $\mathbf{q}_1$, and $\mathbf{q}_2$ are outgoing π^- momenta, and $\mathbf{q}_3$ is the outgoing π^+ momentum. In Eq. (48) the dependence on the squared invariant mass of the three-body system s is suppressed; isospin indices are provided below; the squared invariant mass of the ρ , denoted by $\sigma(|\mathbf{p}|)$, depends only on the size of the spectator momentum $\mathbf{p}$ and is abbreviated as $\sigma(p) \equiv \sigma(|\mathbf{p}|)$ in the following; confusion with four-vector notation

should be excluded from the context. Furthermore,

$\sigma(p) = (P_3 - p)^2 = s + m_\pi^2 - 2\sqrt{s} E_p$ with $E_p^2 = m_\pi^2 + \mathbf{p}^2$ and $P_3 = (\sqrt{s}, \mathbf{0})$ being the total four-momentum of three pions. Note that throughout we are working in the three-body center-of-mass frame unless stated otherwise. Due to Bose symmetry, the amplitude is symmetric under exchange of the two π^- because they are in relative P -wave. Some of the terms appearing in Eq. (48) are shown in Fig. 38 and are defined in the following. Note also that this three-dimensional relativistic equation with all pions on their respective mass shells emerges after carrying out the integration of zeroth dimension, applying the delta-distribution of the spectator from the isobar-spectator propagation, see Ref. [17].

The elementary process $a_1 \rightarrow \pi\rho$ is indicated as $D_{\Lambda\lambda}$. It can have scalar or derivative character,

$$D_{\Lambda\lambda}(\mathbf{q}_1) = \epsilon_{\Lambda,\mu} \epsilon_\lambda^{*\mu}(\mathbf{q}_1) (-m_{a_1} g_s I_{a_1\rho\pi} + \dots), \quad (49)$$

i.e., the $a_1\rho\pi$ coupling is given by a tower of Lagrangians which we do not aim to explicitly use in this study, including masses (m_{a_1}) coupling constants (g_s) and isospin factors ($I_{a_1\rho\pi}$). Instead, we know that symmetry allows for even $\rho\pi$ partial waves each with their own energy and momentum dependence, which we simply parametrize in the JLS basis directly. This is performed and discussed further in the next section. The vertex $v_\lambda^\pm$ in Eq. (48) is an elementary part of the ρ^0 -decay into a $\pi^+\pi^-$ pair with

four-momenta q_2 and q_3 ,

$$\begin{aligned} v_\lambda^\pm(q_2, q_3) &= I' v_\lambda(q_2, q_3), \\ v_\lambda(q_2, q_3) &= -ig_1 \epsilon_\lambda^\mu(\mathbf{q}_1) (q_2 - q_3)_\mu F((q_2 + q_3)^2, (q_2 - q_3)^2), \end{aligned} \quad (50)$$

where g_1 is the $\rho\pi\pi$ coupling, v_λ is the isospin-projected decay vertex, and I' describes the transition from isospin to particle basis as needed only in the final ρ decay. Note that the latter factor is irrelevant as long as there is only one isobar (ρ^0).

Then this factor can be reabsorbed into the overall normalization of the a_1 decay. Furthermore, due to the azimuthal symmetry of the isobar momentum, the imaginary part of ϵ_λ does not contribute to the partial wave projected $v_\lambda(q_2, q_3)$. Thus, in this case, the vertex is strictly real, i.e., $v_\lambda(q_2, q_3) = v_\lambda^*(q_2, q_3)$. Any additional momentum dependence of the isobar decay is encoded in the covariant form-factor F that is introduced to regularize ultraviolet divergences in the two- and three-body sector. Note that three-body unitarity imposes a consistent use of the form-factor in the self-energy τ and exchange B , which requires it to be covariant [17]. The choice of this real-valued functions is not unique. Explicit expressions used in this work are discussed in Appendix A.

The three-body unitarity underlying this formalism implements two-body unitarity automatically [17]. This constrains the imaginary part of the inverse of the isobar-spectator propagator. The remaining freedom allows one to pick a form of the isobar propagator τ that is suitable for the problem at hand. For example, a two-body scattering amplitude motivated by Chiral Perturbation Theory was implemented into the three-body scattering equation in Refs. [38, 204]. This form is very useful for repulsive or weakly attractive channels (such as, e.g., $\pi\pi$ scattering in the channel of the atypical $f_0(500)$ -resonance). For the present work, dealing

Figure 39: A diagrammatic depiction of the isobar propagator from Eqs. (51,52), representing the re-summation of $\pi\pi$ interactions within the isobar to all orders. Double and single lines denote the isobar and stable particle propagators, respectively.

exclusively with the ρ -resonance, it is, however, justified to simply choose

$$\tau(\sigma(p)) = \frac{1}{\sigma(p) - m_\rho^2 - \Sigma(\sigma(p))} \quad (51)$$

as represented in Fig. 39. Here, m_ρ is a real-valued free parameter not fixed by unitarity. We fit it, together with the coupling g_1 from Eq. (50) and the form-factor to two-body $\pi\pi$ phase-shift data [219, 220]. Note that the τ in Eq. (51) is the same as S in Ref. [17]. In Eq. (51), Σ is referred to as the self-energy. We use the explicit

expression

$$\Sigma(\sigma(p)) = \int_0^\infty \frac{dk k^2}{(2\pi)^3 E_k} \frac{\tilde{v}(\sigma(p), k) \tilde{v}^*(\sigma(p), k)}{\sigma(p) - 4E_k^2 + i\epsilon}. \quad (52)$$

This can be evaluated in the isobar rest frame with the two-body 4-momentum $P_2 = (\sqrt{\sigma(p)}, \mathbf{0})$. The tilde on $\tilde{v}$ indicates that this vertex is projected to the total angular momentum of the ρ -meson as

$$\tilde{v}(\sigma(p), k) = \sqrt{\frac{16\pi}{3}} g_1 k \tilde{F}(\sigma(p), k), \quad (53)$$

which is derived from the $\rho\pi\pi$ vertex in Eq. (50). The form-factor $\tilde{F}$ used here is of the same analytic structure as the F in Eq. (50) for the on-shell region. It is, however, modified for technical reasons in the off-shell region, which does not violate unitarity. See Appendix A for further details.

The plane wave solution for the isobar-spectator amplitude is obtained as described in

Ref. [17], i.e., by solving the Bethe-Salpeter type relativistic integral equation

$$T_{\lambda'\lambda}(\mathbf{p}, \mathbf{q}_1) = (B_{\lambda'\lambda}(\mathbf{p}, \mathbf{q}_1) + C) + \sum_{\lambda''} \int \frac{d^3\mathbf{l}}{(2\pi)^3 2E_l} (B_{\lambda'\lambda''}(\mathbf{p}, \mathbf{l}) + C) \tau(\sigma(l)) T_{\lambda''\lambda}(\mathbf{l}, \mathbf{q}_1), \quad (54)$$

where λ' , λ and λ'' are the helicities of the incoming, outgoing, and intermediate ρ -isobar, and $\mathbf{p}$, $\mathbf{q}_1$, and $\mathbf{l}$ are the incoming, outgoing and intermediate spectator momenta, respectively.

The first part of the driving term of Eq. (54), the so-called B -term, can be visualized as the one-pion exchange process depicted in the first term inside the parentheses in

Fig. 37. In the helicity basis it reads

$$B_{\lambda'\lambda}(\mathbf{p}, \mathbf{q}_1) = -\frac{v_{\lambda'}(P_3 - p - q_1, q_1) v_{\lambda}^*(p, P_3 - p - q_1)}{(\sqrt{s} - E_p - E_{q_1})^2 - (m_{\pi}^2 + p^2 + q_1^2 + 2pq_1 z) + i\epsilon}. \quad (55)$$

The equation emerges from the sum of a forward and backward (in time) pion exchange between two $\rho\pi\pi$ vertices. We denote the angle between the incoming and outgoing isobars by θ_1 and its cosine, $z = \cos \theta_1$. This expression already contains the isospin factor of -1 corresponding to the isospin-1 pion and rho coupling to the isospin-1 of the $a_1(1260)$

The second part of the driving term of the integral equation (54) is a real-valued function C of spectator momenta and s , i.e., $C \equiv C(\mathbf{p}, \mathbf{q}, s)$. It arises from the fact that only imaginary parts of the amplitude are fixed as discussed in Ref. [17].

Physically, this function is related to the three-body contact term via a decay of the isobar into two incoming and outgoing pions, see e.g. Ref. [217]. To capture the $\rho\pi$ contact interaction efficiently, we will model this function in the next section after projecting the whole integral equation to the JLS basis.

Figure 40: Graphical representation of the isobar-spectator contact term C via a propagation of a bare $a_1(1260)$. The coupling to $\rho\pi$ states is chosen consistently to that appearing in Fig. 38. See Eq. (56) and text below it for more details.

5.1.2 Partial Wave Amplitudes

The Bethe-Salpeter type integral equation given in Eq. (54) is part of the production amplitude given in Eq. (48). Analytic solutions of such dynamical equations are only known for driving terms consisting of contact interactions only, see e.g., Refs. [9, 68]. Therefore, the equation will be solved numerically here, by discretizing momenta, and thus, transforming the integral equation into algebraic ones.

The technical challenge in doing so is that the B -term, and with it T , depend on the incoming and outgoing spectator three-momenta, $\mathbf{p}$ and $\mathbf{q}_1$, making numerical inversions resource intensive. However, only one of the terms of the partial wave decomposition in Eq. (57) enters the production amplitude Eq. (48) for the quantum numbers of the $a_1(1260)$ -resonance, i.e., the one for $J = 1$.

Besides, the implementation of physical constraints on the three-body force induced by the term C in Eq. (54) can be made most effectively in the JLS basis. The latter basis encodes the total-, relative (between isobar and spectator) and intrinsic (spin) angular momenta, respectively. In this basis we assume a general form of $C_{L'L}$ accounting for the partial wave dependence. Expressing the contact-term in terms of a propagating $a_1(1260)$ connecting initial and final isobar and spectator, as shown in Fig. 40, one may write

$$C_{L'L}(p, q_1) = - \left(\frac{p}{m_\pi} \right)^{L'} \left(\frac{q_1}{m_\pi} \right)^L \frac{m_\pi^2 g_{fL} g_{fL'} H(p) H(q_1)}{s - m_{\text{fit}}^2}. \quad (56)$$

Here g_{fL} , $g_{fL'}$ are the bare couplings that characterize the strength of the decay vertex and m_{fit} is the bare mass of the $a_1(1260)$, respectively. These parameters are fixed to reproduce physical data on the $a_1(1260)$ lineshape. The form-factor H is discussed in Appendix A. Note that factors of m_π are included such that the above contact term is dimensionless.

The term $T_{\lambda\lambda'}$ in Eq. (48) describes the isobar-spectator interaction symbolized in Fig. 37. For the purpose of the present chapter only the part projected to total angular momentum $J = 1$, the quantum number of $a_1(1260)$, is required. Taking into account the azimuthal symmetry the plane wave isobar-spectator amplitude is related to the partial wave amplitudes as

$$A_{\lambda'\lambda}(\mathbf{p}, \mathbf{q}_1) = \sum_J \frac{2J+1}{4\pi} d_{\lambda'\lambda}^J(z) A_{\lambda'\lambda}^J(p, q_1), \quad (57)$$

where $A \in \{T, B\}$ and $d_{\lambda\lambda'}^J(\cos \theta)$ denotes the small Wigner-d function.

In the JLS basis and with respect to the incoming and outgoing orbital angular momenta, L' and L , Eq. (54) becomes

$$T_{L'L}^J(p, q_1) = \left(B_{L'L}^J(p, q_1) + C_{L'L}(p, q_1) \right) + \sum_{L''} \int_0^\infty \frac{dl l^2}{(2\pi)^3 2E_l} \left(B_{L'L''}^J(p, l) + C_{L'L''}(p, l) \right) \tau(\sigma(l)) T_{L''L}^J(l, q_1). \quad (58)$$

The $a_1(1260)$ is constrained by parity conservation and conservation of angular momentum to decay into $\rho\pi$ with angular momentum $L = 0$ or $L = 2$. Thus, we work in a basis in which the T -matrix can have two JLS states, namely 121 and 101 for both incoming and outgoing states.

To obtain the B -term in the partial wave (JLS) basis we employ a two-step procedure. First, from the B -term (55) in helicity basis the relevant partial wave is

extracted, exploiting orthonormality of Wigner-d functions,

$$B_{\lambda'\lambda}^J(p, q_1) = 2\pi \int_{-1}^{+1} dz d_{\lambda'\lambda}^J(z) B_{\lambda\lambda}^J(\mathbf{p}, \mathbf{q}_1). \quad (59)$$

Then, the expression in partial wave basis is obtained from a linear transformation

$$B_{L'L}^J(p, q_1) = U_{L'\lambda'} B_{\lambda'\lambda}^J(p, q_1) U_{L\lambda}^T, \quad (60)$$

where the superscript $(..)^T$ denotes the transposition operation. The transformation matrix U is given by

$$U_{L\lambda} = \sqrt{\frac{2L+1}{2J+1}} (L01\lambda|J\lambda)(1\lambda00|1\lambda), \quad (61)$$

expressed in usual Clebsch-Gordan coefficients [221], while the summation over identical indices in Eq. (60) is understood.

The above equations (58,59,56) complete the main part of the final state interaction to be solved numerically below. Ultimately, the solution of Eq. (58) in the JLS basis needs to enter the decay amplitude (48) as follows. First, the $a_1(1260) \rightarrow \rho\pi$ vertex, D is modeled to include the correct spectator momentum dependence for each partial wave, i.e., $D_L(p) \sim p^L$. Furthermore, the production process itself may contain first-order singularities. The necessity of this contribution is explained at the end of

Sec. 5.2.1. Overall, the vertex is parameterized as

$$D_{L'}(p) = D_{fL'} H(p) \left(\frac{p}{m_\pi}\right)^{L'} + \frac{m_\pi^2 g_{fL'} D_{\bar{f}} H(p)}{s - m_{\text{fit}}^2} \left(\frac{p}{m_\pi}\right)^{L'}, \quad (62)$$

where $D_{fL'}$ for $L' = 0, 2$ and $D_{\bar{f}}$ are free parameters that are fit to the lineshape data as described below. The parametrization of Eq. (62) is similar to the one used in dynamical coupled-channel approaches for the photo-excitation of resonances, see,

e.g., Ref. [222], where it allows to excite resonances and background independently without spoiling Watson's theorem. This theorem [223] is a consequence of unitarity and time-reversal symmetry which can impose a constraint on the interaction kernel for the photo-excitation of resonances. In photo-excitation it fixes the pole-term in the kernel in way that prevents the amplitude from diverging. It does not automatically extend to the three-body sector.

Including the contact interactions C and D as in Eqs. (62, 56), we construct the decay amplitude in the JLS basis. We use the breve symbol on $\check{\Gamma}$ to denote the inclusion of all terms represented diagrammatically in Fig. 38 except the final vertex (v), i.e., the diagram to the right of the parentheses. It can be separated into the contribution from the connected and the disconnected part with $\check{\Gamma} = \check{\Gamma}^C + \check{\Gamma}^D$, where

$$\check{\Gamma}_{L'L}(q_1) = D_{L'}(q_1)\tau(\sigma(q_1))\delta_{L'L} \int \frac{dp p^2}{(2\pi)^3 2E_p} D_{L'}(p)\tau(\sigma(p))T_{L'L}^J(p, q_1)\tau(\sigma(q_1)).$$

Finally, this term is related to the amplitude in Eq. (48) by transforming into the helicity basis and multiplying the final $\rho\pi\pi$ vertex

$$\Gamma_{\Lambda\lambda}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \sqrt{\frac{3}{4\pi}} \mathfrak{D}_{\Lambda\lambda}^{1*}(\phi_1, \theta_1, -\phi_1) U_{L'\lambda'}^T \check{\Gamma}_{L'L}(q_1) U_{L\lambda} v_\lambda(\mathbf{q}_2, \mathbf{q}_3), \quad (63)$$

with sums over λ' , L , and L' and where $\mathfrak{D}_{\lambda'\lambda}^J(\phi_1, \theta_1, -\phi_1)$ are the capital Wigner-D function with angles θ_1 and ϕ_1 giving the polar and azimuthal angles of $\mathbf{q}_1$. We use the convention of Jacob and Wick [224] for the arguments of the function,

$\mathfrak{D}_{\lambda'\lambda}^{J=1}(\phi, \theta, -\phi)$, rather than the alternative convention $\mathfrak{D}_{\lambda'\lambda}^{J=1}(\phi, \theta, 0)$. The latter convention implies an additional phase-factor.

Note also that, while the small Wigner-d function may be used for the evaluation of the angular integration appearing in the partial wave decomposition of Eq. (57), one

must use the capital Wigner-D functions and their ϕ dependence for the back-transformation to plain waves because in the symmetrized decay $a_1 \rightarrow \pi^+\pi^-\pi^-$

of Eq. (48) the final ρ^0 isobars can be produced in different directions. In the next section we describe how the observables can be obtained from the production amplitude given above.

5.1.3 Relation to Observables

After having specified the analytic expressions leading to the production amplitude, Γ , we will demonstrate in the following how it can be related to three-body observables.

In particular, this will allow us later to fix free parameters of the framework, i.e., g_{fL} , $D_{fL'}$, m_{fit} , and $D_{\bar{f}}$. Recall that parameters g_1 , Λ , m_ρ are already fixed to the two-body data – the $\pi\pi$ phase-shifts in the isovector channel. The three-body results considered in this paper are Dalitz plots, projected Dalitz plots and the lineshape. The lineshape is a one dimensional scalar function of the total three-body energy $\sqrt{s}$.

It is given by integrating over all three pion four-momenta in the final state

$$\mathcal{L}(\sqrt{s}) = \frac{1}{\sqrt{s}} \int \frac{d^3\mathbf{q}_1}{(2\pi)^3} \frac{d^3\mathbf{q}_2}{(2\pi)^3} \frac{d^3\mathbf{q}_3}{(2\pi)^3} \frac{1}{2E_{q_1} 2E_{q_2} 2E_{q_3}} (2\pi)^4 \delta^4(P_3 - q_1 - q_2 - q_3) \overline{|\Gamma(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)|^2}. \quad (64)$$

Here the bar over the production amplitude Γ denotes the usual summation over helicity indices, i.e., $\overline{|\Gamma|}^2 \equiv 1/3 \sum_\Lambda \sum_\lambda |\Gamma_{\Lambda\lambda}|^2$. We have chosen to work with the parametrization shown shown in Fig. 41.

The Dalitz plots are calculated in a similar fashion, taking, however, the phase-space integral for fixed invariant masses σ_{23} and σ_{13} made up of the two outgoing pions given in the subscript, with 3 labeling the π^+ and 1, 2 the two π^- ,

$$\mathcal{D}(\sqrt{s}, \sigma_{23}, \sigma_{13}) = \frac{1}{(2\pi)^5} \frac{1}{32\sqrt{s}^3} \int d\Omega_{q_1} d\phi_{12} \overline{|\Gamma(\sigma_{23}, \sigma_{13}, \Omega_{q_1}, \phi_{12})|^2}. \quad (65)$$

Note that the delta function, $\delta^4(P - q_1 - q_2 - q_3)$ has been evaluated, which accounts for the elimination of the differential $d^3\mathbf{q}_3$ and one of the angles. The latter angle is

Figure 41: The kinematics of three body decay in the parametrization of this work.

chosen to be the angle between $\mathbf{q}_1$ and $\mathbf{q}_2$, denoted by θ_{12} . The azimuthal angle between the same momenta, denoted by ϕ_{12} , remains in the integration. Finally, taking the Dalitz distribution of Eq. (65) and integrating over one of the invariant masses, the projected Dalitz plot is obtained with

$$\mathcal{D}_p(\sqrt{s}, \sigma_{23}) = \frac{1}{(2\pi)^5} \frac{1}{32\sqrt{s}^3} \int d\Omega_{q_1} d\phi_{12} d\sigma_{13} |\overline{\Gamma(\sigma_{23}, \sigma_{13}, \Omega_{q_1}, \phi_{12})}|^2. \quad (66)$$

The numerical treatment for these integrals is discussed in Sec. 5.2.3 and results are shown in Sec. 5.3.

5.2 Numerical implementation

5.2.1 Inversion of the Bethe-Salpeter Equation

The Bethe-Salpeter type integral equation given in the plane wave basis in Eq. (54) and the partial wave basis in Eq. (58) must be solved in order to calculate the observables discussed in Sec. 5.1.3. This is done numerically by discretizing the

Figure 42: Discretized version of $\Re B_{p,q_1}^J$ from Eq. (67). X- and Y-axes correspond to the incoming and outgoing combined momentum and angular momentum index as explained in Sec. 5.2.1. Note that the momenta are not uniformly distributed.

Figure 43: Illustration of the singularity structure of the B -term. The red and blue lines mark the integration contour for the momentum p and the moving singularities in the B -term, respectively. Each of the blue lines is fixed by one complex q_1 for various values $\cos \theta \in (-1, 1)$. The former values are representative for the problem at hand, while $\sqrt{s} = 1.260$ GeV. Note that changing the integration contour affects both, the red and blue lines.

analytic functions of incoming and outgoing momenta in the T , τ , B - and C -terms. We start the discretization by introducing indices p and q_1 (not to be confused with fourmomenta). They both sample the region $(0, \infty)$ on a suitably mapped Gaussian quadrature, consecutively for $L = 0$ and $L = 2$, see Fig. 42 for an example. The number of Gauss points determines the precision of this method, and is chosen to be 30 for each of the partial waves.

This mapping allows one to write the terms $B_{LL'}^J(p, q_1)$ and $T_{LL'}^J(p, q_1)$ in a compact matrix form $B_{pq_1}^J$ and $T_{pq_1}^J$, respectively, preserving the dependence on L and L' . In

this notation Eq. (58) reads

$$T_{pq_1}^J = (B_{pq_1}^J + C_{pq_1}) + (B_{pl}^J + C_{pl}) \tilde{\tau}_l T_{lq_1}^J, \quad (67)$$

where the summation over equal indices is understood. To simplify the notation,

factors of $l^2/((2\pi)^3 2E_l)$ and Gaussian integration weights are absorbed into the isobar propagator, indicated by the diagonal expression $\tilde{\tau}$. The solution of this equation reads

$$T_{pq_1}^J = \left([\mathbb{1} - (B^J + C) \tilde{\tau}]^{-1} \right)_{pl} (B_{lq_1}^J + C_{lq_1}) , \quad (68)$$

with $\mathbb{1}$ denoting the unit matrix in the above defined space, i.e., the direct product of momenta and angular momenta. We introduce a numericized version of $\check{\Gamma}$ following

Eq. (63)

$$\check{\Gamma}_{L',q_1} = \left(D_{q_1} \tau_{q_1} + D_p \tilde{\tau}_p T_{pq_1}^J \tau_{q_1} \right) \Big|_{L'} , \quad (69)$$

where on the right hand side, we use a somewhat sloppy notation to show that the incoming angular momentum index is fixed to L' , cf., Eq. (63). Note that $\check{\Gamma}_{L',q_1}$ depends on both incoming and outgoing relative angular momenta, L and L' . The L dependence is implicit in the q_1 index.

Before proceeding to further details of the numerical implementation we wish to come back to the discussion of the second term in Eq. (62). Consider first the three-body contact term C_{pq_1} , which diverges for $\sqrt{s} \rightarrow m_{\text{fit}}$. Thus, and following Eq. (68), in this limit $T_{pq_1}^J \rightarrow -1/\tilde{\tau}_{q_1} \delta_{pq_1}$. Subsequently, this yields for the production amplitude Eq. (69), $\check{\Gamma}_{L',q_1} \rightarrow (D_{q_1} \tau_{q_1} - D_p \delta_{pq_1} \tau_{q_1}) = 0$. There is, however, no physical reason for the amplitude to vanish at this specific point. Adding a contribution, singular at $s = m_{\text{fit}}^2$, such as the second term in Eq. (62), solves this issue.

5.2.2 Integration Contour

While the quantities of interest $\check{\Gamma}_{L',q_1}$ and $T_{pq_1}^J$ read quite straightforwardly in Eqs. (68,69), there is a complication hidden in the analytic structure of its building blocks. The main issue in this respect deals with the cuts of the B -matrix and

self-energy given by Eq. (59) and Eq. (52). Utilizing analyticity, we avoid these cuts by deforming the integration contour, i.e., sampling all momentum magnitudes on a line in the complex plane as, e.g., $\Im k/\Re k = -\alpha$ [225]. In order to solve Eq. (67) with respect to T , one has to choose both momenta p and q_1 along the same deformed integration contour. This contour, used for the momentum integration in Eq. (63), is depicted by the red line in Fig. 43. The figure also illustrates the singularities of the B -matrix (blue lines) which are circumvented by the deformed integration contour. The form-factors regularizing the T matrix, see Eqs. (50, 56), also develops similar poles. We have made sure for all moving singularities appearing in the problem that they do not cross the integration contour, and that these singularities respect the $+i\epsilon$ prescription of Eq. (55) for energies $\sqrt{s}$ above threshold. Note that in order for the poles to avoid singularities, both spectator must have an imaginary part. If one remains real, cuts will cross the contour we choose.

The fact that the integration contour avoids poles guaranties analyticity of the three-body scattering amplitude T . In principle, the integration contour is not unique and other choices are possible, but there are certain constraints on it. On the one hand, α needs to be small enough, such that the analytic extrapolation (see below) is a good approximation of $\check{\Gamma}_{q_1}$ for real q_1 . On the other hand, it needs to be large enough to maintain numerical stability. After extensive exploration, we choose $\alpha = 0.2$ fulfilling the above constraints. Recall that the pole-structure of the self-energy in Eq. (52) is not as intricate. For simplicity, we choose the same integration contour for it as for the B -term. This also ensures that the correct Riemann sheet for the self-energy is picked.

The integration for the self-energy and the integration over the internal spectator momentum, p in Eq. (48), are taken over all momentum. Thus, the integral calculated on our chosen contour will be equal to the integral calculated along the real axis. This is ensured by the form-factors which cause the integrand to vanish at

large momenta. However, the outgoing spectator momentum, q_1 is real. Because q_1 can be observed, $\check{\Gamma}$ is integrated to the physical limit determined by $\sqrt{s}$ in the calculation of observables. In order to relate the results for momenta on the chosen contour to those on the real axis we fit a Padé-approximant to the numerically obtained values of $\check{\Gamma}$ at complex q_1 . We then extrapolate this function to the real axis. Note that this is possible because there are no further non-analyticities between the complex contour and the real q_1 axis. Furthermore, the incoming momentum p of the T -matrix is always complex and we never encounter the situation with both p and q_1 real, which could induce further singularities into the B -matrix [225].

To efficiently parametrize $\check{\Gamma}$, the asymptotic ($q_1 \rightarrow 0$, $q_1 \rightarrow \infty$) behaviour can be explicitly incorporated into the Padé-approximant as

$$\check{\Gamma}_{L'L}(q_1) = \left(\frac{q_1}{m_\pi}\right)^L H(q_1) \frac{\sum_{j=0}^{m=3} a_j^{L'L} (q_1/m_\pi)^j}{\sum_{k=0}^{n=3} b_k^{L'L} (q_1/m_\pi)^k}, \quad (70)$$

which approximates the right-hand side of Eq. (63). The complex coefficients, $a_j^{L'L}$ and $b_k^{L'L}$ are fit to the values of $\check{\Gamma}_{L',q_1}$, defined in Eq. (69), and calculated for complex q_1 . The Padé-approximant is then extrapolated to real values of q_1 . We have extensively checked the validity of this procedure, e.g., by ensuring that the extrapolated function is independent of the choice for α with sufficient precision.

5.2.3 Monte Carlo Integration

After having calculated the production amplitude Γ we need to perform the phase-space integrations of Eqs. (64,65), and (66), leading to the lineshape and Dalitz plots. We do this numerically using Monte Carlo (MC) integration which involves summation over the function for randomly generated values of relevant kinematic variables.

The number of integration variables in Eq. (64) is reduced from nine to five,

accounting for conservation of 4-momentum encoded in the delta function. In doing so, we replace the differential from $d^3\mathbf{q}_1 d^3\mathbf{q}_2 d^3\mathbf{q}_3$ to $dq_1 dq_2 d\phi_1 d\cos\theta_1 d\phi_{12}$. The angles ϕ_1 and θ_1 are the angles, which q_1 forms with the z -direction. The angle ϕ_{12} is the azimuthal angle between q_1 and q_2 .

The integral is then calculated using the Monte Carlo method, implemented similarly to Ref. [226]. This involves N randomly generated sets each consisting in the corresponding integration variables. Each value is denoted with a subscript, for example, the i -th value of ϕ_{12} is denoted ϕ_{12i} . Furthermore, the allowed values for q_1 and q_2 must satisfy the condition $|\cos\theta_{12}| < 1$, where $\cos\theta_{12}$ is fixed by the delta function to be

$$\cos\theta_{12}(q_1, q_2) = \frac{(\sqrt{s} - E_{q_1} - E_{q_2})^2 - m_\pi^2 - \mathbf{q}_1^2 - \mathbf{q}_2^2}{2q_1 q_2}. \quad (71)$$

The explicit formula for the MC integration of the lineshape reads

$$\mathcal{L}(\sqrt{s}) = \frac{1}{(2\pi)^4} \frac{P_L}{N} \frac{1}{8\sqrt{s}} \sum_{i=1}^N \frac{q_{1i} q_{2i}}{E_{q_{1i}} E_{q_{2i}}} \overline{|\Gamma(q_{1i}, q_{2i}, \cos\theta_{1i}, \phi_{1i}, \phi_{12i})|^2}, \quad (72)$$

where N is the number of MC points and P_s is the phase space factor, given by integration over the volume. For the lineshape it is

$$P_L = \int dq_1 dq_2 d\cos\theta_1 d\phi_1 d\phi_{12} \Theta(1 - \cos^2\theta_{12}(q_1, q_2)), \quad (73)$$

where Θ is the Heaviside function. In a similar way the Dalitz plot

$$\mathcal{D}(\sqrt{s}, \sigma_{23}, \sigma_{13}) = \frac{1}{(2\pi)^4} \frac{P_D}{N} \frac{1}{32\sqrt{s}^3} \sum_i \overline{|\Gamma(\sigma_{23}, \sigma_{13}, \cos\theta_{1i}, \phi_{1i}, \phi_{12i})|^2}, \quad (74)$$

Figure 44: The lineshape plot fit to data from $\tau^- \rightarrow \pi^- \pi^- \pi^+ \nu_\tau$. We also include the component of the fit that comes only from the S -wave. As expected, the S -wave channel dominates. We also see that the D -wave provides a larger contribution at higher values of $\sqrt{s}$.

and the projected Dalitz plot

$$\mathcal{D}_P(\sqrt{s}, \sigma_{23}) = \frac{1}{(2\pi)^4} \frac{P_P}{N} \frac{1}{32\sqrt{s}^3} \sum_i \overline{|\Gamma(\sigma_{23}, \sigma_{13i}, \cos \theta_{1i}, \phi_{1i}, \phi_{12i})|^2}, \quad (75)$$

are calculated. The phase space factor for the Dalitz plot and projected Dalitz plots are $P_D = \int d\cos\theta_1 d\phi_1 d\phi_{12}$ and $P_P = \int d\sigma_{13} d\cos\theta_1 d\phi_1 d\phi_{12} \Theta(1 - \cos^2\theta_{12}(\sigma_{13}, \sigma_{23}))$ respectively.

5.3 Results

In the following, we present the results of the approach in terms of Dalitz plot, projections thereof, and lineshape, i.e., $\mathcal{D}$, $\mathcal{D}_p$, and $\mathcal{L}$ calculated from Eqs. (74), (75), and (72), respectively.

The free parameters of the framework are m_ρ , g_1 , Λ , g_{fL} , D_{fL} , $D_{\tilde{f}}$, m_{fit} , where three former are fixed by a fit to the two-body data, i.e., isovector $\pi\pi$ phase-shifts. They

Figure 45: Upper Row: Dalitz plots for various three-body invariant masses. Each Dalitz plot has been re-scaled separately to emphasize qualitative features. The blue/red colors represent lower/higher values, respectively. The light bands at $\sigma_{13} \sim 0.59 \text{ GeV}^2$ are the outline of the ρ -meson. Middle Row: Projected Dalitz plots. An effect of this integration is the kinematic reflection of the peak from the ρ meson at a value of σ_{13} lower than the mass of the ρ . Lower Row: The ratio between $\mathcal{D}_p$ and $\mathcal{D}'_p$, the equivalent projected Dalitz plots calculated without the rescattering terms normalized to the integral over $\mathcal{D}_p$.

read: $m_\rho = 1.38 \text{ GeV}$, $g_1 = 7.26$, and $\Lambda = 1.04$. As discussed below, there is not much sense in determining statistical uncertainties for these quantities for the present purpose. Note that Λ , appearing in form-factor F in the $\rho\pi\pi$ vertex of Eq. (50), regularizes not only the isovector two-body amplitude but also the three-body amplitude through the appearance in the B -term of Eq. (55). This induces the usual and unavoidable regulator dependence in the three-body amplitude which is absorbed in the contact term (56), that by itself contains a form-factor to render the three-body equation well-defined.

The remaining six parameters (note the index $L = 0, 2$ above) are determined from a fit to the lines-shape from the ALEPH [5] experiment. The best fit parameters read:

$D_{f0} = 3.4 \times 10^{-4} \text{ GeV}^{-1/2}$, $D_{f2} = 4.9 \times 10^{-4} \text{ GeV}^{-1/2}$ for the non-resonant $\rho\pi$ production vertices of Eq. (62), $D_{\bar{f}} = 1.7 \times 10^{-3} \text{ GeV}^{-1/2}$ for the strength of the bare a_1 component in the production process in Eq. (62), $g_{f0} = 5.3 \times 10^2$, $g_{f2} = 1.2$, and $m_{\text{fit}} = 0.51 \text{ GeV}$ for the bare a_1 decays into S - and D -wave $\rho\pi$ and the a_1 bare mass of Eqs. (56) and (62), respectively. The result of the fit is presented in Fig. 44

together with the data.

According to the PDG [2], the $a_1(1260)$ mostly decays into the $\rho\pi$ and $\sigma\pi$ channels.

The $\sigma\pi$ branching ratio is determined in Ref. [19] to be $\Gamma(\sigma\pi)/\Gamma_{\text{total}} = 0.1876$, whereas relative to the $\rho\pi$ channel it is quoted in Ref. [227] as $\Gamma(\sigma\pi)/\Gamma(\rho\pi) = 0.06$. Both, these results show that the $\rho\pi$ channel is largely dominant, but the $\sigma\pi$ channel is non-negligible. Since the present analysis does not include the $\sigma\pi$ channel, a χ^2 close to 1 would only indicate that the free parameters are capable of accounting for effects that are probably not due to the physical processes they describe. Therefore, we do not perform a statistical analysis on our fit of the lineshape data and accept that the fit starts diverging from the lineshape data at high energies as shown in

Fig. 44.

We can make a very rough estimate of the D -wave branching ratio (BR) from the lineshape at the resonance position $\sqrt{s} \approx 1.26 \text{ GeV}$. We obtain $\Gamma(\rho\pi)_D/\Gamma_{\text{tot}} \approx 1.5 \%$. This is comparable to the PDG value [2], i.e., $\Gamma(\rho\pi)_D/\Gamma_{\text{tot}} = (1.30 \pm 0.60 \pm 0.22) \%$.

Also, comparing to other determinations of BRs is difficult because in BRs for sequential decays, integrals are often performed over the mother and/or daughter resonances. This contribution depends on integration limits, the non-resonant background and the parameterization of the lineshapes of mother and daughter resonances. A unique and background-independent definition is provided by the residue at the pole [64, 216], which for unstable daughter resonances will be a function

of the spectator momentum; the analytic continuation of the amplitude to the a_1 pole, as performed in Ref. [23], is beyond the scope of this work because the moving three-body singularities appearing in our unitary formalism require special attention for complex energies [64].

In the upper row of Fig. 45, we show our calculation of the Dalitz plots using the parameters from the fit to the lineshape. In those, we note the clear outline of the ρ -meson at $\sigma \sim 0.59 \text{ GeV}^2$ for the plots with large enough available phase-space. The maxima of the plots lie at the intersection of the bands, meaning that the amplitudes exhibit a constructive interference. As such, this is consistent with the symmetry of

the amplitude $\hat{\Gamma}_{\Lambda\lambda}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ under the exchange of the two π^- due to Bose symmetry, see Eq. (48) and discussion below it.

The central row of Fig. 45 shows the projected Dalitz plots. We once again see peaks in each plot at the ρ -mass as well as its kinematic reflections. To understand the importance of the connected contributions, we show in the bottom row of the figure the quotient of the full and disconnected contributions, cf., Eq. (48). We observe a contribution from rescattering at the order of $\sim 5\%$. We note that this contribution is small but of similar order as the branching ratios to some inelastic channels (not included in this analysis), see Ref. [2]. Thus, the incomplete inclusion of rescattering might lead to incorrect extraction of the resonance parameters in these channels.

5.4 Summary

In this chapter, we have adopted a relativistic unitary formalism for three-to-three scattering to address the decay of the $a_1(1260)$ -resonance to three pions. The free parameters of the approach, related to the production mechanism, two-body sub-channel interaction and the three-body contact term were fixed to the two- and three-body data.

The key technical part of the approach is the relativistic Bethe-Salpeter type

equation, which depends on the momenta of the incoming and outgoing pions. First, this integral equation was reduced by a partial wave decomposition. This coupled-channel equation has then been numericized for momenta on a complex contour, leading to an algebraic equation. Solutions of the latter have been extrapolated to real momenta, for which the amplitudes were then related to observables. These include lineshape, Dalitz plots, and projected Dalitz plots. Fitting the free parameters of the approach to the lineshape from the ALEPH experiment, we observe a dominant S -wave contribution. The Dalitz plots and projected Dalitz plots show the outline of the ρ -meson and its kinematic reflections. The rescattering effects are small but non-negligible for this particular system, and could affect the extraction of resonance parameters in a more complete description. The inclusion of $\sigma\pi$ and other channels and the direct analysis of Dalitz plots is work in progress. This would allow for an accurate determination of the pole position of the $a_1(1260)$ through analytic continuation to the complex energy plane.

6 Summary and Outlook

In this thesis, we have used unitarity and the Bethe-Salpeter equation to make calculation for two- and three-body scattering. This principle and this equation allow one to approach the challenges that the non-perturbative nature of low-energy QCD presents. Using this method, we have studied the properties of certain resonances manifested in data for particle scattering and decay processes.

Specifically, our research consists of two major projects. The first looks at the $\Lambda(1405)$ and the $\Sigma(1385)$ through $S = -1$ meson-baryon scattering. Building on the established model, we were able to address differential cross section data. The inclusion of such data allowed the extension of the model to analyze the P -waves. In this study, we found that the data demand a P -wave resonance to account for the effects of the $\Sigma(1385)$ on the differential cross section data. However, this pole does not have the same quantum numbers as the $\Sigma(1385)$; in fact, the partial wave in the model with those quantum numbers is not complete for the calculation of a Lagrangian up to next to leading chiral order, i.e., one would need to go to higher chiral orders to provide meaningful dynamics for this partial wave. Thus, the model could not dynamically generate the $\Sigma(1385)$ resonance, but it generates a pole that accounts for its effects.

The fit in this study uses 20 free parameters, namely 14 LECs and 6 scale constants. This research will be continued by using the LASSO method, cross validation and information criteria to test the robustness of these parameters. Depending on the results, one could remove extraneous degrees of freedom and refit, allowing for an even more accurate description of the data.

The second project that constitutes this thesis involved the development of a model for $\tau^- \rightarrow \pi^- \pi^- \pi^+ \nu_\tau$ decay. This model uses a coupled-channel calculation for $\rho\pi$ in the S -wave and D -wave, based on a manifestly unitary dispersive three-body formalism.. After developing necessary techniques and employing numerical methods,

we presented the line-shape and Dalitz plot calculations. Because we have not included the small but non-negligible $\pi\sigma$ channel, our results are only a proof of concept. We show that the dominant $\pi\rho$ S -wave can roughly describe the data, but without the $\pi\sigma$ channel, the fit is not good enough for a statistical analysis to make sense.

We intend for the work done on this subject to be the first of many studies. We intend to continue this research by the implementation of the $\pi\sigma$ channel which would allow for a complete fit of the data and a measurement of the parameters of the $a_1(1260)$. Once this step has been accomplished, there are a number of further developments and applications of this research. There is an ongoing project to calculate three-pion interactions using lattice QCD [39, 185, 204, 228]. This work can be extended to accommodate subsystems with spin so that the present results can be adapted to relate finite volume calculations to infinite volume ones, with the ultimate goal to analyze lattice QCD data on the $a_1(1260)$, which are currently being produced by different groups.

Furthermore, successful three-body calculations for the simplest case opens the door to further three-body analyses. These are of great interest for the study of exotic states, but the isolation of such states requires a partial wave analysis, to which the current results can provide unitary final-state interactions, once suitably extended to other quantum numbers than the $a_1(1260)$. The application of this method to such states can therefore be of use for the study of explicit gluon dynamics. This will further the goal of providing analysis for the many experiments designed for the study of QCD.

Appendix

A Form-Factors

The integral equations in Sec. 5 are regularized using form-factors. Apart from suppressing large spectator or loop momenta, the requirements on the form-factor in the $\rho \rightarrow \pi\pi$ decay vertex v are

1. As a consequence of three-body unitarity, the decay vertex in the self-energy of Eq. (52), in the π exchange of Eq. (55), and in the final decay of Eq. (48) must be self-consistent [17]. In particular, the form-factor must be Lorentz invariant because it is evaluated in the three-body rest frame but also the isobar rest frames.
2. To preserve unitarity, the previous requirement must be fulfilled if the pions of the ρ decay are on-shell, but may be dropped if they are off-shell.
3. For the B -term, it cannot have poles on the chosen integration contour; this requirement needs to be fulfilled for all scattering angles, spectator momenta, and three-body energies $\sqrt{s}$. Also, it cannot contain poles in the region around the contour and the real-momentum axis, due to the extrapolation procedure to obtain the amplitude for real outgoing momenta from complex ones described in Sec. 5.2.2.

Obviously, the above requirements do not fix the form-factor uniquely. For the B -term and the final $\rho \rightarrow \pi\pi$ decay, we choose a form-factor for Eq. (50) that reads

$$F(\sigma, Q^2) = \frac{\Lambda^4}{\Lambda^4 + e^{1+(Q^2/4-(\sigma-4m_\pi^2))/(1\text{GeV}^2)}} , \quad (76)$$

for $\sigma = (q_2 + q_3)^2$, $Q^2 = (q_2 - q_3)^2$. As demanded by condition 1 in the above list, a consistent choice needs to be made for the form-factor in the self-energy (52).

However, since the latter only includes vertices projected to P -wave, the form-factor will have a different momentum dependence.

$$\tilde{F}(\sigma, k) = \frac{\Lambda^4}{\Lambda^4 + e^{1 + ((4(\sqrt{\sigma} - 2E_k)^2 - 4k^2)/4 - (\sigma - 4m_\pi^2))/(1\text{GeV}^2)}}. \quad (77)$$

Note, that the term $(\sqrt{\sigma} - 2E_k)^2$ vanishes for the on-shell momentum, $k = \sqrt{\sigma - m_\pi^2}/2$, which is the value leading to an imaginary part of the self-energy integral (52). Therefore, we can multiply this term by an arbitrary factor and still satisfy condition 1 for the form-factors. We multiply it by a factor of four to allow the form-factor to suppress large momenta k for the case when $\sigma(p) = 0$.

Another form-factor appears in the three-body term C of Eq. (56) and similarly in the $a_1(1260) \rightarrow \rho\pi$ vertex of Eq. (62). Both these contributions are real-valued and thus respect the unitarity requirement and furthermore enter only in a fixed (center-of-mass) reference frame. Hence, they do not have to follow the above constraints, and are chosen for simplicity as

$$H(p) = \frac{\Lambda'^4}{\Lambda'^4 + p^4}. \quad (78)$$

In principle this factor allows for an additional parameter Λ' . We choose the cutoff

$$\Lambda' = 1.0 \text{ GeV}.$$

In order to explain why we have chosen the given function for the $\rho\pi\pi$ form factor, we show several simpler functions and how they fail to meet these conditions. The form factor must depend on an invariant for the vertex. The two simplest invariants for a $\rho\pi\pi$ vertex with outgoing pion momentum q_2 and q_3 will be the momentum exchange, $Q^2 = (q_2 - q_3)^2$, and the invariant mass, $\sigma = (q_2 + q_3)^2$.

Figure 46: A plot of the complex plane in p for a given value $q = 1.8(1 - 0.36i)$. The red line gives the contour and the blue points are values at which the form factor given in Eq. (79) has a pole. (Each one is calculated for a different $\cos\theta$ between -1 and 1). The fact that the blue points cross the red line indicates that calculation with this form factor on this contour would give numerically unstable results.

One of the simplest cases for a form factor, (employed by Ref. [106]) is

$$F(\sigma, Q^2) = \frac{\Lambda^4}{\Lambda^4 + Q^4/(1\text{GeV}^2)}. \quad (79)$$

This form factor is valid on the real axis, but it has cuts that cross our chosen contour as shown in Fig. 46 making numerical calculation impossible.

Another possible form factor is

$$F(\sigma, Q^2) = \frac{\Lambda^2}{\Lambda^2 - Q^2/(1\text{GeV}^2)}. \quad (80)$$

This form factor does not have cuts that cross the contour but it does not consistently converge as shown in Fig. 47.

To avoid this error, we use an exponential form factor. A simple one might have the form

$$F(\sigma, Q^2) = \frac{\Lambda^4}{\Lambda^4 + e^{1+Q^2/4/(1\text{GeV}^2)}}. \quad (81)$$

Figure 47: A contour plot of the squared form factor, $F^2(Q^2(p, q), \sigma(p, q))$ given in Eq. (80) for OPE as a function of spectator momentum p and q_1 . For smaller values, the form factor exhibits the behavior one would hope for, but we also see a reemergence for values around 3 GeV.

Figure 48: A contour plot of the squared form factor for OPE given in Eq. (81) as a function of spectator momentum p and q_1 . This form factor converges if both incoming and outgoing momentum are large but if one is small and the other is large, it does not converge.

This form factor converges when both spectator momentum are large but not for the case where one is small and one is large as shown in Fig. 48.

We modify it to give the form factor in Eq. (76) which we have chosen for this research. It converges for both spectator momenta and does not cross a contour.

However, there is a problem using this form factor for the self-energy given in Eq. (52). It depends on Q^2 and σ . But the range of values allowed for σ includes $\sigma = 0$. When $\sigma = 0$, the pion momentum difference for a vertex in the self-energy is $Q^2 = m_\pi^2$. Thus, the form factor does not depend on the loop momentum and cannot regularize the function. Instead we modify it as given by Eq. (77). We are allowed to make this modification without violating unitarity because the imaginary part of the inverse of the self-energy is fixed. The imaginary contribution comes from the on-shell point, at which, Eq. (76) and Eq. (77) are the same.

B Angular Kinematics and Phase Space Integration

The $\rho \rightarrow \pi\pi$ vertex in Eq. (50) includes the four-product $\epsilon_\lambda^\mu(\mathbf{q}_1) (q_2 - q_3)_\mu$ with λ indicating the helicity of the isobar. The polarization vector or its complex conjugate is assigned to a given incoming or outgoing isobar. Explicitly, the corresponding polarization vectors read,

$$\epsilon_0 = \frac{1}{m} \begin{pmatrix} p \\ 0 \\ 0 \\ E_p \end{pmatrix}, \quad \epsilon_{\pm 1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mp 1 \\ -i \\ 0 \end{pmatrix}, \quad (82)$$

$$\epsilon_0^* = \frac{1}{m} \begin{pmatrix} k \\ E_{q_1} \sin \theta_1 \cos \phi_1 \\ E_{q_1} \sin \theta_1 \sin \phi_1 \\ E_{q_1} \cos \theta_1 \end{pmatrix}, \quad \epsilon_{\pm 1}^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mp \cos \theta_1 \cos \phi_1 + i \sin \phi_1 \\ -i \cos \phi_1 \mp \cos \theta_1 \sin \phi_1 \\ \pm \sin \theta_1 \end{pmatrix}, \quad (83)$$

where the incoming isobar with momentum $\mathbf{p}$ points in the z -direction, see, e.g., Eq. (54). In order to simplify the partial wave projection of Eq. (59), azimuthal symmetry allows us to choose a reference frame, in which the outgoing isobar at momentum $\mathbf{q}_1$ lies in the xz -plane with scattering angle θ_1 and $\phi_1 = 0$.

The polarization vectors are also used for the calculation of the final vertex in Eq. (48). In this case, we work with a more general expression of the momenta in the phase space as defined in the integral of Eq. (65). We choose to define the angles of $\mathbf{q}_2$ relative to $\mathbf{q}_1$, i.e., the angle θ_{12} and the azimuthal angle ϕ_{12} . Thus, one defines the

explicit components of $\mathbf{q}_2$ in terms of θ_{12} , ϕ_{12} , and the angles of the rotation of $\mathbf{q}_1$ from the z -axis in its actual direction. We introduce the rotation matrices $R_{\mathbf{q}_1}(\phi_{12})$ and $R_{\mathbf{q}_{1\perp}}(\theta_{12})$, where the subscript specifies the vector about which they are rotated and the argument gives the angle by which they are rotated. The vector $\mathbf{q}_{1\perp}$ is calculated by $R_z(\phi_1)R_y(\theta_1)R_z(-\phi_1)\hat{y}$. Note that this is the convention by Jacob and Wick, in which an initial rotation about the z -axis takes place [224]. It can be easily

shown that this corresponds to a rotation about an axis given by

$\mathbf{n} = \mathbf{e}_z \times \mathbf{q}_1 / |\mathbf{e}_z \times \mathbf{q}_1|$ It is important to use the same convention for the polarization vectors as for the capital Wigner-D functions employed for the transformation from the partial wave basis back to the plane wave basis in Eq. (63). Explicitly, one has

$$\mathbf{q}_1 = -q_1 \left(R_z(\phi_1)R_y(\theta_1)R_z(-\phi_1)\hat{z} \right), \quad \mathbf{q}_2 = q_2 \left(R_{\mathbf{q}_1}(\phi_{12})R_{\mathbf{q}_{1\perp}}(\theta_{12})R_{\mathbf{q}_1}(-\phi_{12})\hat{\mathbf{q}}_1 \right), \quad (84)$$

whereas $\mathbf{q}_3$ is fixed to be $-\mathbf{q}_3 = \mathbf{q}_1 + \mathbf{q}_2$ in the overall center of mass frame. The labeling of momenta changes through the symmetrization of the two π^- indicated in Eq. (48). The components of the polarization vectors change depending on whether

q_1 or q_2 is designated to be the spectator momentum. If q_1 is the spectator momentum, the polarization vectors will have the same components as the right two equations in (82), whereas if q_2 is the spectator momentum, the equations read

$$\epsilon_0 = \frac{1}{m} \left(\begin{array}{c} k \\ R_{\mathbf{q}_1}(\phi_{12})R_{\mathbf{q}_{1\perp}}(\theta_{12})R_{\mathbf{q}_1}(-\phi_{12})R_z(\phi_1)R_y(\theta_1)R_z(-\phi_1) \begin{pmatrix} 0 \\ 0 \\ E_p \end{pmatrix} \end{array} \right), \quad (85)$$

and

$$\epsilon_{\pm 1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ R_{\mathbf{q}_1}(\phi_{12})R_{\mathbf{q}_{1\perp}}(\theta_{12})R_{\mathbf{q}_1}(-\phi_{12})R_z(\phi_1)R_y(\theta_1)R_z(-\phi_1) \begin{pmatrix} \mp 1 \\ -i \\ 0 \end{pmatrix} \end{pmatrix}. \quad (86)$$

To relate our amplitude to data, we must integrate over all possible variables. There are three outgoing pions that are on-shell. Hence we will integrate over 9 variables because each one has three momentum components. But conservation of momentum and our freedom of choice of frame provide some constraints that will be implemented in the form of δ distributions. The integral is

$$\int d^3\mathbf{q}_1 \frac{1}{E_{q_1}} d^3\mathbf{q}_2 \frac{1}{E_{q_2}} d^3\mathbf{q}_3 \frac{1}{E_{q_3}} \sum_{\lambda} |\Gamma|^2 \delta^3(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \delta(\sqrt{s} - E_{q_1} - E_{q_2} - E_{q_3}). \quad (87)$$

We will call q_1 the spectator momentum, and q_2 and q_3 the momenta from pions that decay from the isobar. The first delta function can be written to break out of the integration over $\mathbf{q}_3$.

The momenta can be written in Cartesian or spherical coordinates with angles θ_i and ϕ_i corresponding to q_i :

$$\int d^3\mathbf{q}_1 \frac{1}{E_{q_1}} d^3\mathbf{q}_2 \frac{1}{E_{q_2}} \frac{1}{E_{q_3}} \sum_{\lambda} |\Gamma|^2 \delta(\sqrt{s} - E_{q_1} - E_{q_2} - E_{q_3}). \quad (88)$$

Here $\mathbf{q}_3 = -\mathbf{q}_1 - \mathbf{q}_2$ and $E_{q_3} = \sqrt{m^2 + (-\mathbf{q}_1 - \mathbf{q}_2)^2} = \sqrt{m^2 + q_1^2 + q_2^2 + 2q_1q_2 \cos \theta_{12}}$.

θ_{12} is the angle between $\mathbf{q}_1$ and $\mathbf{q}_2$.

We now want to evaluate the second δ function in the integral. We choose to evaluate it over θ_{12} . This means we need to write $d^3\mathbf{q}_2$ in a way that allows it. We write it as

$dq_2 d\cos\theta_{12} d\phi_{12}$. Note that the factor of $\sin\theta_{12}$ is absorbed to give us $d\cos\theta$ rather than just $d\theta$.

We rewrite the integral

$$\int d^3\mathbf{q}_1 \frac{1}{E_{q_1}} dq_2 d\cos\theta_{12} d\phi_{12} q_2^2 \frac{1}{E_{q_2}} \frac{1}{E_{q_3}} \sum_{\lambda} |\Gamma|^2 \times \delta(\sqrt{s} - E_{q_1} - E_{q_2} - \sqrt{m^2 + q_1^2 + q_2^2 + 2q_1 q_2 \cos\theta_{12}}). \quad (89)$$

There is a δ function identity that reads $\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$ where x_i are the simple roots of the δ function. In this case, $\cos\theta_{12}$ is the integration variable over which we evaluate the δ function. If it can equal zero, then

$$g'(x_i) = \frac{dg}{d\cos\theta} = \frac{2q_1 q_2}{\sqrt{m^2 + q_1^2 + q_2^2 + 2q_1 q_2 \cos\theta_{12}}}$$

We apply this function to evaluate the integral over θ_{12} which

$$\begin{aligned} & \int d^3\mathbf{q}_1 \frac{1}{E_{q_1}} dq_2 d\phi_{12} q_2^2 \frac{1}{E_{q_2}} \frac{1}{E_{q_3}} \sum_{\lambda} |\Gamma|^2 \frac{\sqrt{m^2 + q_1^2 + q_2^2 + 2q_1 q_2 \cos\theta_{12}}}{2q_1 q_2} \Theta(1 - \cos^2\theta_{12}) \\ &= \int d^3\mathbf{q}_1 \frac{q_1 q_2}{E_{q_1} E_{q_2}} dq_2 d\phi_{12} \sum_{\lambda} |\Gamma|^2 \Theta(1 - \cos^2\theta_{12}), \end{aligned} \quad (90)$$

where

$$\cos\theta_{12} = ((\sqrt{s} - E_1 - E_2)^2 - m^2 - q_1^2 - q_2^2)/(2q_1 q_2)$$

The Heaviside function accounts for the fact that the integral will vanish if $\cos\theta_{12}$ is not between -1 and 1. The full integration is over 5 independent variables: q_1 , q_2 , θ_1 , ϕ_1 , and ϕ_{12} . The kinematics for these variables are depicted in Fig. 41

Nota Bene: We integrate over 5 variables: q_1 , q_2 , θ_1 , ϕ_1 , and ϕ_{12} . However, when one exploits the azimuthal symmetry of the specific calculation we perform, one could

potentially simplify the calculation. Specifically this could be done by picking an outgoing reference frame in which all three outgoing three-momenta lie on the $x - z$ plane.

C Extrapolation

In Sec. 5, we have chosen to calculate $\check{\Gamma}(q_1)$ for q_1 on a contour with a slope in the complex plane of $-\alpha$. (See Sec. 5.2.2). This method avoids the singularities that would otherwise make numerical integration impossible. However the results for $\check{\Gamma}(q_1)$ as a function of complex momenta are not the same as for physically meaningful, real momenta. To relate the former result to the latter we perform extrapolation with a modified Pade-Approximant with the form

$$\check{\Gamma}_{L'L}(q_1) = \left(\frac{q_1}{m_\pi}\right)^L H(q_1) \frac{\sum_{j=0}^m a_j^{L'L} (q_1/m_\pi)^j}{\sum_{k=0}^n b_k^{L'L} (q_1/m_\pi)^k}. \quad (91)$$

This extrapolation can be done for $\check{\Gamma}(q_1)$ calculated for values of q_1 along a single line in the complex plane. If $\check{\Gamma}$ were an analytic complex function of two real variables, this would not be possible. In that case, the function could only give information about the derivatives along the direction of the line. But $\check{\Gamma}$ is an analytic function of a single variable with a real and imaginary part which means it must satisfy the Cauchy relation for analyticity: For a function $f(z) = u(x, y) + iv(x, y)$ with $z = x + iy$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \quad (92)$$

From this relation it follows that knowledge of the derivative at a point in one direction gives knowlege of the derivative in all directions.

Because we only calculate the function at a finite number of points, we can make an approximation that is only accurate close to the contour on which our momenta lie,

but if this contour is close enough to the real axis, the approximation will be good. However, if the contour is too close to the real axis, the poles that our integration avoids will make numerical calculations unstable. Thus, there is an upper and a lower bound to the slope, α , in the complex plane.

There is an easy way to test whether α falls within these bounds. If we make the extrapolation for different α , we must see a similar value for the function extrapolated to the real axis. We perform this test and find that it is satisfied for lower values of s but not for higher values of s . The lower values of s are shown for $\alpha = 0.15, 0.2, 0.25$ in Fig. 49. The extrapolated functions for α in the range we show are quite similar, far more similar than the data from which they were extrapolated. This reassures us that if we pick $\alpha = 0.2$ the extrapolation is effective. There is one small blemish in $\alpha = 0.15$, $w = 1.2$ but it does not make a very big difference and disappears if the number of Gauss points used for the integration is increased. Of course, the stability of the extrapolation depends on the number of Gauss points. A compromise between numerical stability and speed is given by 30 Gauss points used to evaluate all results shown in Sec. 5.

Fig. 50 shows α too low and too high resulting in a bad approximation and a numerical instability, respectively. Thus, α outside the range 0.15-0.25 do not necessarily yield accurate result if the number of Gauss points is kept low enough to guarantee a speedy evaluation.

Our plots and analysis in this section have been done with 30 values of $\check{\Gamma}$. If we calculated more values, we would have more accurate information about higher order derivatives, meaning that the upper bound of alpha would increase. The function would also be better able to handle numerical instabilities so the lower bound would decrease. Thus increasing the precision of $\check{\Gamma}$ increase the range of α in both directions.

Figure 49: A comparison of the data calculated on the complex contour for various α in blue, ($\alpha = 0.15$), yellow, ($\alpha = 0.2$), and green ($\alpha = 0.25$), and the extrapolated functions for the same α . The solid lines are extrapolated functions that coincide well for real momenta, even if the points from which they are extrapolated (dots) are quite different, as alpha is different. Here, W refers to the total three-body energy called $\sqrt{s}$ in other parts of the thesis.

Figure 50: A comparison of the data calculated on the complex contour for α too high or too low to provide a good extrapolation. The red plot on the left shows an extreme case in which α is too low ($\alpha = 0.05$) in comparison with the adequate values of alpha shown previously. In this case, the calculation is numerically unstable and the fit function becomes unstable when it is fit to the data. The red plot on the right shows an α that is too high. In this case, because we are extrapolating a great distance, the fit function produces a pole near the real axis which does not have much effect on the data in the range where it is calculated but has a tremendous effect when we extrapolate all the way to the real axis.

D Symmetrization

The full decay calculation in Sec. 5 involves the sum over all possible pions

$$\frac{1}{3!} \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \langle q_i, q_j, q_k | \Gamma | \tau \rangle . \quad (93)$$

There are 6 possible outgoing states we sum over. $\langle q_1, q_2, q_3 |$, $\langle q_2, q_1, q_3 |$, $\langle q_3, q_2, q_1 |$, $\langle q_1, q_3, q_2 |$, $\langle q_2, q_3, q_1 |$, and $\langle q_3, q_1, q_2 |$. But q_3 , which corresponds to the π^+ , cannot be the spectator. If it were the spectator, the isobar would have to be formed by the two π^- , which would have the wrong quantum numbers for the ρ . If the first momentum in the bra state corresponds to the spectator, then the amplitudes for $\langle q_3, q_1, q_2 |$ and

$\langle q_3, q_2, q_1 |$ vanish leaving: $\langle q_1, q_2, q_3 |$, $\langle q_2, q_1, q_3 |$, $\langle q_1, q_3, q_2 |$, and $\langle q_2, q_3, q_1 |$. The amplitude is identical if one switches the two spectator pions. Thus, the first and second states are identical to the third and fourth states respectively. This effectively

means the sum is over only two states, $\langle q_1, q_2, q_3 |$, and $\langle q_2, q_1, q_3 |$.

$$\frac{1}{3!} \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \langle q_i, q_j, q_k | \Gamma | \tau \rangle = \frac{1}{3!} (\langle q_1, q_2, q_3 | \Gamma | \tau \rangle + \langle q_2, q_1, q_3 | \Gamma | \tau \rangle) . \quad (94)$$

The vertices $a_1(1260) \rightarrow \pi_1 \rho$ and $a_1(1260) \rightarrow \pi_2 \rho$ will have different Clebsch-Gordan coefficients. For $\langle q_2, q_1, q_3 |$ it is $1/\sqrt{2}$ and for $\langle q_1, q_2, q_3 |$ is $-1/\sqrt{2}$. Thus, the two states differ by a relative minus sign which gives us the minus sign in Eq. (48).

E Comparison of Various Components

In this section, we compare the strength of different terms. Fig. 51 compares the strength of the B -term from Eq. (60) and τ term from Eq. (51). The latter depends on $\sqrt{s}$ as well a single spectator momentum which is constrained to be the same incoming as outgoing whereas the former depends of $\sqrt{s}$ and different incoming and outgoing spectator momentum. In order to compare the two terms, we consider only values of the B -matrix with identical ingoing and outgoing momentum. Furthermore we compare B to $1/\tau$ to compare quantities of the same dimension. The projection of the B -term is smaller in D -wave than in S -wave. Note that the spectator momentum p is chosen complex along the integration contour (see discussion in Sec. 5.2.2). We compare only the real parts in the figure. The rapid change in B as a function of p indicates the region where three particles can go onshell. If p were real momenta, the maximum and minimum in $\text{Re } B$ would show as log-like structures from partial-wave projection but they are smoothed because we take a function of complex momentum.

Fig. 52 compares the B -matrix from Eq. (60) and C -matrix from Eq. (56) as functions of incoming and outgoing momenta at different energies. We observe that for the D -wave the two terms are around the same size whereas for the S -wave, the C -term is generally larger. It should be mentioned that the size of the C -term is regularization dependent and not a physical quantity. The C will change as the form

Figure 51: A comparison between $1/\tau$ and B for S -wave (Upper Row) and D -wave (Lower Row). τ will be the same for each partial wave, but B is not. Even though B can return values for different incoming and outgoing spectator momentum, we show only values where they are identical to make a valid comparison.

Figure 52: A comparison of the B -matrix for OPE and the C -matrix for three-body contact. For the S -wave, the C -term almost entirely dominates.

factor of Eq. (78) changes to describe the same lineshape, or in general, and other physical quantity. It would require further effort to study the regularization dependence of C beyond the scope of this thesis.

F Bibliography

References

- [1] Sarah Campbell. Photon production from gluon-mediated quark–anti-quark annihilation at confinement. *Phys. Rev.*, C92(1):014907, 2015, 1504.01654.
- [2] M. Tanabashi et al. Review of Particle Physics. *Phys. Rev.*, D98(3):030001, 2018.
- [3] Christine Davies. Lattice QCD: A Guide for people who want results. In *Proceedings, 58th Scottish Universities Summer School in physics, SUSSP58, St. Andrews, UK, August 22-28, 2004*, pages 233–272, 2005, hep-lat/0509046.
- [4] D.J. Griffiths and P.D.J. Griffiths. *Introduction to Quantum Mechanics*. Pearson international edition. Pearson Prentice Hall, 2005.
- [5] S. Schael et al. Branching ratios and spectral functions of tau decays: Final ALEPH measurements and physics implications. *Phys. Rept.*, 421:191–284, 2005, hep-ex/0506072.
- [6] M. Alekseev et al. Observation of a $J^{*}PC = 1^{-+}$ exotic resonance in diffractive dissociation of 190-GeV/c π^{-} into $\pi^{-} \pi^{-} \pi^{+}$. *Phys.Rev.Lett.*, 104:241803, 2010, 0910.5842.
- [7] Jonathan Zarling. The GlueX Experiment: Recent Results and Future Plans. *JPS Conf. Proc.*, 26:022002, 2019, 1911.11239.
- [8] P. del Amo Sanchez et al. Dalitz plot analysis of $D_s^{+} \rightarrow K^{+} K^{-} \pi^{+}$. *Phys. Rev.*, D83:052001, 2011, 1011.4190.
- [9] Maxim Mai. *From meson-baryon scattering to meson photoproduction*. PhD thesis, Bonn U., HISKP, 2013.

- [10] K. Moriya et al. Measurement of the $\Sigma\pi$ photoproduction line shapes near the $\Lambda(1405)$. *Phys. Rev.*, C87(3):035206, 2013, 1301.5000.
- [11] Maxim Mai and Ulf-G. Meißner. Constraints on the chiral unitary $\bar{K}N$ amplitude from $\pi\Sigma K^+$ photoproduction data. *Eur. Phys. J.*, A51(3):30, 2015, 1411.7884.
- [12] B. Borasoy, U.-G. Meissner, and R. Nissler. K- p scattering length from scattering experiments. *Phys. Rev. C*, 74:055201, 2006, hep-ph/0606108.
- [13] Zhi-Hui Guo and J. A. Oller. Meson-baryon reactions with strangeness -1 within a chiral framework. *Phys. Rev.*, C87(3):035202, 2013, 1210.3485.
- [14] Yoichi Ikeda, Tetsuo Hyodo, and Wolfram Weise. Chiral SU(3) theory of antikaon-nucleon interactions with improved threshold constraints. *Nucl. Phys. A*, 881:98–114, 2012, 1201.6549.
- [15] Maxim Mai and Ulf-G. Meißner. New insights into antikaon-nucleon scattering and the structure of the Lambda(1405). *Nucl. Phys.*, A900:51 – 64, 2013, 1202.2030.
- [16] L. Roca and E. Oset. $\Lambda(1405)$ poles obtained from $\pi i^0\Sigma^0$ photoproduction data. *Phys. Rev.*, C87(5):055201, 2013, 1301.5741.
- [17] M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak. Three-body Unitarity with Isobars Revisited. *Eur. Phys. J.*, A53(9):177, 2017, 1706.06118.
- [18] H. Albrecht et al. Analysis of the decay $\tau^- \rightarrow \ell^- \pi^- \pi^0 \pi^+$ tau-neutrino and determination of the $a_1(1260)$ resonance parameters. *Z. Phys.*, C58:61–70, 1993.
- [19] D. M. Asner et al. Hadronic structure in the decay. *Phys. Rev.*, D61:012002, 2000, hep-ex/9902022.

- [20] Michel Davier, Andreas Höcker, Bogdan Malaescu, Chang-Zheng Yuan, and Zhiqing Zhang. Update of the ALEPH non-strange spectral functions from hadronic τ decays. *Eur. Phys. J.*, C74(3):2803, 2014, 1312.1501.
- [21] G. Janssen, J. W. Durso, K. Holinde, B. C. Pearce, and J. Speth. π ρ scattering and the π N N form-factor. *Phys. Rev. Lett.*, 71:1978–1981, 1993.
- [22] H. Kamano, S. X. Nakamura, T. S. H. Lee, and T. Sato. Unitary coupled-channels model for three-mesons decays of heavy mesons. *Phys. Rev.*, D84:114019, 2011, 1106.4523.
- [23] M. Mikhasenko, A. Pilloni, M. Albaladejo, C. Fernández-Ramírez, A. Jackura, V. Mathieu, J. Nys, A. Rodas, B. Ketzer, and A. P. Szczepaniak. Pole position of the $a_1(1260)$ from τ -decay. *Phys. Rev.*, D98(9):096021, 2018, 1810.00016.
- [24] R. J. Nowak et al. Charged Sigma Hyperon Production by K- Meson Interactions at Rest. *Nucl. Phys.*, B139:61–71, 1978.
- [25] Mason B. Watson, Massimiliano Ferro-Luzzi, and Robert D. Tripp. Analysis of $Y-0^*$ (1520) and Determination of the Sigma Parity. *Phys. Rev.*, 131:2248–2281, 1963.
- [26] J. Ciborowski et al. KAON SCATTERING AND CHARGED SIGMA HYPERON PRODUCTION IN K- P INTERACTIONS BELOW 300-MEV/C. *J. Phys.*, G8:13–32, 1982.
- [27] William E. Humphrey and Ronald R. Ross. Low-Energy Interactions of K-Mesons in Hydrogen. *Phys. Rev.*, 127:1305–1323, 1962.
- [28] Terry S. Mast, Margaret Alston-Garnjost, Roger O. Bangerter, Angela S. Barbaro-Galtieri, Frank T. Solmitz, and Robert D. Tripp. Elastic, Charge

- Exchange, and Total K^-p Cross-Sections in the Momentum Range 220-MeV/c to 470-MeV/c. *Phys. Rev.*, D14:13, 1976.
- [29] Yoichi Ikeda, Tetsuo Hyodo, and Wolfram Weise. Improved constraints on chiral SU(3) dynamics from kaonic hydrogen. *Phys. Lett.*, B706:63–67, 2011, 1109.3005.
- [30] R. J. Hemingway. Production of $\Lambda(1405)$ in K^-p Reactions at 4.2-GeV/c. *Nucl. Phys.*, B253:742–752, 1985.
- [31] Michael Edward Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Westview Press, 1995. Reading, USA: Addison-Wesley (1995) 842 p.
- [32] F. Gross and R. Minehart, editors. *CONTINUOUS ELECTRON BEAM ACCELERATOR FACILITY. PROCEEDINGS, CEBAF/SURA SUMMER WORKSHOP, NEWPORT NEWS, USA, JUNE 23-27, 1986*, 1986.
- [33] L.S. Cardman. Jefferson Lab. *Nucl. Phys. News*, 6N4:18–25, 1996.
- [34] T. Horn. Introduction to EIC/detector concept. In *4th Workshop on Exclusive Reactions at High Momentum Transfer*, pages 385–394, 2011.
- [35] Peter Ostroumov, Robert Janssens, Michael Kelly, Sergei Kondrashev, Brahim Mustapha, Richard Pardo, and Guy Savard. Efficiency and Intensity Upgrade of the ATLAS Facility. In *25th International Linear Accelerator Conference*, page MOP045, 2011.
- [36] Rajan Gupta. Introduction to lattice QCD: Course. In *Probing the standard model of particle interactions. Proceedings, Summer School in Theoretical Physics, NATO Advanced Study Institute, 68th session, Les Houches, France, July 28-September 5, 1997. Pt. 1, 2*, pages 83–219, 1997, hep-lat/9807028.

- [37] M. Mai and M. Döring. Three-body Unitarity in the Finite Volume. *Eur. Phys. J. A*, 53(12):240, 2017, 1709.08222.
- [38] Maxim Mai. 3-body quantization condition in unitary isobar formalism. *PoS, LATTICE2018:050*, 2018, 1810.00604.
- [39] M. Mai, M. Döring, C. Culver, and A. Alexandru. Three-body unitarity versus finite-volume $\pi^+\pi^+\pi^+$ spectrum from lattice QCD. *Phys. Rev. D*, 101(5):054510, 2020, 1909.05749.
- [40] M. Döring, H. W. Hammer, M. Mai, J. Y Pang, § A. Rusetsky, and J. Wu. Three-body spectrum in a finite volume: the role of cubic symmetry. *Phys. Rev.*, D97(11):114508, 2018, 1802.03362.
- [41] Martin Lüscher. Two particle states on a torus and their relation to the scattering matrix. *Nucl. Phys.*, B354:531–578, 1991.
- [42] M. Lüscher. Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States. *Commun. Math. Phys.*, 105:153–188, 1986.
- [43] E. Oset. Unitarized chiral perturbation theory of hadrons. *Nucl. Phys.*, A721:58–65, 2003, nucl-th/0211077.
- [44] A. Chodos, R. L. Jaffe, K. Johnson, Charles B. Thorn, and V. F. Weisskopf. A New Extended Model of Hadrons. *Phys. Rev.*, D9:3471–3495, 1974.
- [45] Nathan Isgur and Gabriel Karl. Hyperfine Interactions in Negative Parity Baryons. *Phys. Lett.*, 72B:109, 1977.
- [46] S. N. Jena and M. R. Behera. Electromagnetic form-factors of nucleons in a relativistic square-root potential model of independent quarks. *Pramana*, 47:233–248, 1996.

- [47] C. Quigg and Jonathan L. Rosner. Quantum Mechanics with Applications to Quarkonium. *Phys. Rept.*, 56:167–235, 1979.
- [48] John L. Richardson. The Heavy Quark Potential and the Upsilon, J/psi Systems. *Phys. Lett.*, 82B:272–274, 1979.
- [49] R. H. Dalitz and S. F. Tuan. A possible resonant state in pion-hyperon scattering. *Phys. Rev. Lett.*, 2:425–428, 1959.
- [50] Khin Swe Myint, Yoshinori Akaishi, Maryam Hassanvand, and Toshimitsu Yamazaki. Single-pole Nature of the Detectable Lambda(1405). *PTEP*, 2018(7):073D01, 2018, 1804.08240.
- [51] Curtis Meyer. *Light and Exotic Mesons*, <http://curtismeyer.com/material/lecture.pdf>.
- [52] Bernhard Ketzer, Boris Grube, and Dmitry Ryabchikov. Light-Meson Spectroscopy with COMPASS. 9 2019, 1909.06366.
- [53] Florian M. Kaspar, Boris Grube, Fabian Krinner, Stephan Paul, and Stefan Wallner. Wave-Selection Techniques for Partial-Wave Analysis in Light-Meson Spectroscopy. In *18th International Conference on Hadron Spectroscopy and Structure*, 12 2019, 1912.09849.
- [54] F. Krinner. Freed-Isobar Analysis of Light Mesons at COMPASS. In *18th International Conference on Hadron Spectroscopy and Structure*, 11 2019, 1911.12907.
- [55] M. Aghasyan et al. Light isovector resonances in $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ at 190 GeV/c. *Phys. Rev. D*, 98(9):092003, 2018, 1802.05913.
- [56] M.R. Shepherd. The GLUEX experiment. *AIP Conf. Proc.*, 1182(1):816–819, 2009.

- [57] Cristiano Fanelli. The GlueX Experiment: First Results. *Few Body Syst.*, 58(3):135, 2017.
- [58] David B. Kaplan. More effective field theory for nonrelativistic scattering. *Nucl. Phys. B*, 494:471–484, 1997, nucl-th/9610052.
- [59] Paulo F. Bedaque and Harald W. Griebhammer. Quartet S wave neutron deuteron scattering in effective field theory. *Nucl. Phys.*, A671:357–379, 2000, nucl-th/9907077.
- [60] E.E. Salpeter and H.A. Bethe. A Relativistic equation for bound state problems. *Phys. Rev.*, 84:1232–1242, 1951.
- [61] Z.K. Silagadze. Wick-Cutkosky model: An Introduction. 3 1998, hep-ph/9803307.
- [62] Noboru Nakanishi. A General survey of the theory of the Bethe-Salpeter equation. *Prog. Theor. Phys. Suppl.*, 43:1–81, 1969.
- [63] Helmut Habertzettl. *Quantum Mechanics*. 2015.
- [64] M. Döring, C. Hanhart, F. Huang, S. Krewald, and U. G. Meißner. Analytic properties of the scattering amplitude and resonances parameters in a meson exchange model. *Nucl. Phys.*, A829:170–209, 2009, 0903.4337.
- [65] v n gribov. The theory of complex angular momenta : Gribov lectures on theoretical physics. 2003.
- [66] V. Baru, J. Haidenbauer, C. Hanhart, Alexander Evgenyevich Kudryavtsev, and Ulf-G. Meißner. Flatte-like distributions and the $a(0)(980)$ / $f(0)(980)$ mesons. *Eur. Phys. J.*, A23:523–533, 2005, nucl-th/0410099.
- [67] Klaus J. Peters. A Primer on partial wave analysis. *Int. J. Mod. Phys.*, A21:5618–5624, 2006, hep-ph/0412069.

- [68] Peter C. Bruns, Maxim Mai, and Ulf-G. Meißner. Chiral dynamics of the S11(1535) and S11(1650) resonances revisited. *Phys. Lett.*, B697:254–259, 2011, 1012.2233.
- [69] Stefan Scherer and Matthias R. Schindler. *A Primer for Chiral Perturbation Theory*, volume 830. 2012.
- [70] Bin Hu. *Analysis of Two- and Three-Body Systems from Lattice QCD Simulations*. PhD thesis, GWU, Washington, 2018.
- [71] Neramballi Ripunjay Acharya, Feng-Kun Guo, Maxim Mai, and Ulf-G. Meißner. θ -dependence of the lightest meson resonances in QCD. *Phys. Rev.*, D92:054023, 2015, 1507.08570.
- [72] Steven Weinberg. Phenomenological Lagrangians. *Physica A*, 96(1-2):327–340, 1979.
- [73] E. Oset, J.A. Oller, J.R. Pelaez, and A. Ramos. Meson meson and meson - baryon interactions in a chiral nonperturbative approach. *Acta Phys. Polon. B*, 29:3101–3109, 1998, hep-ph/9806460.
- [74] M. Sakitt, T. B. Day, R. G. Glasser, N. Seeman, J. H. Friedman, W. E. Humphrey, and R. R. Ross. LOW-ENERGY K- MESON INTERACTIONS IN HYDROGEN. *Phys. Rev.*, 139:B719, 1965.
- [75] M. Bazzi et al. A New Measurement of Kaonic Hydrogen X-rays. *Phys. Lett.*, B704:113–117, 2011, 1105.3090.
- [76] Ulf-G. Meißner, Udit Raha, and Akaki Rusetsky. Spectrum and decays of kaonic hydrogen. *Eur. Phys. J.*, C35:349–357, 2004, hep-ph/0402261.
- [77] M. Döring and U.-G. Meißner. Kaon–nucleon scattering lengths from kaonic deuterium experiments revisited. *Phys. Lett. B*, 704:663–666, 2011, 1108.5912.

- [78] D. N. Tovee et al. Some properties of the charged sigma hyperons. *Nucl. Phys.*, B33:493–504, 1971.
- [79] Norbert Kaiser, P. B. Siegel, and W. Weise. Chiral dynamics and the S11 (1535) nucleon resonance. *Phys. Lett.*, B362:23–28, 1995, nucl-th/9507036.
- [80] E. Oset and A. Ramos. Nonperturbative chiral approach to s wave anti-K N interactions. *Nucl. Phys.*, A635:99–120, 1998, nucl-th/9711022.
- [81] J. A. Oller and Ulf-G. Meißner. Chiral dynamics in the presence of bound states: Kaon nucleon interactions revisited. *Phys. Lett.*, B500:263–272, 2001, hep-ph/0011146.
- [82] Matthias Frink and Ulf-G. Meissner. On the chiral effective meson-baryon Lagrangian at third order. *Eur. Phys. J. A*, 29:255–260, 2006, hep-ph/0609256.
- [83] A. C. Davison and D. V. Hinkley. *Bootstrap Methods and their Application*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 1997.
- [84] L.D. Faddeev. Scattering theory for a three particle system. *Sov. Phys. JETP*, 12:1014–1019, 1961.
- [85] L.D. Faddeev. The Construction of the resolvent of the Schroedinger operator for a three-particle system, and the scattering problem. *Sov. Phys. Dokl.*, 7:600–602, 1963.
- [86] E.O. Alt, P. Grassberger, and W. Sandhas. Reduction of the three - particle collision problem to multichannel two - particle Lippmann-Schwinger equations. *Nucl. Phys. B*, 2:167–180, 1967.
- [87] R. Aaron, R. D. Amado, and J. E. Young. Relativistic three-body theory with applications to pi-minus n scattering. *Phys. Rev.*, 174:2022–2032, 1968.

- [88] R. Aaron and R. D. Amado. Analysis of three-hadron final states. *Phys. Rev. Lett.*, 31:1157–1159, 1973.
- [89] R. D. Amado. Minimal three-body scattering theory. *Phys. Rev. Lett.*, 33:333–336, 1974.
- [90] Ronald Aaron, Richard H. Thompson, Ralph D. Amado, Richard A. Arndt, Doris C. Teplitz, and Vigdor L. Teplitz. Unitarity Effects in Pion Production Isobar Amplitudes. *Phys. Rev.*, D12:1984, 1975.
- [91] Ralph D. Amado. Minimal three-body equations with finite-range effects. *Phys. Rev.*, C12:1354–1357, 1975.
- [92] R. Aaron and R. D. Amado. Relativistic Isobar Model: Spinless Particles. *Phys. Rev.*, D13:2581, 1976.
- [93] Ronald Aaron, R. D. Amado, Richard A. Arndt, Yogesh Goradia, Doris C. Teplitz, and Vigdor L. Teplitz. Isobar Formalism and One Pion Exchange Partial Wave Cross-Sections in $\pi^- n \rightarrow \pi^- \pi^+ \pi^- n$. *Phys. Rev.*, D16:50, 1977.
- [94] N. N. Khuri and S. B. Treiman. Pion-Pion Scattering and $K^+ \rightarrow \pi^- \pi^+ \pi^0$ Decay. *Phys. Rev.*, 119:1115–1121, 1960.
- [95] Franz Niecknig, Bastian Kubis, and Sebastian P. Schneider. Dispersive analysis of $\omega^- \rightarrow 3\pi$ and $\phi \rightarrow 3\pi$ decays. *Eur. Phys. J.*, C72:2014, 2012, 1203.2501.
- [96] Sebastian P. Schneider, Bastian Kubis, and Franz Niecknig. The $\omega^- \rightarrow \pi^0 \gamma^*$ and $\phi^- \rightarrow \pi^0 \gamma^*$ transition form factors in dispersion theory. *Phys. Rev.*, D86:054013, 2012, 1206.3098.
- [97] I. V. Danilkin, C. Fernández-Ramírez, P. Guo, V. Mathieu, D. Schott, M. Shi, and A. P. Szczepaniak. Dispersive analysis of $\omega/\phi \rightarrow 3\pi, \pi^- \gamma^*$. *Phys. Rev.*, D91(9):094029, 2015, 1409.7708.

- [98] Peng Guo, Igor V. Danilkin, Diane Schott, C. Fernández-Ramírez, V. Mathieu, and Adam P. Szczepaniak. Three-body final state interaction in $\eta \rightarrow 3\pi$. *Phys. Rev.*, D92(5):054016, 2015, 1505.01715.
- [99] Franz Niecknig and Bastian Kubis. Dispersion-theoretical analysis of the $D^+ \rightarrow K^- \pi^+ \pi^+$ Dalitz plot. *JHEP*, 10:142, 2015, 1509.03188.
- [100] P. Guo, I. V. Danilkin, C. Fernández-Ramírez, V. Mathieu, and A. P. Szczepaniak. Three-body final state interaction in $\eta \rightarrow 3\pi$ updated. *Phys. Lett.*, B771:497–502, 2017, 1608.01447.
- [101] M. Albaladejo and B. Moussallam. Extended chiral Khuri-Treiman formalism for $\eta \rightarrow 3\pi$ and the role of the $a_0(980)$, $f_0(980)$ resonances. *Eur. Phys. J.*, C77(8):508, 2017, 1702.04931.
- [102] Tobias Isken, Bastian Kubis, Sebastian P. Schneider, and Peter Stoffer. Dispersion relations for $\eta' \rightarrow \eta \pi \pi$. *Eur. Phys. J.*, C77(7):489, 2017, 1705.04339.
- [103] R. Akers et al. Measurement of the Hadronic Decay Current in tau- $\rightarrow \pi^- \pi^+ \pi^- \pi^+$ tau-neutrino. *Z. Phys. C*, 67:45–56, 1995.
- [104] P. Abreu et al. A Study of the hadronic resonance structure in the decay tau $\rightarrow \pi^- \pi^+ \pi^- \pi^+ \pi^- \pi^+$ neutrino(tau). *Phys. Lett. B*, 426:411–427, 1998.
- [105] D. Buskulic et al. Performance of the ALEPH detector at LEP. *Nucl. Instrum. Meth.*, A360:481–506, 1995.
- [106] Georg Janssen. *Ein mesontheoretisches Modell für die π ρ Wechselwirkung und der $\pi N N$ -Formfaktor*. PhD thesis, Institut für Kernphysik, Forschungszentrum, Jülich, 1993.

- [107] D. Sadasivan, M. Mai, and M. Döring. S- and p-wave structure of $S = -1$ meson-baryon scattering in the resonance region. *Phys. Lett. B*, 789:329–335, 2019, 1805.04534.
- [108] A. Cieplý, M. Mai, Ulf-G. Meißner, and J. Smejkal. On the pole content of coupled channels chiral approaches used for the $\bar{K}N$ system. *Nucl. Phys.*, A954:17–40, 2016, 1603.02531.
- [109] Kenta Miyahara, Tetsuo Hyodo, and Wolfram Weise. Construction of a local $\bar{K}N - \pi\Sigma - \pi\Lambda$ potential and composition of the $\Lambda(1405)$. *Phys. Rev. C*, 98(2):025201, 2018, 1804.08269.
- [110] Zhan-Wei Liu, Jonathan M. M. Hall, Derek B. Leinweber, Anthony W. Thomas, and Jia-Jun Wu. Structure of the $\Lambda(\mathbf{1405})$ from Hamiltonian effective field theory. *Phys. Rev.*, D95(1):014506, 2017, 1607.05856.
- [111] A. Ramos, A. Feijoo, and V. K. Magas. The chiral $S = -1$ meson-baryon interaction with new constraints on the NLO contributions. *Nucl. Phys.*, A954:58–74, 2016, 1605.03767.
- [112] Yuki Kamiya, Kenta Miyahara, Shota Ohnishi, Yoichi Ikeda, Tetsuo Hyodo, Eulogio Oset, and Wolfram Weise. Antikaon-nucleon interaction and $\Lambda(1405)$ in chiral SU(3) dynamics. *Nucl. Phys.*, A954:41–57, 2016, 1602.08852.
- [113] A. Feijoo, V. K. Magas, and A. Ramos. The $\bar{K}N \rightarrow K\Xi$ reaction in coupled channel chiral models up to next-to-leading order. *Phys. Rev.*, C92(1):015206, 2015, 1502.07956.
- [114] R. Molina and M. Döring. Pole structure of the $\Lambda(1405)$ in a recent QCD simulation. *Phys. Rev.*, D94(5):056010, 2016, 1512.05831. [Addendum: *Phys. Rev.* D94, no. 7, 079901 (2016)].

- [115] M. Döring, J. Haidenbauer, Ulf-G. Meißner, and A. Rusetsky. Dynamical coupled-channel approaches on a momentum lattice. *Eur. Phys. J.*, A47:163, 2011, 1108.0676.
- [116] A. Cieply and J. Smejkal. Chirally motivated $\bar{K}N$ amplitudes for in-medium applications. *Nucl. Phys.*, A881:115–126, 2012, 1112.0917.
- [117] Aleš Cieplý and Vojtěch Krejčířík. Effective model for in-medium $\bar{K}N$ interactions including the $L = 1$ partial wave. *Nucl. Phys.*, A940:311–330, 2015, 1501.06415.
- [118] D. Jido, E. Oset, and A. Ramos. Chiral dynamics of p wave in K- p and coupled states. *Phys. Rev.*, C66:055203, 2002, nucl-th/0208010.
- [119] M. Döring, D. Jido, and E. Oset. Helicity Amplitudes of the Lambda(1670) and two Lambda(1405) as dynamically generated resonances. *Eur. Phys. J.*, A45:319–333, 2010, 1002.3688.
- [120] Jose A. Oller. On the strangeness -1 S-wave meson-baryon scattering. *Eur. Phys. J.*, A28:63–82, 2006, hep-ph/0603134.
- [121] E. Oset, A. Ramos, and C. Bennhold. Low lying $S = -1$ excited baryons and chiral symmetry. *Phys. Lett.*, B527:99–105, 2002, nucl-th/0109006. [Erratum: *Phys. Lett.* B530,260(2002)].
- [122] C. Fernandez-Ramirez, I. V. Danilkin, D. M. Manley, V. Mathieu, and A. P. Szczepaniak. Coupled-channel model for $\bar{K}N$ scattering in the resonant region. *Phys. Rev.*, D93(3):034029, 2016, 1510.07065.
- [123] Cesar Fernandez-Ramirez, Igor V. Danilkin, Vincent Mathieu, and Adam P. Szczepaniak. Understanding the Nature of $\Lambda(1405)$ through Regge Physics. *Phys. Rev.*, D93(7):074015, 2016, 1512.03136.

- [124] H. Kamano, S. X. Nakamura, T. S. H. Lee, and T. Sato. Dynamical coupled-channels model of K^-p reactions. II. Extraction of Λ^* and Σ^* hyperon resonances. *Phys. Rev.*, C92(2):025205, 2015, 1506.01768. [Erratum: *Phys. Rev.*C95,no.4,049903(2017)].
- [125] H. Kamano, S. X. Nakamura, T. S. H. Lee, and T. Sato. Dynamical coupled-channels model of K^-p reactions: Determination of partial-wave amplitudes. *Phys. Rev.*, C90(6):065204, 2014, 1407.6839.
- [126] H. Zhang, J. Tulpan, M. Shrestha, and D. M. Manley. Multichannel parametrization of $\bar{K}N$ scattering amplitudes and extraction of resonance parameters. *Phys. Rev.*, C88(3):035205, 2013, 1305.4575.
- [127] H. Zhang, J. Tulpan, M. Shrestha, and D. M. Manley. Partial-wave analysis of $\bar{K}N$ scattering reactions. *Phys. Rev.*, C88(3):035204, 2013, 1305.3598.
- [128] Jonathan M. M. Hall, Waseem Kamleh, Derek B. Leinweber, Benjamin J. Menadue, Benjamin J. Owen, and Anthony W. Thomas. Light-quark contributions to the magnetic form factor of the Lambda(1405). *Phys. Rev.*, D95(5):054510, 2017, 1612.07477.
- [129] Jonathan M. M. Hall, Waseem Kamleh, Derek B. Leinweber, Benjamin J. Menadue, Benjamin J. Owen, Anthony W. Thomas, and Ross D. Young. Lattice QCD Evidence that the $\Lambda(1405)$ Resonance is an Antikaon-Nucleon Molecule. *Phys. Rev. Lett.*, 114(13):132002, 2015, 1411.3402.
- [130] Georg P. Engel, C. B. Lang, Daniel Mohler, and Andreas Schäfer. QCD with Two Light Dynamical Chirally Improved Quarks: Baryons. *Phys. Rev.*, D87(7):074504, 2013, 1301.4318.

- [131] Robert G. Edwards, Nilmani Mathur, David G. Richards, and Stephen J. Wallace. Flavor structure of the excited baryon spectra from lattice QCD. *Phys. Rev.*, D87(5):054506, 2013, 1212.5236.
- [132] Georg P. Engel, C. B. Lang, Markus Limmer, Daniel Mohler, and Andreas Schafer. Meson and baryon spectrum for QCD with two light dynamical quarks. *Phys. Rev.*, D82:034505, 2010, 1005.1748.
- [133] Ju-Jun Xie, Jia-Jun Wu, and Bing-Song Zou. Role of the possible $\Sigma^*(\frac{1}{2}^-)$ state in the $\Lambda p \rightarrow \Lambda p \pi^0$ reaction. *Phys. Rev.*, C90(5):055204, 2014, 1407.7984.
- [134] C. Helminen and D. O. Riska. Low lying q q q q anti-q states in the baryon spectrum. *Nucl. Phys.*, A699:624–648, 2002, nucl-th/0011071.
- [135] Robert L. Jaffe and Frank Wilczek. Diquarks and exotic spectroscopy. *Phys. Rev. Lett.*, 91:232003, 2003, hep-ph/0307341.
- [136] Feng-Kun Guo, Christoph Hanhart, Ulf-G. Meißner, Qian Wang, Qiang Zhao, and Bing-Song Zou. Hadronic molecules. *Rev. Mod. Phys.*, 90(1):015004, 2018, 1705.00141.
- [137] S. Ajimura et al. ” K^-pp ”, a $\bar{K}$ -Meson Nuclear Bound State, Observed in ${}^3\text{He}(K^-, \Lambda p)n$ Reactions. *Phys. Lett. B*, 789:620–625, 2019, 1805.12275.
- [138] Y. Sada et al. Structure near K^-+p+p threshold in the in-flight ${}^3\text{He}(K^-, \Lambda p)n$ reaction. *PTEP*, 2016(5):051D01, 2016, 1601.06876.
- [139] Takayasu Sekihara, Eulogio Oset, and Angels Ramos. On the structure observed in the in-flight ${}^3\text{He}(K^-, \Lambda p)n$ reaction at J-PARC. *PTEP*, 2016(12):123D03, 2016, 1607.02058.
- [140] D. B. Kaplan and A. E. Nelson. Strange Goings on in Dense Nucleonic Matter. *Phys. Lett.*, B175:57–63, 1986.

- [141] Subrata Pal, Debades Bandyopadhyay, and Walter Greiner. Anti- K^{*0} condensation in neutron stars. *Nucl. Phys.*, A674:553–577, 2000, astro-ph/0001039.
- [142] J. Abadie et al. A Gravitational wave observatory operating beyond the quantum shot-noise limit: Squeezed light in application. *Nature Phys.*, 7:962–965, 2011, 1109.2295.
- [143] A. Krause. Baryon Matrix Elements of the Vector Current in Chiral Perturbation Theory. *Helv. Phys. Acta*, 63:3–70, 1990.
- [144] Matthias Frink and Ulf-G. Meißner. Chiral extrapolations of baryon masses for unquenched three flavor lattice simulations. *JHEP*, 07:028, 2004, hep-lat/0404018.
- [145] G. Hohler. *NUMERICAL DATA AND FUNCTIONAL RELATIONSHIPS*. Berlin, Germany: Springer (1983) 601 P. (Landolt-boernstein. New Series, I/9B2), 1984.
- [146] Trevor Hastie, Trevor Hastie, Robert Tibshirani, and J. H. Friedman. *The elements of statistical learning: data mining, inference, and prediction*. Springer, 2001.
- [147] D. Rönchen, M. Döring, F. Huang, H. Haberzettl, J. Haidenbauer, C. Hanhart, S. Krewald, U.-G. Meißner, and K. Nakayama. Coupled-channel dynamics in the reactions $\pi N \rightarrow \pi N$, ηN , $K\Lambda$, $K\Sigma$. *Eur. Phys. J.*, A49:44, 2013, 1211.6998.
- [148] Malcolm N. Butler, Martin J. Savage, and Roxanne P. Springer. Strong and electromagnetic decays of the baryon decuplet. *Nucl. Phys.*, B399:69–88, 1993, hep-ph/9211247.

- [149] Daniel Sadasivan, Maxim Mai, Hakan Akdag, and Michael Döring. Dalitz plots and lineshape of $a_1(1260)$ from a relativistic three-body unitary approach, 2020, arXiv:2002.12431.
- [150] M. Ripani et al. Measurement of $e p \rightarrow e' p \pi^+ \pi^-$ and baryon resonance analysis. *Phys. Rev. Lett.*, 91:022002, 2003, hep-ex/0210054.
- [151] I. G. Aznauryan et al. Electroexcitation of the Roper resonance for $1.7 \leq Q^2 \leq 4.5$ -GeV² in $\text{vec-ep} \rightarrow e' n \pi^+ \pi^-$. *Phys. Rev.*, C78:045209, 2008, 0804.0447.
- [152] V. Sokhoyan et al. High-statistics study of the reaction $\gamma p \rightarrow p 2\pi^0$. *Eur. Phys. J.*, A51(8):95, 2015, 1507.02488. [Erratum: *Eur. Phys. J.*A51,no.12,187(2015)].
- [153] J. Ahrens et al. Helicity dependence of the $\gamma(\text{pol.}) p(\text{pol.}) \rightarrow n \pi^+ \pi^-$ reaction in the second resonance region. *Phys. Lett.*, B551:49–55, 2003.
- [154] T. Inoue, E. Oset, and M. J. Vicente Vacas. Chiral unitary approach to S wave meson baryon scattering in the strangeness $S = 0$ sector. *Phys. Rev.*, C65:035204, 2002, hep-ph/0110333.
- [155] J. Nieves and E. Ruiz Arriola. The $S(11) - N(1535)$ and $- N(1650)$ resonances in meson baryon unitarized coupled channel chiral perturbation theory. *Phys. Rev.*, D64:116008, 2001, hep-ph/0104307.
- [156] D. Mark Manley, Richard A. Arndt, Yogesh Goradia, and Vigdor L. Teplitz. An Isobar Model Partial Wave Analysis of $\pi N \rightarrow \pi\pi N$ in the Center-of-mass Energy Range 1320-MeV to 1930-MeV. *Phys. Rev.*, D30:904, 1984.
- [157] R. A. Arndt, W. J. Briscoe, I. I. Strakovsky, and R. L. Workman. Extended partial-wave analysis of πN scattering data. *Phys. Rev.*, C74:045205, 2006, nucl-th/0605082.

- [158] Viktor I. Mokeev, Volker D. Burkert, Tsung-Shung H. Lee, Latifa Elouadrhiri, Gleb V. Fedotov, and Boris S. Ishkhanov. Model Analysis of the $p \pi^+ \pi^-$ Electroproduction Reaction on the Proton. *Phys. Rev.*, C80:045212, 2009, 0809.4158.
- [159] A. V. Anisovich, R. Beck, E. Klempt, V. A. Nikonov, A. V. Sarantsev, and U. Thoma. Properties of baryon resonances from a multichannel partial wave analysis. *Eur. Phys. J.*, A48:15, 2012, 1112.4937.
- [160] H. Kamano, B. Julia-Diaz, T.-S. H. Lee, A. Matsuyama, and T. Sato. Dynamical coupled-channels study of $\pi N \rightarrow \pi \pi N$ reactions. *Phys. Rev.*, C79:025206, 2009, 0807.2273.
- [161] V. Shklyar, H. Lenske, and U. Mosel. 2π production in the Giessen coupled-channel model. *Phys. Rev.*, C93(4):045206, 2016, 1409.7920.
- [162] H. Al Ghouli et al. Measurement of the beam asymmetry Σ for π^0 and η photoproduction on the proton at $E_\gamma = 9$ GeV. *Phys. Rev.*, C95(4):042201, 2017, 1701.08123.
- [163] D. M. Asner et al. Physics at BES-III. *Int. J. Mod. Phys.*, A24:S1–794, 2009, 0809.1869.
- [164] C Adolph et al. Resonance Production and $\pi\pi$ S-wave in $\pi^- + p \rightarrow \pi^- \pi^- \pi^+ + p_{recoil}$ at 190 GeV/c. *Phys. Rev.*, D95(3):032004, 2017, 1509.00992.
- [165] M. Aghasyan et al. Light isovector resonances in $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ at 190 GeV/c. *Phys. Rev.*, D98(9):092003, 2018, 1802.05913.
- [166] B. S. Zou and D. V. Bugg. Covariant tensor formalism for partial wave analyses of ψ decay to mesons. *Eur. Phys. J.*, A16:537–547, 2003, hep-ph/0211457.

- [167] M. Ablikim et al. Resonances in $J/\psi \rightarrow \phi \pi^+ \pi^-$ and $\phi K^+ K^-$. *Phys. Lett.*, B607:243–253, 2005, hep-ex/0411001.
- [168] M. Albrecht et al. Coupled Channel Analysis of $\bar{p}p \rightarrow \pi^0 \pi^0 \eta$, $\pi^0 \eta \eta$ and $K^+ K^- \pi^0$ at 900 MeV/c and of $\pi\pi$ -Scattering Data. 2019, 1909.07091.
- [169] Bastian Kubis and Sebastian P. Schneider. The Cusp effect in η' $\rightarrow \pi^+ \pi^-$ decays. *Eur. Phys. J.*, C62:511–523, 2009, 0904.1320.
- [170] Sebastian P. Schneider, Bastian Kubis, and Christoph Ditsche. Rescattering effects in $\eta \rightarrow 3\pi$ decays. *JHEP*, 02:028, 2011, 1010.3946.
- [171] Peng Guo, Ryan Mitchell, Matthew Shepherd, and Adam P. Szczepaniak. Amplitudes for the analysis of the decay $J/\psi \rightarrow K^+ K^- \pi^0$. *Phys. Rev.*, D85:056003, 2012, 1112.3284.
- [172] Peng Guo, I. V. Danilkin, and Adam P. Szczepaniak. Dispersive approaches for three-particle final state interaction. *Eur. Phys. J.*, A51(10):135, 2015, 1409.8652.
- [173] Martin Hoferichter, Bai-Long Hoid, Bastian Kubis, Stefan Leupold, and Sebastian P. Schneider. Dispersion relation for hadronic light-by-light scattering: pion pole. *JHEP*, 10:141, 2018, 1808.04823.
- [174] A. Jackura, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, J. Nys, A. Pilloni, K. Saldaña, N. Sherrill, and A. P. Szczepaniak. Phenomenology of Relativistic $\mathbf{3} \rightarrow \mathbf{3}$ Reaction Amplitudes within the Isobar Approximation. *Eur. Phys. J.*, C79(1):56, 2019, 1809.10523.
- [175] Juerg Gasser and Akaki Rusetsky. Solving integral equations in $\eta \rightarrow 3\pi$. *Eur. Phys. J.*, C78(11):906, 2018, 1809.06399.

- [176] V. Mathieu, M. Albaladejo, C. Fernández-Ramírez, A. W. Jackura, M. Mikhasenko, A. Pilloni, and A. P. Szczepaniak. Moments of angular distribution and beam asymmetries in $\eta\pi^0$ photoproduction at GlueX. *Phys. Rev.*, D100(5):054017, 2019, 1906.04841.
- [177] M. Albaladejo, D. Winney, I.V. Danilkin, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, J.A. Silva-Castro, and A.P. Szczepaniak. Khuri-Treiman equations for 3π decays of particles with spin. *Phys. Rev. D*, 101(5):054018, 2020, 1910.03107.
- [178] M. Mikhasenko et al. Dalitz-plot decomposition for three-body decays. *Phys. Rev. D*, 101(3):034033, 2020, 1910.04566.
- [179] M. Mikhasenko, Y. Wunderlich, A. Jackura, V. Mathieu, A. Pilloni, B. Ketzer, and A. P. Szczepaniak. Three-body scattering: Ladders and Resonances. *JHEP*, 08:080, 2019, 1904.11894.
- [180] Silas R. Beane, William Detmold, Thomas C. Luu, Kostas Orginos, Martin J. Savage, and Aaron Torok. Multi-Pion Systems in Lattice QCD and the Three-Pion Interaction. *Phys. Rev. Lett.*, 100:082004, 2008, 0710.1827.
- [181] C. B. Lang, L. Leskovec, M. Padmanath, and S. Prelovsek. Pion-nucleon scattering in the Roper channel from lattice QCD. *Phys. Rev.*, D95(1):014510, 2017, 1610.01422.
- [182] C. B. Lang, Luka Leskovec, Daniel Mohler, and Sasa Prelovsek. Axial resonances $a_1(1260)$, $b_1(1235)$ and their decays from the lattice. *JHEP*, 04:162, 2014, 1401.2088.
- [183] Ben Hörz and Andrew Hanlon. Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD. *Phys. Rev. Lett.*, 123(14):142002, 2019, 1905.04277.

- [184] Antoni J. Woss, Christopher E. Thomas, Jozef J. Dudek, Robert G. Edwards, and David J. Wilson. b_1 resonance in coupled $\pi\omega$, $\pi\phi$ scattering from lattice QCD. *Phys. Rev.*, D100(5):054506, 2019, 1904.04136.
- [185] Chris Culver, Maxim Mai, Ruairí Brett, Andrei Alexandru, and Michael Döring. Three body spectrum from lattice QCD. 2019, 1911.09047.
- [186] Stephen R. Sharpe. Testing the threshold expansion for three-particle energies at fourth order in ϕ^4 theory. *Phys. Rev. D*, 96(5):054515, 2017, 1707.04279. [Erratum: Phys.Rev.D 98, 099901 (2018)].
- [187] H. W. Hammer, J. Y. Pang, and A. Rusetsky. Three particle quantization condition in a finite volume: 2. general formalism and the analysis of data. *JHEP*, 10:115, 2017, 1707.02176.
- [188] Hans-Werner Hammer, Jin-Yi Pang, and A. Rusetsky. Three-particle quantization condition in a finite volume: 1. The role of the three-particle force. *JHEP*, 09:109, 2017, 1706.07700.
- [189] Peng Guo and Vladimir Gasparian. An solvable three-body model in finite volume. 2017, 1701.00438.
- [190] Peng Guo. One spatial dimensional finite volume three-body interaction for a short-range potential. *Phys. Rev.*, D95(5):054508, 2017, 1607.03184.
- [191] Maxwell T. Hansen and Stephen R. Sharpe. Applying the relativistic quantization condition to a three-particle bound state in a periodic box. *Phys. Rev.*, D95(3):034501, 2017, 1609.04317.
- [192] Maxwell T. Hansen and Stephen R. Sharpe. Threshold expansion of the three-particle quantization condition. *Phys. Rev.*, D93(9):096006, 2016, 1602.00324. [Erratum: Phys. Rev.D96,no.3,039901(2017)].

- [193] Maxwell T. Hansen and Stephen R. Sharpe. Perturbative results for two and three particle threshold energies in finite volume. *Phys. Rev.*, D93:014506, 2016, 1509.07929.
- [194] Maxwell T. Hansen and Stephen R. Sharpe. Expressing the three-particle finite-volume spectrum in terms of the three-to-three scattering amplitude. *Phys. Rev.*, D92(11):114509, 2015, 1504.04248.
- [195] Ulf-G. Meißner, Guillermo Ríos, and Akaki Rusetsky. Spectrum of three-body bound states in a finite volume. *Phys. Rev. Lett.*, 114(9):091602, 2015, 1412.4969. [Erratum: *Phys. Rev. Lett.* 117, no. 6, 069902 (2016)].
- [196] Maxwell T. Hansen and Stephen R. Sharpe. Relativistic, model-independent, three-particle quantization condition. *Phys. Rev.*, D90(11):116003, 2014, 1408.5933.
- [197] Raúl A. Briceño and Zohreh Davoudi. Three-particle scattering amplitudes from a finite volume formalism. *Phys. Rev.*, D87(9):094507, 2013, 1212.3398.
- [198] Simon Kreuzer and Harald W. Grißhammer. Three particles in a finite volume: The breakdown of spherical symmetry. *Eur. Phys. J.*, A48:93, 2012, 1205.0277.
- [199] L. Roca and E. Oset. Scattering of unstable particles in a finite volume: the case of $\pi\rho$ scattering and the $a_1(1260)$ resonance. *Phys. Rev.*, D85:054507, 2012, 1201.0438.
- [200] K. Polejaeva and A. Rusetsky. Three particles in a finite volume. *Eur. Phys. J.*, A48:67, 2012, 1203.1241.
- [201] Simon Kreuzer and H. W. Hammer. Efimov physics in a finite volume. *Phys. Lett.*, B673:260–263, 2009, 0811.0159.

- [202] Maxwell T. Hansen and Stephen R. Sharpe. Lattice QCD and Three-particle Decays of Resonances. *Ann. Rev. Nucl. Part. Sci.*, 69:65–107, 2019, 1901.00483.
- [203] Tyler D. Blanton, Fernando Romero-López, and Stephen R. Sharpe. $I = 3$ three-pion scattering amplitude from lattice QCD. *Phys. Rev. Lett.*, 124(3):032001, 2020, 1909.02973.
- [204] Maxim Mai and Michael Döring. Finite-Volume Spectrum of $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$ Systems. *Phys. Rev. Lett.*, 122(6):062503, 2019, 1807.04746.
- [205] M. F. M. Lutz and E. E. Kolomeitsev. On meson resonances and chiral symmetry. *Nucl. Phys.*, A730:392–416, 2004, nucl-th/0307039.
- [206] L. Roca, E. Oset, and J. Singh. Low lying axial-vector mesons as dynamically generated resonances. *Phys. Rev.*, D72:014002, 2005, hep-ph/0503273.
- [207] Markus Wagner and Stefan Leupold. Tau decay and the structure of the $a(1)$. *Phys. Lett.*, B670:22–26, 2008, 0708.2223.
- [208] Markus Wagner and Stefan Leupold. Information on the structure of the $a(1)$ from tau decay. *Phys. Rev.*, D78:053001, 2008, 0801.0814.
- [209] M. F. M. Lutz and S. Leupold. On the radiative decays of light vector and axial-vector mesons. *Nucl. Phys.*, A813:96–170, 2008, 0801.3821.
- [210] H. Nagahiro, K. Nawa, S. Ozaki, D. Jido, and A. Hosaka. Composite and elementary natures of $a_1(1260)$ meson. *Phys. Rev.*, D83:111504, 2011, 1101.3623.
- [211] Xu Zhang and Ju-Jun Xie. The three-pion decays of the $a_1(1260)$. *Commun. Theor. Phys.*, 70(1):060, 2018, 1712.05572.
- [212] L. R. Dai, L. Roca, and E. Oset. τ decay into a pseudoscalar and an axial-vector meson. *Phys. Rev.*, D99(9):096003, 2019, 1811.06875.

- [213] Maxim Mai, Bin Hu, Michael Döring, Alessandro Pilloni, and Adam Szczepaniak. Three-body scattering in isobar ansatz. *PoS*, page 140, 2018.
- [214] Richard John Eden, Peter V. Landshoff, David I. Olive, and John Charlton Polkinghorne. *The analytic S-matrix*. Cambridge Univ. Press, Cambridge, 1966.
- [215] A. Matsuyama, T. Sato, and T. S. H. Lee. Dynamical coupled-channel model of meson production reactions in the nucleon resonance region. *Phys. Rept.*, 439:193–253, 2007, nucl-th/0608051.
- [216] M. Döring, C. Hanhart, F. Huang, S. Krewald, and U.-G. Meißner. The Role of the background in the extraction of resonance contributions from meson-baryon scattering. *Phys. Lett.*, B681:26–31, 2009, 0903.1781.
- [217] A. W. Jackura, S. M. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, S. R. Sharpe, and A. P. Szczepaniak. Equivalence of three-particle scattering formalisms. *Phys. Rev.*, D100(3):034508, 2019, 1905.12007.
- [218] G. Janssen, K. Holinde, and J. Speth. A Meson exchange model for pi rho scattering. *Phys. Rev.*, C49:2763–2776, 1994.
- [219] S. D. Protopopescu, M. Alston-Garnjost, A. Barbaro-Galtieri, Stanley M. Flatte, J. H. Friedman, T. A. Lasinski, G. R. Lynch, M. S. Rabin, and F. T. Solmitz. $\pi\pi$ Partial Wave Analysis from Reactions $\pi^+p \rightarrow \pi^+\pi^-\Delta^{++}$ and $\pi^+p \rightarrow K^+K^-\Delta^{++}$ at 7.1-GeV/c. *Phys. Rev.*, D7:1279, 1973.
- [220] P. Estabrooks and Alan D. Martin. pi pi Phase Shift Analysis Below the K anti-K Threshold. *Nucl. Phys.*, B79:301–316, 1974.
- [221] Suh Urk Chung. SPIN FORMALISMS. <http://cds.cern.ch/record/186421>, 1971.
- [222] D. Rönchen, M. Döring, F. Huang, H. Haberzettl, J. Haidenbauer, C. Hanhart, S. Krewald, U.-G. Meißner, and K. Nakayama. Photocouplings at the Pole from

- Pion Photoproduction. *Eur. Phys. J.*, A50(6):101, 2014, 1401.0634. [Erratum: *Eur. Phys. J.*A51,no.5,63(2015)].
- [223] Kenneth M. Watson. The effect of final state interactions on reaction cross sections. *Phys. Rev.*, 88:1163–1171, Dec 1952.
- [224] M. Jacob and G. C. Wick. On the General Theory of Collisions for Particles with Spin. *Annals Phys.*, 7:404–428, 1959. [Annals Phys.281,774(2000)].
- [225] J. H. Hetherington and L. H. Schick. Exact Multiple-Scattering Analysis of Low-Energy Elastic K–d Scattering with Separable Potentials. *Phys. Rev.*, 137:B935–B948, 1965.
- [226] M. Döring, E. Oset, and U. G. Meißner. Evaluation of the polarization observables I^S and I^C in the reaction $\gamma p \rightarrow \pi^0 \eta p$. *Eur. Phys. J.*, A46:315–323, 2010, 1003.0097.
- [227] P. Salvini et al. anti-p p annihilation into four charged pions at rest and in flight. *Eur. Phys. J.*, C35:21–33, 2004.
- [228] C. Culver, M. Mai, A. Alexandru, M. Döring, and F.X. Lee. Pion scattering in the isospin $I = 2$ channel from elongated lattices. *Phys. Rev. D*, 100(3):034509, 2019, 1905.10202.