General Tensor Structure for Inclusive and Semi-inclusive Electron Scattering from Polarized Spin-1/2 Targets

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Abstract

The general structure of semi-inclusive polarized electron scattering from polarized spin-1/2 targets is developed for use at all energy scales, from modest-energy nuclear physics applications to use in very high energy particle physics. The leptonic and hadronic tensors that enter in the formalism are constructed in a general covariant way in terms of kinematic factors that are frame dependent but model independent and invariant response functions which contain all of the model-dependent dynamics. In the process of developing the general problem the relationships to the conventional responses expressed in terms of the helicity components of the exchanged virtual photon are presented. For semi-inclusive electron scattering with polarized electrons and polarized spin-1/2 targets one finds that 18 invariant response functions are required, each depending on four Lorentz scalar invariants. Additionally it is shown how the semi-inclusive cross

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sections are related via integrations over the momentum of the selected coincidence particle and sums over open channels.

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1. Introduction

In this study we place our main focus on semi-inclusive polarized electron scattering from polarized spin-1/2 targets, shown schematically in Fig. 1. That is, we consider reactions of the type $\overrightarrow{e} + \overrightarrow{A}(1/2) \rightarrow e' + x + B$ where the incident electron may be polarized, the spin-1/2 target A may be polarized and where we assume that, in addition to the scattered electron, some (unpolarized) particle x is detected in coincidence. The sum of all open channels that make up the final state is denoted B and is assumed not to be detected. Employing notation commonly used in nuclear physics the reaction may be written $\overrightarrow{A}(1/2)(\overrightarrow{e}, e'x)B$. We shall discuss how such semi-inclusive reactions are related to the inclusive cross section, *i.e.,* for reactions of the type $\overrightarrow{e} + \overrightarrow{A}(1/2) \rightarrow e' + X$ or $\overrightarrow{A}(1/2)(\overrightarrow{e}, e')X$, where X denotes the complete (undetected) final state. We develop the formalism in a general frame as we wish to be able to relate the response in different frames of reference, in particular, in the target rest frame and in a frame where the incident electrons and the spin-1/2 target are both moving and colliding. Importantly, we show how the cross sections may be written in a general way in terms of invariant response functions (see below for an introductory discussion of what motivates this strategy).

Before entering into the polarized semi-inclusive developments, here we discuss in general terms a simple, well-known example to help in understanding the basic motivation for the present study, namely, we consider the case of unpolarized, inclusive electron scattering from unpolarized targets. The conventions employed in this work are summarized in Appendix A. The electron tensor $\eta_{\mu\nu}$ takes on its standard form; this will be introduced in Sec. 2 and here we take it as given. It is symmetric under interchange of μ and ν, *viz.,* $\eta_{\mu\nu} = \eta_{\nu\mu}$. Accordingly, in forming the contraction of the leptonic and hadronic tensors,

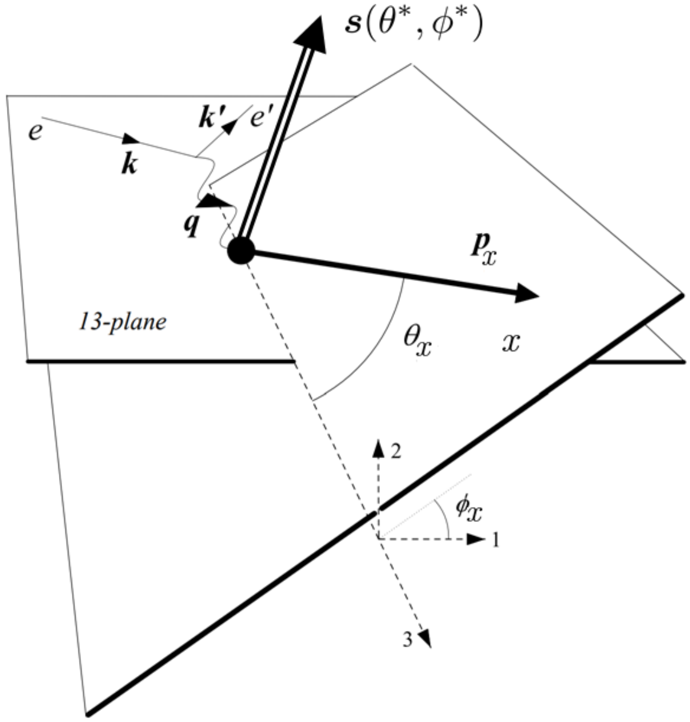

Figure 1: Schematic representation of semi-inclusive electron scattering. The coordinate system is chosen such that the electron scattering occurs in the 13-plane and has the 3-momentum transfer along the 3-axis. The particle x detected in coincidence with the scattered electron has 3-momentum $\mathbf{p}_x$ which lies in a plane in general inclined at an azimuthal angle ϕ_x with respect to the electron scattering plane and has polar angle θ_x with respect to $\mathbf{q}$. The polarization of the spin-1/2 target involves the spin 3-vector $\mathbf{s}$ with polar and azimuthal angles θ^* and ϕ^*, respectively, in the chosen coordinate system.

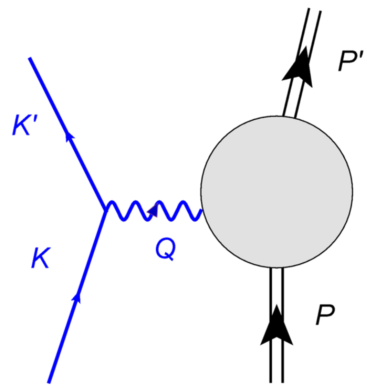

Figure 2: Feynman diagram for inclusive electron scattering. The 4-momenta here are discussed in the text.

$\eta_{\mu\nu}W^{\mu\nu}$ to obtain the invariant quantity that yields the cross section for this situation we require only the symmetric part of the hadronic tensor $W^{\mu\nu}$. The hadronic piece of the problem is indicated in Fig. 2: here the virtual photon having 4-momentum Q^μ interacts with the target having 4-momentum P^μ, leading to a final state with 4-momentum P'^μ. Since we are assuming that the process involves inclusive scattering, nothing in the hadronic final state is detected. Momentum conservation allows us to eliminate the total final-state momentum, $P'^\mu = Q^\mu + P^\mu$, and hence we have two independent 4-momenta with which the hadronic tensor is to be constructed, namely, Q^μ and P^μ. The Lorentz scalars that can be built from these two are Q^2, $Q \cdot P$ and P^2; since $P^2 = M^2$, with M the mass of the target, is presumed to be known we have only the two remaining dynamical Lorentz scalars upon which the hadronic tensor can depend. Typically one uses other (perhaps not invariant) quantities such as (Q^2, x) or (q, ω) for the dynamical variables — these variables will be introduced in due course.

The final step in building the hadronic tensor is then to determine the most general form it can take, given the type of reaction being assumed. Certainly one uses the two dynamical 4-vectors to do this. It proves convenient to use Q^μ

with, instead of P^μ, a linear combination of P^μ with Q^μ

$$U^\mu \equiv \frac{1}{M}\left(P^\mu - \left(\frac{Q\cdot P}{Q^2}\right)Q^\mu\right), \tag{1}$$

since it has the property that $Q \cdot U = 0$, by construction. Any choice of two independent 4-vectors will work, although experience shows that such projected quantities help in simplifying the arguments. The symmetric hadronic tensor for unpolarized, inclusive scattering may then be written in terms of $Q^\mu Q^\nu$, $U^\mu U^\nu$ and $Q^\mu U^\nu + Q^\nu U^\mu$ together with $g^{\mu\nu}$:

$$W^{\mu\nu} = X_1 g^{\mu\nu} + X_2 Q^\mu Q^\nu + X_3 U^\mu U^\nu + X_4\left(Q^\mu U^\nu + Q^\nu U^\mu\right) \tag{2}$$

which contains four contributions involving these basis tensors each multiplied by an invariant response function that depends on the two dynamical Lorentz scalars, *i.e.*, $X_i = X_i\left(Q^2, Q\cdot P\right)$ where $i = 1,\ldots,4$. Since the electromagnetic current is conserved the constraint

$$Q_\mu W^{\mu\nu} = 0 \tag{3}$$

must be satisfied, which leads to

$$\left(X_1 + X_2 Q^2\right)Q^\nu + \left(X_4 Q^2\right)U^\nu = 0. \tag{4}$$

The theorem of linear algebra and the fact that Q^ν and U^ν are linearly independent 4-vectors then immediately yields the following:

$$X_1 + X_2 Q^2 = X_4 = 0. \tag{5}$$

Changing notation to the more conventional one, $X_1 \equiv -W_1$ and $X_3 \equiv W_2$, one has then proven the well-known result

$$W^{\mu\nu} = -W_1\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + W_2 U^\mu U^\nu, \tag{6}$$

namely, the hadronic response for this particular situation has two terms involving two invariant response functions, each a function of two Lorentz invariants. Upon contracting with the leptonic tensor one then recovers the standard result

$$\sigma \sim W_2 + 2W_1 \tan^2\theta_e/2 \tag{7}$$

where θ_e is the electron scattering angle. Note that $W_{1,2}$ are invariant response functions, but that the factor $\tan^2 \theta_e/2$ depends on the particular frame of reference. Moreover, this result is often recast in a form where the helicity projections of the virtual photon are made manifest (see later examples of why this can be important). For this situation one finds that the cross section may be written

$$\sigma \sim v_L W^L + v_T W^T \tag{8}$$

where the quantities v_L and v_T are the well-known leptonic (Rosenbluth) factors and the W^L and W^T the corresponding hadronic responses. All of these quantities, however, are not Lorentz invariants and accordingly are different in different frames. A strong motivation in the present study is to establish the relationships between the two ways of representing the hadronic response and thereby to provide a way to relate the hadronic physics between any two frames.

This simplest example is a textbook case (see, for example, [1]). While constituting relatively straightforward extensions to what has been summarized here, for the general problem that involves polarized electrons, polarized spin-1/2 targets and semi-inclusive reactions the developments are much more complicated.

To summarize, some of the basic motivations for this study are the following:

- As in the simple example discussed above, in this study we will develop the general formalism for semi-inclusive electron scattering of polarized electrons on polarized spin-1/2 targets. We anticipate applications to both particle and nuclear physics and to both relatively low energies and to the high-energy regime (HER).

- We shall see that there are four sectors which may be separated by employing the polarizations. When unpolarized electrons are involved only symmetric tensors enter, whereas when the incident electrons are longitudinally polarized only anti-symmetric tensors occur.

- The four types of polarization (electrons polarized or not with target polarized or not) may be separated using those polarizations. We shall see that there are four symmetric invariant responses for the fully unpolarized case, one anti-symmetric invariant response when the electron is polarized but the target is not, eight symmetric invariant responses when the electron is unpolarized but the target is polarized, and five anti-symmetric invariant responses when both electrons and target are polarized.

- These 18 invariant response functions will be shown to be functions of four Lorentz scalar invariants. The 18 responses may be sub-divided into two sets of nine according to their properties under parity and time-reversal; these two sets typically behave quite differently.

- We also detail how the hadronic response may be characterized using the helicity decomposition of the virtual photon to label the various contributions. We shall detail how this representation relates to the decomposition in terms of invariant response functions.

- A prime motivation for this study is to have the semi-inclusive cross section written in a completely general frame of reference. This then allows one to relate the results in (say) the collider frame to the target rest frame, or to relate the results in the rest frame to those in the photon-target center-of-momentum frame. This can prove to be essential when models are being developed for the hadronic physics that are non-relativistic and hence cannot be boosted — polarized ^{3}He would be one such example — since only in the target rest frame will such models make sense.

- Finally, we provide some discussion of how inclusive (polarized) scattering emerges via specific integrals over semi-inclusive cross sections with appropriate sums over all open channels.

With the above basic motivations for the present study we briefly discuss two examples where the ideas are relevant, one from particle physics and one from nuclear physics. For the former consider charged pion production from a polarized proton target. For single-pion production one then has the (exclusive) reaction $\overrightarrow{e} + \overrightarrow{p} \rightarrow e' + n + \pi^+$ with a neutron and a positive pion in the final state. As a semi-inclusive reaction one then has either $\overrightarrow{p}(\overrightarrow{e}, e'n)\pi^+$ where particle x is a neutron and the pion is undetected or $\overrightarrow{p}(\overrightarrow{e}, e'\pi^+)n$ where particle x is a π^+ and the neutron is undetected. In fact these are the same reaction and accordingly they constitute a single channel. Clearly there are experimental considerations involved in which particle is the one detected in coincidence; however, theoretically they are not distinguishable. For higher-energy kinematics one reaches a threshold where additional channels open. For instance, once the relevant threshold is reached, two-pion production becomes possible, $\overrightarrow{e} + \overrightarrow{p} \rightarrow e' + n + \pi^+ + \pi^0$ and then $\overrightarrow{e} + \overrightarrow{p} \rightarrow e' + p + \pi^- + \pi^+$, and so one, with more and more particles in the final state. Of those a given semi-inclusive reaction is to be taken as having some given particle detected in coincidence with the scattered electron and all other particles undetected.

A second example, taken from nuclear physics, is where the polarized electron is scattered from a polarized ^{3}He target. Let us focus on the reaction $^3\overrightarrow{\mathrm{He}}(\overrightarrow{e}, e'p)$ where a proton is assumed to be detected in coincidence with the scattered electron. The unobserved part of the final state depends on the specific kinematics of the reaction. At threshold one has the (exclusive) two-body reaction $\overrightarrow{e} +^3 \overrightarrow{\mathrm{He}} \rightarrow e' + p + d$ and then for slightly higher missing energies the three-body breakup reaction $\overrightarrow{e} +^3 \overrightarrow{\mathrm{He}} \rightarrow e' + p + p + n$. Alternatively one could have a neutron as the particle detected in coincidence with the scattered electron, $^3\overrightarrow{\mathrm{He}}(\overrightarrow{e}, e'n)$. In this case the two-body channel does not occur, although the three-body breakup channel does. In fact, for the latter the final state is the same and this will have consequences later when we discuss the issue of avoiding double counting. As in the particle physics example above, as the energy increases a threshold is reached where pion production can occur and the final state becomes even more complicated. Nevertheless, the semi-inclusive reaction is well defined: the point is that a specific particle is assumed to be the one called x, namely, the one that is detected, whereas all other particles in the final state must be summed while avoiding double counting.

The paper is organized in the following way: in Sec. 2 some general developments are summarized which involve the contraction of the leptonic and hadronic tensors and include the specific forms for the electron scattering tensors in the Extreme Relativistic Limit (ERL$_e$). This is followed in Sec. 3 with the detailed construction of the general hadronic tensors for the semi-inclusive reaction. In Sec. 3.1 the basic 4-vectors used in building the hadronic tensors are introduced, followed in Sec. 3.2 with the 18 types of tensors that constitute the problem, and in Sec. 3.3 with specific components of responses categorized by the projections of the exchanged virtual photon's helicity. In Sec. 4 the semi-inclusive cross section is given for a general situation where the polarized spin-1/2 target is moving in some arbitrary direction — this for use in collider physics. For completeness the simpler situation of polarized inclusive electron scattering from a (moving) polarized spin-1/2 target is presented in Sec. 5. These general developments are then specialized to the target rest frame in Sec. 6. To conclude the body of the paper a summary is given in Sec. 7 and to extend some aspects of the problem six appendices are included detailing the conventions used (A), expressing the contraction of the tensors entirely in terms of invariants (B), inverting the invariant response representations in terms of photon helicity projections (C), detailing the nature of the cross section as the available phase-space increases and more channels become open (D), including some connections with conventional kinematic variables (E) and discussing inclusive scattering in more detail to make connections with (more) familiar material (F).

2. General Developments

We begin with some general developments that are common to all electron scattering formalism at the level of the plane-wave Born approximation. The general cross section is proportional to the contraction of the leptonic and hadronic tensors $\eta_{\mu\nu}$ and $W^{\mu\nu}$, respectively

$$\eta_{\mu\nu} W^{\mu\nu}. \tag{9}$$

Being composed of bilinear products of the corresponding leptonic and hadronic current matrix elements $(j_{fi})_\mu$ and $(J_{fi})^\mu$, respectively, in the forms

$$\eta_{\mu\nu} \sim \overline{\sum_{if}} (j_{fi})^*_\mu (j_{fi})_\nu \tag{10}$$

$$W^{\mu\nu} \sim \overline{\sum_{if}} (J_{fi})^{\mu *} (J_{fi})^\nu , \tag{11}$$

with appropriate averages over initial and sums over final states, one has immediately that

$$\eta_{\nu\mu} = (\eta_{\mu\nu})^* \tag{12}$$

$$W^{\nu\mu} = (W^{\mu\nu})^* . \tag{13}$$

Instead of $\eta_{\mu\nu}$ we employ the following convention for the leptonic tensor (see [2])

$$\chi^{\mu\nu} \equiv 4m_e^2 \eta^{\mu\nu} \tag{14}$$

$$= \chi^{\mu\nu}_{unpol} + \chi^{\mu\nu}_{pol} . \tag{15}$$

Also, since the electromagnetic current is conserved,

$$Q^\mu (j_{fi})_\mu = Q_\mu (J_{fi})^\mu = 0, \tag{16}$$

one has that

$$Q^\mu \chi_{\mu\nu} = \chi_{\mu\nu} Q^\nu = Q_\mu W^{\mu\nu} = W^{\mu\nu} Q_\nu = 0. \tag{17}$$

Since one can decompose the tensors into symmetric and anti-symmetric contributions (*i.e.,* under exchange of μ and ν), namely,

$$\chi^s_{\mu\nu} \equiv \frac{1}{2} (\chi_{\mu\nu} + \chi_{\nu\mu}) \tag{18}$$

$$\chi^a_{\mu\nu} \equiv \frac{1}{2} (\chi_{\mu\nu} - \chi_{\nu\mu}) \tag{19}$$

$$W^{\mu\nu}_s \equiv \frac{1}{2} (W^{\mu\nu} + W^{\nu\mu}) \tag{20}$$

$$W^{\mu\nu}_a \equiv \frac{1}{2} (W^{\mu\nu} - W^{\nu\mu}) \tag{21}$$

with

$$\chi^{\mu\nu} = \chi^{\mu\nu}_s + \chi^{\mu\nu}_a \tag{22}$$

$$W^{\mu\nu} = W^{\mu\nu}_s + W^{\mu\nu}_a . \tag{23}$$

Clearly one has the individual continuity equation relationships

$$Q_\mu W_s^{\mu\nu} = Q_\mu W_a^{\mu\nu} = 0, \tag{24}$$

and also only symmetric (anti-symmetric) leptonic tensors will contract with symmetric (anti-symmetric) hadronic tensors when forming the cross section, the last going as

$$\chi_{\mu\nu} W^{\mu\nu} = \chi_{\mu\nu}^s W_s^{\mu\nu} + \chi_{\mu\nu}^a W_a^{\mu\nu}. \tag{25}$$

We also have from Eqs. (12) and (13) that

$$\chi_s^{\mu\nu} = \mathrm{Re}\chi^{\mu\nu} \tag{26}$$

$$\chi_a^{\mu\nu} = i\mathrm{Im}\chi^{\mu\nu} \tag{27}$$

$$W_s^{\mu\nu} = \mathrm{Re}W^{\mu\nu} \tag{28}$$

$$W_a^{\mu\nu} = i\mathrm{Im}W^{\mu\nu}; \tag{29}$$

we shall make use of this when constructing explicit forms for the tensors by including the factor i in the appropriate places.

Furthermore, one can isolate contributions that contain the target spin from those that do not by forming the unpolarized (spin sum) terms and polarized (spin difference) terms, so that the total becomes

$$\chi_{s,a}^{\mu\nu} = \left(\chi_{s,a}^{\mu\nu}\right)_{unpol} + \left(\chi_{s,a}^{\mu\nu}\right)_{pol} \tag{30}$$

$$W_{s,a}^{\mu\nu} = \left(W_{s,a}^{\mu\nu}\right)_{unpol} + \left(W_{s,a}^{\mu\nu}\right)_{pol} \tag{31}$$

with all four contributions individually satisfying the continuity equation constraint:

$$Q_\mu \left(\chi_{s,a}^{\mu\nu}\right)_{unpol} = Q_\mu \left(\chi_{s,a}^{\mu\nu}\right)_{pol} = 0 \tag{32}$$

$$Q_\mu \left(W_{s,a}^{\mu\nu}\right)_{unpol} = Q_\mu \left(W_{s,a}^{\mu\nu}\right)_{pol} = 0. \tag{33}$$

When only the incident electrons may be polarized but the scattered electron's polarization is assumed not to be measured one can show that the leptonic tensor contributions that do not involve the electron polarization are only symmetric, while those that do involve the electron polarization are only anti-symmetric (see [2]).

We shall adopt the convention where $\mathbf{q}$ points along the 3-direction so that the 4-vector momentum transfer is

$$Q^\mu = (\omega, 0, 0, q) \tag{34}$$

with energy transfer $\omega = \nu$ (the former is commonly employed in nuclear physics while the latter is almost always chosen for use in particle physics; we use the two interchangeably) and 3-momentum transfer $q = |\mathbf{q}|$. One can show that for electron scattering the 4-momentum transfer must be spacelike:

$$Q^2 = \omega^2 - q^2 \leq 0; \tag{35}$$

(see the comment in Appendix A). We shall define the following dimensionless quantities that prove to be useful later

$$\nu' \equiv \frac{\omega}{q} = \frac{\nu}{q} \tag{36}$$

$$\rho \equiv \frac{-Q^2}{q^2} = 1 - \nu'^2 \tag{37}$$

$$\rho' \equiv \frac{q}{\epsilon + \epsilon'} \tag{38}$$

and have from Eq. (35) that

$$0 \leq \nu' \leq 1 \tag{39}$$

$$0 \leq \rho \leq 1. \tag{40}$$

$$0 \leq \rho' \leq 1. \tag{41}$$

The continuity equation constraints above then imply that

$$\left(\chi^{3\nu}_{s,a}\right)_{unpol} = \nu' \left(\chi^{0\nu}_{s,a}\right)_{unpol} \tag{42}$$

$$\left(\chi^{3\nu}_{s,a}\right)_{pol} = \nu' \left(\chi^{0\nu}_{s,a}\right)_{pol} \tag{43}$$

$$\left(W^{3\nu}_{s,a}\right)_{unpol} = \nu' \left(W^{0\nu}_{s,a}\right)_{unpol} \tag{44}$$

$$\left(W^{3\nu}_{s,a}\right)_{pol} = \nu' \left(W^{0\nu}_{s,a}\right)_{pol}. \tag{45}$$

2.1. Contraction of Tensors

We now proceed to contract the leptonic and hadronic tensors involving the separated symmetric (10) and anti-symmetric (6) contractions

$$\begin{aligned}
(\chi_s)_{\mu\nu}\, W_s^{\mu\nu} &= (\chi_s)_{00}\, W_s^{00} + 2\,(\chi_s)_{03}\, W_s^{03} + (\chi_s)_{33}\, W_s^{33} \\
&\quad + (\chi_s)_{11}\, W_s^{11} + (\chi_s)_{22}\, W_s^{22} + 2\,(\chi_s)_{12}\, W_s^{12} \\
&\quad + 2\left\{(\chi_s)_{01}\, W_s^{01} + (\chi_s)_{31}\, W_s^{31}\right. \\
&\quad \left. + (\chi_s)_{02}\, W_s^{02} + (\chi_s)_{32}\, W_s^{32}\right\} \qquad (46) \\
(\chi_a)_{\mu\nu}\, W_a^{\mu\nu} &= 2\left\{(\chi_a)_{03}\, W_a^{03} + (\chi_a)_{12}\, W_a^{12}\right. \\
&\quad + (\chi_a)_{01}\, W_a^{01} + (\chi_a)_{31}\, W_a^{31} \\
&\quad \left. + (\chi_a)_{02}\, W_a^{02} + (\chi_a)_{32}\, W_a^{32}\right\} \qquad (47)
\end{aligned}$$

where we have employed the symmetries under $\mu \leftrightarrow \nu$. Also, using the continuity equation constraints we find that

$$\begin{aligned}
(\chi_s)_{\mu\nu}\, W_s^{\mu\nu} &= \rho^2\,(\chi_s)_{00}\, W_s^{00} + (\chi_s)_{11}\, W_s^{11} + (\chi_s)_{22}\, W_s^{22} \\
&\quad + 2\left\{(\chi_s)_{12}\, W_s^{12} + \rho\,(\chi_s)_{01}\, W_s^{01} + \rho\,(\chi_s)_{02}\, W_s^{02}\right\} \qquad (48) \\
(\chi_a)_{\mu\nu}\, W_a^{\mu\nu} &= 2\left\{(\chi_a)_{12}\, W_a^{12} + \rho\,(\chi_a)_{01}\, W_a^{01} + \rho\,(\chi_a)_{02}\, W_a^{02}\right\}, \qquad (49)
\end{aligned}$$

namely, 6 symmetric and 3 anti-symmetric contributions for a total of 9, as expected. Note that $(\chi_a)_{03}\, W_a^{03} = 0$, since $(\chi_a)_{03} = -\nu'\,(\chi_a)_{00} = 0$. We have from past work [2] that equivalently the contractions may be re-written in terms of the following real leptonic kinematic factors and responses:

$$\chi_s^{00} = \chi_{unpol}^{00} \equiv \frac{1}{2} v_0 \qquad (50)$$

we have

$$
\begin{aligned}
V_L &\equiv \rho^2 && (51)\\
V_T &\equiv \frac{2}{v_0}\cdot\frac{1}{2}\left(\chi_s^{22}+\chi_s^{11}\right) && (52)\\
V_{TT} &\equiv \frac{2}{v_0}\cdot\frac{1}{2}\left(\chi_s^{22}-\chi_s^{11}\right) && (53)\\
V_{TL} &\equiv \frac{2}{v_0}\cdot\frac{1}{\sqrt{2}}\rho\left(-\chi_s^{01}\right) && (54)\\
V_{T'} &\equiv \frac{2}{v_0}\cdot\left(-i\chi_a^{12}\right) && (55)\\
V_{TL'} &\equiv \frac{2}{v_0}\cdot\frac{1}{\sqrt{2}}\rho\left(i\chi_a^{02}\right) && (56)\\
V_{\underline{TT}} &\equiv \frac{2}{v_0}\cdot\left(\chi_s^{12}\right) && (57)\\
V_{\underline{TL}} &\equiv \frac{2}{v_0}\cdot\frac{1}{\sqrt{2}}\rho\left(-\chi_s^{02}\right) && (58)\\
V_{\underline{TL}'} &\equiv \frac{2}{v_0}\cdot\frac{1}{\sqrt{2}}\rho\left(-i\chi_a^{01}\right) && (59)
\end{aligned}
$$

for the leptonic factors, and

$$
\begin{aligned}
W^L &\equiv \left(W_{fi}^{00}\right)_s && (60)\\
W^T &\equiv \left(W_{fi}^{22}\right)_s+\left(W_{fi}^{11}\right)_s && (61)\\
W^{TT} &\equiv \left(W_{fi}^{22}\right)_s-\left(W_{fi}^{11}\right)_s && (62)\\
W^{TL} &\equiv 2\sqrt{2}\left(W_{fi}^{01}\right)_s = 2\sqrt{2}\mathrm{Re}W_{fi}^{01} && (63)\\
W^{T'} &\equiv 2i\left(W_{fi}^{12}\right)_a = -2\mathrm{Im}W_{fi}^{12} && (64)\\
W^{TL'} &\equiv 2\sqrt{2}i\left(W_{fi}^{02}\right)_a = -2\sqrt{2}\mathrm{Im}W_{fi}^{02} && (65)\\
W^{\underline{TT}} &\equiv 2\left(W_{fi}^{12}\right)_s = 2\mathrm{Re}W_{fi}^{12} && (66)\\
W^{\underline{TL}} &\equiv 2\sqrt{2}\left(W_{fi}^{02}\right)_s = 2\sqrt{2}\mathrm{Re}W_{fi}^{02} && (67)\\
W^{\underline{TL}'} &\equiv -2\sqrt{2}i\left(W_{fi}^{01}\right)_a = 2\sqrt{2}\mathrm{Im}W_{fi}^{01} && (68)
\end{aligned}
$$

for the hadronic parts of the response. As in the cited work the notation here is the following: the quantities labelled L refer to contributions involving the $\mu\nu = 00$ parts of the tensors; those labelled T, TT, T' and $\underline{TT}$ involve only transverse components of the tensors; and those labelled TL, TL', $\underline{TL}$ and $\underline{TL}'$

involve interferences having real or imaginary parts of the $\mu\nu = 01$ and 02 components of the tensors. Unprimed quantities arise from symmetric tensors, *viz.,* those that do not involve polarized electrons, whereas those with primes only occur when electron polarizations enter. The underlined quantities labelled $\underline{TT}$ and $\underline{TL}$ occur only when the electron beam is polarized *and* the polarization of the scattered electron is measured (see [2]); since we will not consider this situation in the present study, these contributions are henceforth dropped. Finally, the sector labelled $\underline{TL}'$ does occur when only the electron beam is polarized, although at high energies these can also safely be ignored since they go as $1/\gamma$ where γ is the usual ratio of energy to mass for the electron and thus are also neglected in the present work leaving 6 classes of response. Accordingly, for the situation of interest in the present study the full contraction of the leptonic and hadronic tensors may then be written in terms of these real quantities, 4 involving symmetric contributions and 2 involving anti-symmetric contributions:

$$\begin{aligned} \mathcal{C} &= v_0 \left[V_L W^L + V_T W^T + V_{TL} W^{TL} + V_{TT} W^{TT} \right. \\ &\qquad \left. + V_{T'} W^{T'} + V_{TL'} W^{TL'} \right], \end{aligned} \tag{69}$$

where $\mathcal{C}$ is a Lorentz invariant. We again note that, while the entire right-hand side of the equation forms a Lorentz invariant, the individual factors are all frame-dependent.

We next proceed to develop the various tensors and related parts of the response.

2.2. Leptonic Tensors

The leptonic tensor may be built from the 4-momenta of the incident electron beam and of the scattered electron, K^μ and K'^μ, respectively. The incident electron has 3-momentum k and on-shell energy $\epsilon = \sqrt{m_e^2 + k^2}$, the scattered electron has 3-momentum k' and energy $\epsilon' = \sqrt{m_e^2 + k'^2}$, and θ_e denotes the electron scattering angle. Alternatively, it is often convenient to re-express the tensor in terms of two other 4-vectors, the 4-momentum transfer

$$Q^\mu = K^\mu - K'^\mu \tag{70}$$

and

$$R^\mu \equiv \frac{1}{2}\left(K^\mu + K'^\mu\right). \tag{71}$$

The leptonic tensor for electron scattering in the plane-wave Born approximation is well-known from previous work. Here we draw on the developments in [2] where a general form was presented that allows for the electron mass to be retained and where both the incident and scattered electrons can be polarized. Hereafter we will restrict our attention to the situation where the electrons have energies that are much greater than their mass, namely, the so-called Extreme Relativistic Limit (ERL$_e$). Accordingly, we take $\epsilon \approx k$ and $\epsilon' \approx k'$. Additionally, we shall assume that only the incident beam is polarized (in fact longitudinally; see [2]) and then some simplifications for the leptonic tensor are seen to occur. In particular, the cases $V_{\underline{TT}}$, $V_{\underline{TL}}$ and $V_{\underline{TL'}}$ in Eqs. (57-59) are absent and for the six cases that remain one has (see [2])

$$V_L \underset{ERL_e}{\longrightarrow} v_L \tag{72}$$

$$V_T \underset{ERL_e}{\longrightarrow} v_T \tag{73}$$

$$V_{TT} \underset{ERL_e}{\longrightarrow} v_{TT} \tag{74}$$

$$V_{TL} \underset{ERL_e}{\longrightarrow} v_{TL} \tag{75}$$

$$V_{T'} \underset{ERL_e}{\longrightarrow} h v_{T'} \tag{76}$$

$$V_{TL'} \underset{ERL_e}{\longrightarrow} h v_{TL'} \tag{77}$$

where $h \equiv \pm 1$ is the incident electron's helicity. In detail we have

$$\chi^{\mu\nu} \equiv \chi^{\mu\nu}_{unpol} + \chi^{\mu\nu}_{pol}, \tag{78}$$

where

$$\begin{aligned}
\chi_{unpol}^{\mu\nu} &= \chi_s^{\mu\nu} = K^\mu K'^\nu + K'^\mu K^\nu + \frac{1}{2}Q^2 g^{\mu\nu} && (79)\\
&\equiv \chi_{1,s}^{\mu\nu} + \chi_{2,s}^{\mu\nu} && (80)\\
\chi_{1,s}^{\mu\nu} &= \frac{1}{2}Q^2\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) && (81)\\
\chi_{2,s}^{\mu\nu} &= 2R^\mu R^\nu && (82)\\
\chi_{pol}^{\mu\nu} &= \chi_a^{\mu\nu} = -ih\epsilon^{\mu\nu\alpha\beta}K_\alpha K'_\beta && (83)\\
&= -ih\epsilon^{\mu\nu\alpha\beta}Q_\alpha R_\beta. && (84)
\end{aligned}$$

As noted above, typically we work in a coordinate system where the 3-momentum transfer lies in the 3-direction, and hence

$$Q^\mu = (\omega, 0, 0, q) = q\,(\nu', 0, 0, 1) \tag{85}$$

with $\nu' \equiv \omega/q$, as usual. This implies that

$$\chi_m^{3\alpha} = \nu' \chi_m^{0\alpha} \tag{86}$$

for $m = (1, s)\,,\ (2, s)$ or (a). We have

$$\begin{aligned}
-Q^2 &= \left|Q^2\right| = q^2\rho = 4kk'\sin^2\theta_e/2 && (87)\\
v_0 &\equiv (\epsilon + \epsilon')^2 - q^2 = q^2\left(\frac{1}{(\rho')^2} - 1\right) = 4kk'\cos^2\theta_e/2. && (88)
\end{aligned}$$

Using these identities one finds for the required components of the symmetric (electron unpolarized) and anti-symmetric (electron longitudinally polarized) tensors

$$\begin{aligned}
\chi_s^{00} - 2\nu\chi_s^{03} + \nu^2\chi_s^{33} &\equiv \frac{1}{2}v_0 \times v_L && (89)\\
\frac{1}{2}\left(\chi_s^{22} + \chi_s^{11}\right) &\equiv \frac{1}{2}v_0 \times v_T && (90)\\
\frac{1}{2}\left(\chi_s^{22} - \chi_s^{11}\right) &\equiv \frac{1}{2}v_0 \times v_{TT} && (91)\\
\frac{1}{\sqrt{2}}\left(\chi_s^{01} - \nu\chi_s^{31}\right) &\equiv -\frac{1}{2}v_0 \times v_{TL} && (92)\\
\chi_a^{12} &\equiv ih\frac{v_0}{2} \times v_{T'} && (93)\\
\frac{1}{\sqrt{2}}\left(\chi_a^{02} - \nu\chi_a^{32}\right) &\equiv -ih\frac{v_0}{2} \times v_{TL'}, && (94)
\end{aligned}$$

which yields the standard results:

$$
\begin{aligned}
v_L &= \rho^2 && (95)\\
v_T &= \frac{1}{2}\rho + \tan^2\theta_e/2 && (96)\\
v_{TT} &= -\frac{1}{2}\rho && (97)\\
v_{TL} &= -\frac{1}{\sqrt{2}}\rho\sqrt{\rho + \tan^2\theta_e/2} && (98)\\
v_{T'} &= \tan\theta_e/2\sqrt{\rho + \tan^2\theta_e/2} && (99)\\
v_{TL'} &= -\frac{1}{\sqrt{2}}\rho\tan\theta_e/2 && (100)
\end{aligned}
$$

One may also re-write the leptonic factors in a way that involves the so-called photon longitudinal polarization. One begins with the transverse term in Eq. (96)

$$
\begin{aligned}
v_T &= \frac{1}{2}\rho + \tan^2\theta_e/2 && (101)\\
&= \frac{1}{2}\rho\left[1 + \frac{2}{\rho}\tan^2\theta_e/2\right], && (102)
\end{aligned}
$$

thereby defining the photon longitudinal polarization

$$\mathcal{E} \equiv \left[1 + \frac{2}{\rho}\tan^2\theta_e/2\right]^{-1}, \tag{103}$$

which implies that

$$\tan^2\theta_e/2 = \frac{\rho}{2}\left(\mathcal{E}^{-1} - 1\right). \tag{104}$$

If one defines the ratios

$$u_X \equiv \frac{v_X}{v_T} \tag{105}$$

with $X = L, T, TT, TL, T'$ and TL' and substitutes in the above equations for

v_X for the factor $\tan\theta_e/2$ one finds that

$$
\begin{aligned}
u_L &= 2\rho\mathcal{E} \\
u_T &= 1 \\
u_{TT} &= -\mathcal{E} \\
u_{TL} &= -\sqrt{\rho}\sqrt{\mathcal{E}\,(1+\mathcal{E})} \\
u_{T'} &= \sqrt{1-\mathcal{E}^2} \\
u_{TL'} &= -\sqrt{\rho}\sqrt{\mathcal{E}\,(1-\mathcal{E})}.
\end{aligned}
\tag{106}
$$

The invariant in Eq. (69) in this notation in the ERL$_e$ then becomes

$$
\begin{aligned}
\mathcal{C} = v_0 v_T \Big[& 2\rho\mathcal{E}W^L + W^T - \mathcal{E}W^{TT} - \sqrt{\rho}\sqrt{\mathcal{E}\,(1+\mathcal{E})}W^{TL} \\
& + \sqrt{1-\mathcal{E}^2}W^{T'} - \sqrt{\rho}\sqrt{\mathcal{E}\,(1-\mathcal{E})}W^{TL'} \Big].
\end{aligned}
\tag{107}
$$

3. Hadronic Tensors

In this section we proceed to build the most general tensors for semi-inclusive electron scattering from polarized spin-1/2 targets. A general frame will be assumed to begin with and later the special choice of the rest frame will be discussed. The strategy is to write the tensors in terms of invariant response functions. Accordingly, if one (say) has a model for the cross section in the rest frame, then the set of invariant response functions can be deduced and one immediately has the corresponding cross section in a general frame, for instance, in the collider frame that will provide a focus for some of our discussions. This way there is no need to deal with the difficulties of requiring modeling that is covariant, something that is rarely possible.

We begin by introducing the basic 4-vectors upon which the general hadronic tensor is built.

3.1. Basic Hadronic 4-Vectors

We build the hadronic tensors in an arbitrary frame using the basic 4-vectors that characterize semi-inclusive electron scattering from (possibly) polarized

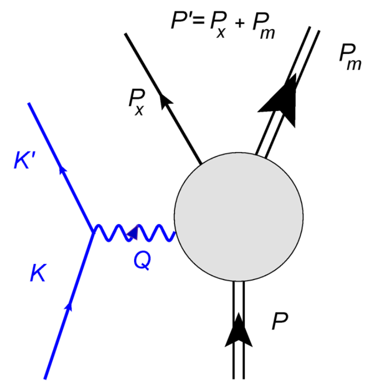

Figure 3: Feynman diagram for semi-inclusive electron scattering. The 4-momenta here are discussed in the text. In particular, particle x is assumed to be detected in coincidence with the scattered electron and thus P_x^μ is assumed to be known. Since the total final-state momentum P'^μ is known (see Fig. 2 for inclusive scattering) this implies that the missing 4-momentum is also known via the relationship $P_m^\mu = P'^\mu - P_x^\mu$.

spin-1/2 targets. The strategy in the following is to expand the general second-rank tensors that are needed in the four sectors symmetric/unpolarized, anti-symmetric/unpolarized, symmetric/polarized and anti-symmetric/polarized, and impose the continuity equation in each sector – note that, since each sector may be isolated by controlling the electron and target spins, they may be considered independently. Upon contracting with Q^μ in each case one may expand in a basis set of four independent 4-vectors such as those below to determine which contributions enter and which do not.

We begin this discussion of the 4-vectors with a specific choice of coordinate system; see Fig. 1. Since we want to retain the usual meaning for the leptonic and hadronic factors discussed in Sec. 2.1, it is important to employ this system for the developments to follow. In this system we have the following 4-vectors:

$$Q^\mu = (\omega, \mathbf{q}) \tag{108}$$

$$P^\mu = (E_p, \mathbf{p}) \tag{109}$$

$$P_x^\mu = (E_x, \mathbf{p}_x) \tag{110}$$

$$S^\mu = \left(S^0, \mathbf{s}\right) \tag{111}$$

with 3-vectors

$$
\begin{aligned}
\mathbf{q} &= q\mathbf{u}_3 && (112)\\
\mathbf{p} &= p\left(\sin\theta\cos\phi\mathbf{u}_1+\sin\theta\sin\phi\mathbf{u}_2+\cos\theta\mathbf{u}_3\right) && (113)\\
\mathbf{p}_x &= p_x\left(\sin\theta_x\cos\phi_x\mathbf{u}_1+\sin\theta_x\sin\phi_x\mathbf{u}_2+\cos\theta_x\mathbf{u}_3\right) && (114)\\
\mathbf{s} &= s\left(\sin\theta^*\cos\phi^*\mathbf{u}_1+\sin\theta^*\sin\phi^*\mathbf{u}_2+\cos\theta^*\mathbf{u}_3\right), && (115)
\end{aligned}
$$

where $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are unit vectors (see Fig. 1) defined such that $\mathbf{u}_3$ is along the direction of the 3-momentum transfer, the lepton scattering plane is the 13-plane and $\mathbf{u}_2$ is normal to that plane. The target (mass M) and particle detected in coincidence with the scattered electron (mass M_x) are both on-shell and thus

$$
\begin{aligned}
E_p &= \sqrt{p^2+M^2} && (116)\\
E_x &= \sqrt{p_x^2+M_x^2}. && (117)
\end{aligned}
$$

The target spin 4-vector may be developed further by exploiting the two conditions it must satisfy, namely

$$P\cdot S=0 \tag{118}$$

and

$$S^2=\left(S^0\right)^2-s^2=-1, \tag{119}$$

which may be verified by going to the target rest frame. If we define

$$\boldsymbol{\beta}_p\equiv\mathbf{p}/E_p \tag{120}$$

so that

$$\gamma_p=\frac{1}{\sqrt{1-\beta_p^2}}=E_p/M \tag{121}$$

and let χ be the angle between $\mathbf{p}$ and $\mathbf{s}$, then Eq. (118) implies that

$$S^0=\boldsymbol{\beta}_p\cdot\mathbf{s}=\beta_p s\cos\chi, \tag{122}$$

where

$$\cos\chi=\cos\theta\cos\theta^*+\sin\theta\sin\theta^*\cos\left(\phi-\phi^*\right). \tag{123}$$

Equation (119) implies that

$$s^2\left(1-\beta_p^2\cos^2\chi\right)=1 \tag{124}$$

which yields

$$\begin{aligned} s &\equiv |\mathbf{s}| = \frac{1}{\sqrt{1-\beta_p^2\cos^2\chi}} && (125)\\ \mathbf{s} &\equiv h^* s\left(\sin\theta^*\cos\phi^*\mathbf{u}_1+\sin\theta^*\sin\phi^*\mathbf{u}_2+\cos\theta^*\mathbf{u}_3\right) && (126)\\ S^0 &= h^*\beta_p s\cos\chi && (127)\end{aligned}$$

where we have now introduced $h^*=\pm$, namely, a convenient factor that allows the target spin to be flipped while keeping the axis of quantization for the spin fixed. Accordingly, the target spin 4-vector may be written

$$S^\mu=\frac{h^*}{\sqrt{1-\beta_p^2\cos^2\chi}}\left(\beta_p\cos\chi,\sin\theta^*\cos\phi^*,\sin\theta^*\sin\phi^*,\cos\theta^*\right), \tag{128}$$

Thus we see that as building blocks we may employ the 4-momentum transfer Q^μ, the target 4-momentum P^μ, the 4-momentum of some particle detected in the final state P_x^μ, and the 4-vector that characterizes the target spin, S^μ. As usual, it is convenient to replace the last three with projected 4-vectors, *i.e.,* vectors that are by construction orthogonal to Q^μ. When the spin is not involved, namely the target is unpolarized (see below for the polarized case), we use the following as basic polar vectors

$$\begin{aligned} &Q^\mu && (129)\\ &U^\mu\equiv\frac{1}{M}\left(P^\mu-\left(\frac{Q\cdot P}{Q^2}\right)Q^\mu\right) && (130)\\ &V^\mu\equiv\frac{1}{M}\left(P_x^\mu-\left(\frac{Q\cdot P_x}{Q^2}\right)Q^\mu\right), && (131)\end{aligned}$$

where the factors M in Eqs. (130) and (131) are included for convenience to make the 4-vectors dimensionless (see below) and where, by construction, one has

$$Q\cdot U=Q\cdot V=0 \tag{132}$$

and

$$\begin{aligned} U^2 &= 1 - \frac{(Q \cdot P)^2}{M^2 Q^2} && (133) \\ V^2 &= \frac{1}{M^2}\left(M_x^2 - \frac{(Q \cdot P_x)^2}{Q^2}\right) && (134) \\ U \cdot V &= \frac{1}{M^2}\left(P \cdot P_x - \frac{(Q \cdot P)(Q \cdot P_x)}{Q^2}\right). && (135) \end{aligned}$$

Note that we have chosen to use the target mass M above and not the mass of particle x, namely M_x, since we want to allow the latter to be general enough to include the photon. Furthermore, we can replace V^μ with a 4-vector that is orthogonal not only to Q^μ but to U^μ as well:

$$X^\mu \equiv V^\mu - \left(\frac{U \cdot V}{U^2}\right) U^\mu, \tag{136}$$

where then

$$Q \cdot U = Q \cdot X = U \cdot X = 0 \tag{137}$$

and

$$X^2 = V^2 - \frac{(U \cdot V)^2}{U^2}. \tag{138}$$

We can also define a fourth 4-vector via

$$D^\mu \equiv \frac{1}{M}\epsilon^{\mu\alpha\beta\gamma} Q_\alpha U_\beta X_\gamma = \frac{1}{M^3}\epsilon^{\mu\alpha\beta\gamma} Q_\alpha P_\beta P_{x\gamma} \tag{139}$$

which is dual to the above set, behaves as an axial-vector and satisfies

$$Q \cdot D = U \cdot D = X \cdot D = 0. \tag{140}$$

This yields a set of four 4-vectors that can be used to span 4-dimensional space.

We can define the following invariants

$$\begin{aligned} I_1 &\equiv \frac{Q \cdot P}{Q^2} = (\omega E_p - qp\cos\theta)/Q^2 && (141) \\ I_2 &\equiv \frac{Q \cdot P_x}{Q^2} = (\omega E_x - qp_x \cos\theta_x)/Q^2 && (142) \\ I_3 &\equiv \frac{P \cdot P_x}{Q^2} \\ &= (E_p E_x - pp_x(\sin\theta\sin\theta_x\cos(\phi - \phi_x) + \cos\theta\cos\theta_x)/Q^2, && (143) \end{aligned}$$

where as usual $-Q^2 = 4kk' \sin^2 \theta_e/2$ in the ERL$_e$ (see Sec. 2.2). We note here that for the Lorentz scalars upon which the invariant response functions discussed above depend we have the following: two of them are fixed, namely, $P^2 = M^2$and $P_x^2 = M_x^2$, one can be chosen to be Q^2, and three can be chosen to be those given in Eqs. (141–143), for a total of four dynamical scalars, namely, (Q^2, I_1, I_2 and I_3). Eqs. (130–131) then yield the projected 4-vectors employed above:

$$
\begin{aligned}
U^\mu &\equiv \frac{1}{M}\left(P^\mu - I_1 Q^\mu\right) && (144)\\
V^\mu &\equiv \frac{1}{M}\left(P_x^\mu - I_2 Q^\mu\right). && (145)
\end{aligned}
$$

We also have that

$$
\begin{aligned}
U^2 &= 1 - \frac{Q^2 I_1^2}{M^2} && (146)\\
U \cdot V &= \frac{Q^2}{M^2}\left(I_3 - I_1 I_2\right) && (147)
\end{aligned}
$$

and therefore that

$$
I_4 \equiv \frac{U \cdot V}{U^2} = \frac{Q^2\left(I_3 - I_1 I_2\right)}{M^2 - Q^2 I_!^2}, \qquad (148)
$$

which yields explicit expressions for the basis 4-vectors:

$$
\begin{aligned}
U^0 &= \frac{1}{M}\left(E_p - I_1 \omega\right) && (149)\\
U^i &= \frac{1}{M}\left(\mathbf{p} - I_1 \mathbf{q}\right)^i && (150)\\
X^0 &= \frac{1}{M}\left(E_x - I_4 E_p - \left[I_2 - I_1 I_4\right]\omega\right) && (151)\\
X^i &= \frac{1}{M}\left(\mathbf{p}_x - I_4 \mathbf{p} - \left[I_2 - I_1 I_4\right]\mathbf{q}\right)^i. && (152)
\end{aligned}
$$

The four Lorentz scalars above can be replaced by (frame dependent) variables that are traditionally employed in specific sub-fields. For instance, in high-energy physics one often uses $\left(Q^2, \nu\right)$ where $\nu \equiv \omega$ or $\left(Q^2, x\right)$ for the first two scalars, x being defined by $x \equiv 1/\left(2I_1\right)$. In nuclear physics where the energy and 3-momentum transfer are more natural one typically uses (q, ω) instead. Furthermore, in nuclear physics it is generally better for the remaining two dynamical variables to use the missing energy E_m and missing momentum p_m when studying semi-inclusive electroweak reactions (see later) [3].

When the target spin is involved we can also define another Lorentz invariant

$$I_s \equiv \frac{Q \cdot S}{Q^2} \tag{153}$$

and the corresponding projected 4-vector

$$\Sigma^\mu \equiv S^\mu - I_s Q^\mu \tag{154}$$

where $Q \cdot \Sigma = 0$ and for completeness we note that

$$I_s = \left(\omega S^0 - qs \cos\theta^*\right)/Q^2. \tag{155}$$

Note that the spin 4-vector does not enter as a dynamical Lorentz scalar since it occurs as part of the projection operator

$$\mathcal{P}_{spin} \equiv \frac{1}{2}\left(1 + \gamma_5 \gamma_\mu S^\mu\right) \tag{156}$$

and either does not enter (unpolarized) or occurs explicitly (polarized) where, being part of the projection operator, it only enters linearly.

We can also define two 4-vectors that contain the spin 4-vector linearly and are dual to specific combinations of the others, namely,

$$\overline{X}^\mu \equiv \frac{1}{M}\epsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta U_\gamma = \frac{1}{M^2}\epsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta P_\gamma \tag{157}$$

$$\overline{U}^\mu \equiv \frac{1}{M}\epsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta X_\gamma = \frac{1}{M}\epsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta V_\gamma - \left(\frac{U \cdot V}{U^2}\right)\overline{X}^\mu. \tag{158}$$

One has that

$$\begin{aligned} Q \cdot \overline{U} &= X \cdot \overline{U} = \Sigma \cdot \overline{U} = 0 && (159) \\ Q \cdot \overline{X} &= U \cdot \overline{X} = \Sigma \cdot \overline{X} = 0 && (160) \end{aligned}$$

and additionally that

$$\begin{aligned} U \cdot \overline{U} &= -X \cdot \overline{X} && (161) \\ &= \frac{1}{M}\epsilon^{\alpha\beta\gamma\delta}\Sigma_\alpha Q_\beta U_\gamma X_\delta \equiv I_0 && (162) \\ &= \frac{1}{M^3}\epsilon^{\alpha\beta\gamma\delta} S_\alpha Q_\beta P_\gamma P_{x\delta}, && (163) \end{aligned}$$

namely, a dimensionless invariant. Note that a tensor of the form

$$\overline{Q}^{\mu} \equiv \epsilon^{\mu\alpha\beta\gamma} \Sigma_{\alpha} U_{\beta} X_{\gamma} \tag{164}$$

is redundant, since it can be shown that

$$\overline{Q}^{\mu} = -\frac{M I_0}{Q^2} Q^{\mu} \tag{165}$$

where Q^{μ} will be used instead as a building block.

Next, let us consider the 4-vector $\overline{X}^{\mu}$ defined in Eq. (157). In contracting with $\epsilon^{\mu\alpha\beta\gamma}$ the contributions in Σ_{α} and U_{γ} containing Q_{α} and Q_{γ}, respectively, may be ignored due to the explicit factor Q_{β}, and hence we can write

$$\begin{aligned} \overline{X}^{\mu} &= \frac{1}{M^2} \epsilon^{\mu\alpha\beta\gamma} S_{\alpha} Q_{\beta} P_{\gamma} && (166) \\ &= -\frac{1}{M^2} \left[\omega \epsilon^{\mu 0 \alpha\gamma} - q \epsilon^{\mu 3 \alpha\gamma} \right] S_{\alpha} P_{\gamma}, && (167) \end{aligned}$$

the latter expression in the 123-system. Upon developing this expression one can show that

$$\begin{aligned} \overline{X}^{i} &= \frac{1}{M^2} \left(\left[(\omega \mathbf{p} - E \mathbf{q}) \times \mathbf{s} \right] + S^0 \left(\mathbf{q} \times \mathbf{p} \right) \right)^{i} \ , i = 1,2,3 && (168) \\ \overline{X}^{0} &= \frac{1}{\nu'} \overline{X}^{3}. && (169) \end{aligned}$$

And finally we have the 4-vector $\overline{U}^{\mu}$ defined in Eq. (158)

$$\begin{aligned} \overline{U}^{\mu} &= \frac{1}{M} \epsilon^{\mu\alpha\beta\gamma} S_{\alpha} Q_{\beta} X_{\gamma} && (170) \\ &= \frac{1}{M} \epsilon^{\mu\alpha\beta\gamma} S_{\alpha} Q_{\beta} V_{\gamma} - \left(\frac{U \cdot V}{U^2} \right) \overline{X}^{\mu} && (171) \\ &= \overline{T}^{\mu} - \left(\frac{U \cdot V}{U^2} \right) \overline{X}^{\mu}, && (172) \end{aligned}$$

where

$$\begin{aligned} \overline{T}^{\mu} &\equiv \frac{1}{M^2} \epsilon^{\mu\alpha\beta\gamma} S_{\alpha} Q_{\beta} P_{x\gamma} && (173) \\ &= -\frac{1}{M^2} \left[\omega \epsilon^{\mu 0 \alpha\gamma} - q \epsilon^{\mu 3 \alpha\gamma} \right] S_{\alpha} P_{x\gamma}. && (174) \end{aligned}$$

As above we can develop this expression to find that

$$\begin{aligned} \overline{T}^{i} &= \frac{1}{M^2} \left(\left[(\omega \mathbf{p}_x - E_x \mathbf{q}) \times \mathbf{s} \right] + S^0 \left(\mathbf{q} \times \mathbf{p}_x \right) \right)^{i}, \ i = 1,2,3 && (175) \\ \overline{T}^{3} &= \nu' \overline{T}^{0}. && (176) \end{aligned}$$

This completes the specification of the basis 4-vectors that will be used in the next section to obtain the most general form for the hadronic tensor.

3.2. Hadronic Tensors in a General Reference Frame

3.2.1. Second-Rank Tensors: Symmetric, Unpolarized

Given the above 4-vector building blocks, we now proceed to construct second-rank hadronic tensors with the appropriate symmetries. We begin with the symmetric cases where no target polarization is involved.

$$W_{1,s}^{\mu\nu} \equiv g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \tag{177}$$

$$W_{2,s}^{\mu\nu} \equiv U^\mu U^\nu \tag{178}$$

$$W_{3,s}^{\mu\nu} \equiv X^\mu X^\nu \tag{179}$$

$$W_{4,s}^{\mu\nu} \equiv U^\mu X^\nu + X^\mu U^\nu . \tag{180}$$

Here the motivation for including the factors M becomes clear: all four of the tensors above are dimensionless. Note that no contributions of the form $Q^\mu U^\nu + U^\mu Q^\nu$ or $Q^\mu X^\nu + X^\mu Q^\nu$ are used, since upon contraction with the lepton tensor these would yield zero, and that D^μ does not enter in similar forms since the results would correspond to second-rank tensors that are of vector/axial-vector or axial/axial character rather than vector/vector as required and we have no pseudoscalars to use as multiplicative factors to produce this behavior in the unpolarized situation where the spin does not enter. We have the following upon contracting with Q_μ:

$$Q_\mu W_{m,s}^{\mu\nu} = 0 \tag{181}$$

for $m = 1, 2, 3, 4$. The general tensor of this type is obtained by summing over the 4 contributions, where each is multiplied by a Lorentz scalar, invariant response function, A_m, that in turn depends only on the Lorentz invariants in the problem, namely

$$(W_s^{\mu\nu})_{unpol} = \sum_{m=1}^{4} A_m W_{m,s}^{\mu\nu} \tag{182}$$

and from Eq. (181) one has

$$Q_\mu \left(W_s^{\mu\nu}\right)_{unpol} = 0 \tag{183}$$

as required for the overall symmetric, unpolarized tensor by the continuity equation. Thus the symmetric, unpolarized second-rank hadronic tensor may then be written

$$\begin{aligned}\left(W_s^{\mu\nu}\right)_{unpol} &= -W_1 \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + W_2 U^\mu U^\nu \\ &\quad + W_3 X^\mu X^\nu + W_4 \left(U^\mu X^\nu + X^\mu U^\nu\right),\end{aligned} \tag{184}$$

namely, with *four* contributions involving invariant functions W_m, $m = 1, 2, 3, 4$ (here we have shifted from using invariant functions A_m to more familiar notation, including the minus sign in the W_1 case, which is conventional and useful as will become apparent when inclusive scattering is discussed later). Since the tensors defined above are all dimensionless, the invariant functions here all have the same dimensions.

3.2.2. Second-Rank Tensors: Anti-symmetric, Unpolarized

We have only one anti-symmetric contribution that uses Q^μ, U^μ and X^ν as a basis, namely

$$W_{1,a}^{\mu\nu} \equiv i(U^\mu X^\nu - X^\mu U^\nu), \tag{185}$$

where here and below the factor i has been included following Eq. (29); as above this tensor is dimensionless. Note that again no contributions such as $Q^\mu U^\nu - U^\mu Q^\nu$ or $Q^\mu X^\nu - X^\mu Q^\nu$ are included as these yield zero upon contraction with the lepton tensor, and that contributions of this form using D^μ are invalid since they do not behave as vector/vector, and that contributions such as $\epsilon^{\mu\nu\alpha\beta} Q_\alpha U_\beta$, $\epsilon^{\mu\nu\alpha\beta} Q_\alpha V_\beta$ and $\epsilon^{\mu\nu\alpha\beta} U_\alpha V_\beta$ are invalid for the same reason. Also contributions involving the Levi-Civita symbol with D^μ and one of (Q^μ, U^μ, X^μ) can be shown to be redundant; in fact one has the following identities:

$$i\epsilon^{\mu\nu\alpha\beta}Q_\alpha D_\beta = \frac{1}{M}Q^2 W_{1,a}^{\mu\nu} \tag{186}$$

$$i\epsilon^{\mu\nu\alpha\beta}U_\alpha D_\beta = -\frac{i}{M}U^2\left(Q^\mu X^\nu - X^\mu Q^\nu\right) \tag{187}$$

$$i\epsilon^{\mu\nu\alpha\beta}X_\alpha D_\beta = \frac{i}{M}X^2\left(Q^\mu U^\nu - U^\mu Q^\nu\right). \tag{188}$$

Contracting the valid anti-symmetric tensor with Q_μ yields zero and we find that the anti-symmetric, unpolarized tensor is constructed from the *single* basis tensor of the correct type with an invariant functions here called W_5:

$$\left(W_a^{\mu\nu}\right)_{unpol} = iW_5\left(U^\mu X^\nu - X^\mu U^\nu\right), \tag{189}$$

namely the so-called 5th response (see [4] and references therein). We note in passing that in that same reference the general problem of reactions of the type $A(\overrightarrow{e}, e'x_1x_2\ldots)$ having the target unpolarized, but having any number of particles x_1, x_2, *etc.,* detected in coincidence with the scattered electron was developed.

3.2.3. Second-Rank Tensors: Symmetric, Polarized

Let us begin the symmetric polarized developments by starting with a set of symmetric second-rank tensors that starts with the set of four symmetric tensors obtained in the unpolarized case, $W_{m,s}^{\mu\nu}$, with $m = 1\cdots 4$ as in Eqs. (177–180), multiplied by I_0, namely

$$\begin{aligned}
W_{1,s}'^{\mu\nu} &\equiv \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) I_0 \\
W_{2,s}'^{\mu\nu} &\equiv \left(U^\mu U^\nu\right) I_0 \\
W_{3,s}'^{\mu\nu} &\equiv \left(X^\mu X^\nu\right) I_0 \\
W_{4,s}'^{\mu\nu} &\equiv \left(U^\mu X^\nu + X^\mu U^\nu\right) I_0.
\end{aligned} \tag{190}$$

Here and below the prime is included to denote the fact that the target spin is involved. These all have the desired properties, namely, they behave as vector/vector and are linear in the spin; they are all dimensionless. Contractions with Q^μ yield zero as above. To these we can add another set built from $\bar{U}^\mu$ and $\overline{X}^\mu$ together with the 4-vectors Q^μ, U^μ and X^μ.

For the remaining building blocks constructed from tensors containing the spin we use

$$W'^{\mu\nu}_{5,s} \equiv U^\mu \bar{U}^\nu + U^\nu \bar{U}^\mu \tag{191}$$

$$W'^{\mu\nu}_{6,s} \equiv U^\mu \overline{X}^\nu + U^\nu \overline{X}^\mu \tag{192}$$

$$W'^{\mu\nu}_{7,s} \equiv X^\mu \bar{U}^\nu + X^\nu \bar{U}^\mu \tag{193}$$

$$W'^{\mu\nu}_{8,s} \equiv X^\mu \overline{X}^\nu + X^\nu \overline{X}^\mu, \tag{194}$$

again with no contributions that are proportional to Q^μ or Q^ν as these would yield zero when contracted with the electron tensor. Again these behave as vector/vector and are linear in the spin and all yield zero when contracted with Q^μ. Accordingly, if we expand the symmetric polarized tensor in this set of basis tensors,

$$(W^{\mu\nu}_s)_{pol} = \sum_{m=1}^{8} A'_m W'^{\mu\nu}_{m,s} \tag{195}$$

with general invariant response functions A'_m, and impose the continuity equation constraint $Q_\mu (W^{\mu\nu}_s)_{pol} = 0$ we obtain the following:

$$\begin{aligned}
(W^{\mu\nu}_s)_{pol} \quad = \quad & \left[-W'_1 \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) + W'_2 U^\mu U^\nu \right. \\
& + W'_3 X^\mu X^\nu + W'_4 \left(U^\mu X^\nu + X^\mu U^\nu \right) \big] I_0 \\
& + W'_5 \left(U^\mu \bar{U}^\nu + U^\nu \bar{U}^\mu \right) + W'_6 \left(U^\mu \overline{X}^\nu + U^\nu \overline{X}^\mu \right) \\
& + W'_7 \left(X^\mu \bar{U}^\nu + X^\nu \bar{U}^\mu \right) + W'_8 \left(X^\mu \overline{X}^\nu + X^\nu \overline{X}^\mu \right),
\end{aligned} \tag{196}$$

again shifting from generic invariant functions A'_m to the more conventional notation involving invariant W'_m. Thus, for the symmetric, polarized case we are left with *eight* contributions. All tensors here are dimensionless and consequently all invariant functions have the same dimensions.

3.2.4. Second-Rank Tensors: Anti-symmetric, polarized

In this sector we begin with a basis tensor that involves the Levi-Civita symbol and is linear in spin:

$$W'^{\mu\nu}_{1,a} \equiv \frac{i}{M}\epsilon^{\mu\nu\alpha\beta}\Sigma_\alpha Q_\beta. \tag{197}$$

Note that one has the following identities,

$$Q^2\epsilon^{\mu\nu\alpha\beta}\Sigma_\alpha X_\beta = M\left(Q^\mu\bar{U}^\nu - Q^\nu\bar{U}^\mu\right) \tag{198}$$

$$Q^2\epsilon^{\mu\nu\alpha\beta}\Sigma_\alpha U_\beta = M\left(Q^\mu\overline{X}^\nu - Q^\nu\overline{X}^\mu\right) \tag{199}$$

and hence no terms having the Levi-Civita symbol as here are needed, since they also yield zero upon contraction with the electron tensor. Since we want tensors that are linear in spin and of vector/vector form we can also have the following dimensionless tensors:

$$W'^{\mu\nu}_{2,a} \equiv i(U^\mu\bar{U}^\nu - U^\nu\bar{U}^\mu) \tag{200}$$

$$W'^{\mu\nu}_{3,a} \equiv i(U^\mu\overline{X}^\nu - U^\nu\overline{X}^\mu) \tag{201}$$

$$W'^{\mu\nu}_{4,a} \equiv i(X^\mu\bar{U}^\nu - X^\nu\bar{U}^\mu) \tag{202}$$

$$W'^{\mu\nu}_{5,a} \equiv i(X^\mu\overline{X}^\nu - X^\nu\overline{X}^\mu) \tag{203}$$

with no terms of the form $Q^\mu\bar{U}^\nu - Q^\nu\bar{U}^\mu$ or $Q^\mu\overline{X}^\nu - Q^\nu\overline{X}^\mu$, since, as above, these yield zero when contracted with the lepton tensor. Finally, as in the symmetric case we can use the dimensionless, anti-symmetric contribution above (Eq. (185)) multiplied by the invariant I_0:

$$W'^{\mu\nu}_{6,a} \equiv i(U^\mu X^\nu - X^\mu U^\nu)I_0. \tag{204}$$

however, one can prove the following identity

$$\begin{aligned} -I_0\left(U^\mu X^\nu - U^\nu X^\mu\right) &= \frac{1}{M}U^2X^2\epsilon^{\mu\nu\alpha\beta}\Sigma_\alpha Q_\beta + X^2\left(U^\mu\overline{X}^\nu - U^\nu\overline{X}^\mu\right) \\ &\quad + U^2\left(X^\mu\overline{U}^\nu - X^\nu\overline{U}^\mu\right) \end{aligned} \tag{205}$$

and hence the tensor $W'^{\mu\nu}_{6,a}$ is redundant. The remaining five tensors all yield zero when contracted with Q^μ. Accordingly we have the following *five* independent contributions:

$$
\begin{aligned}
(W_a^{\mu\nu})_{pol} &= i\left[\frac{1}{M}W_9'\epsilon^{\mu\nu\alpha\beta}\Sigma_\alpha Q_\beta\right.\\
&\quad +W_{10}'(U^\mu\bar{U}^\nu - U^\nu\bar{U}^\mu) + W_{11}'(U^\mu\overline{X}^\nu - U^\nu\overline{X}^\mu)\\
&\quad \left.+ W_{12}'(X^\mu\bar{U}^\nu - X^\nu\bar{U}^\mu) + W_{13}'(X^\mu\overline{X}^\nu - X^\nu\overline{X}^\mu)\right] \qquad (206)
\end{aligned}
$$

As above, we have shifted notation to make this sector coherent with the previous ones; all tensors are dimensionless, implying that the invariant functions all have the same dimensions. As an alternative it is also possible to expand the contraction of the leptonic and hadronic tensors in terms of Lorentz scalars rather than employing the 4-vectors as we have here. The resulting form is documented in Appendix B.

Let us end this section with a brief discussion of how the use of time-reversal invariance allows one to separate the four types of contributions into two classes. The basic requirement for the time-reversal operator is to relate a given matrix element to one that describes the process running in the opposite direction, that is to a matrix element where the incoming state now contains all of the particles from the original final state and the final state contains the particles contains the particles from the original initial state. If the original matrix element has a final state with two or more interacting particles this requires that the boundary condition for this state be changed from the incoming boundary condition to the outgoing boundary condition.

The effects of time-reversal on the hadronic tensor have been studied in great detail in the context of multipole expansions for arbitrary target spin (see, for instance, [5] and references therein). The result is that the matrix elements must fall into two classes: one where the transition multipole moment is real and another where it is imaginary. These two classes result in response functions that are either even or odd under time-reversal, TRE or TRO, respectively. Note that time-reversal invariance is assumed throughout this work; being TRE or TRO does not imply violation of this symmetry.

For the case of a spin-1/2 particle in the initial or final state the effects of time-reversal can be greatly simplified by the simultaneous application of both time-reversal and parity [6]. This is particularly useful in the case where the hadronic tensor is written as a linear combination of invariant functions of inner products of the available 4-momenta and second-rank tensors constructed from these four-momenta and the spin vector, such as we have done above. For the purpose of this discussion let

$$\begin{aligned} W^{\mu\nu}(Q,P,P_x,P_m,S,(-)) = &\langle P,S|\, J^{\mu\dagger}(Q)\,|P_x,P_m,S,(-)\rangle \\ &\times \langle P_x,P_m,S,(-)|\, J^{\mu}(Q)(-)\,|P,S\rangle\,, \end{aligned} \tag{207}$$

where $(-)$ denotes the incoming boundary conditions for the final scattering state. This trivially implies that

$$W^{*\mu\nu}(Q,P,P_x,P_m,S,(-)) = W^{\nu\mu}(Q,P,P_x,P_m,S,(-)). \tag{208}$$

Equations (184,189,196,206) are constructed such that $W_i,\ i = 1,\ldots,5$ and $W_i',\ i = 1,\ldots,13$ are real.

The components of the hadronic tensor in Eqs. (184,189,196,206) are parameterized in terms of Lorentz 4-vectors. The result of combining time-reversal and parity causes no change to the momentum 4-vectors while causing the spin 4-vector to change sign. Most importantly, time-reversal causes a change in the boundary condition of the scattering state from incoming ($(-)$) to outgoing ($(+)$). This gives

$$\begin{aligned} W^{\mu\nu}(Q,P,P_x,P_m,S,(-)) \xrightarrow{\mathcal{TP}} &W^{\nu\mu}(Q,P,P_x,P_m,-S,(+)) \\ = &W^{*\mu\nu}(Q,P,P_x,P_m,S,(+))\,. \end{aligned} \tag{209}$$

Since Q^μ, U^μ and X^μ depend only on the momentum 4-vectors one has

$$\begin{aligned} Q^\mu &\xrightarrow{\mathcal{TP}} Q^\mu \\ U^\mu &\xrightarrow{\mathcal{TP}} U^\mu \\ X^\mu &\xrightarrow{\mathcal{TP}} X^\mu\,. \end{aligned} \tag{210}$$

The vectors Σ^μ, $\overline{X}^\mu$ and $\overline{U}^\mu$ are linear in S^μ and thus

$$
\begin{aligned}
\Sigma^\mu &\xrightarrow{\mathcal{TP}} -\Sigma^\mu \\
\overline{X}^\mu &\xrightarrow{\mathcal{TP}} -\overline{X}^\mu \\
\overline{U}^\mu &\xrightarrow{\mathcal{TP}} -\overline{U}^\mu .
\end{aligned} \tag{211}
$$

The scalar I_0 is also linear in S^μ and accordingly

$$
I_0 \xrightarrow{\mathcal{TP}} -I_0\,. \tag{212}
$$

The invariant functions W_i and W'_i are real and the complex conjugation changes the sign of all factors of i.

Applying these rules to Eqs. (184,189,196,206) yields

$$
W_i(-) \xrightarrow{\mathcal{TP}} W_i(+), \qquad i = 1, \ldots, 4 \tag{213}
$$

$$
W_5(-) \xrightarrow{\mathcal{TP}} -W_5(+) \tag{214}
$$

$$
W'_i(-) \xrightarrow{\mathcal{TP}} -W'_i(+), \qquad i = 1, \ldots, 8 \tag{215}
$$

$$
W'_i(-) \xrightarrow{\mathcal{TP}} W'_i(+), \qquad i = 9, \ldots, 13\,. \tag{216}
$$

Under conditions where the boundary condition has no effect, such as the plane-wave impulse approximation, factorization approximations or where the final state is obtained through a single resonance at the energy where only the real part contributes, the invariant functions in Eqs. (214) and (215) must be zero. In such a special case this reduces the number of invariant functions from 18 to 9 with a similar reduction in the number of reponse functions. Generally speaking, however, all 18 play a role. This is the same as would be obtained by applying the multipole analysis with time-reversal only (see [5] and references therein).

In summary we have 18 invariant response functions falling into the four sectors categorized in Table 1, with the symmetric contributions entering when the incident electrons are unpolarized and the anti-symmetric contributions when

		Number	Time-Reversal
Unpolarized	Symmetric	4	Even
	Anti-symmetric	1	Odd
Polarized	Symmetric	8	Odd
	Anti-symmetric	5	Even

Table 1: This table shows the number of invariant functions falling into the four sectors according to polarization and symmetry indicating the time-reversal properties of each sector.

they are polarized, in fact, longitudinally polarized when in the ERL_e. The sectors are otherwise specified by whether or not the spin-1/2 target is unpolarized or polarized.

3.3. Specific Components of the General Hadronic Tensors

We next proceed to write explicit forms for the hadronic tensors defined above. We begin with the **symmetric, unpolarized** case given in Eq. (184) which immediately yields the following for the minimal set of components:

$$\begin{aligned}
\left(W_s^{00}\right)_{unpol} &= -\frac{1}{\rho}W_1 + \left(U^0\right)^2 W_2 + \left(X^0\right)^2 W_3 + \left(2U^0X^0\right) W_4 && (217)\\
\left(W_s^{01}\right)_{unpol} &= \left(U^0U^1\right) W_2 + \left(X^0X^1\right) W_3 + \left(U^0X^1 + X^0U^1\right) W_4 && (218)\\
\left(W_s^{11}\right)_{unpol} &= W_1 + \left(U^1\right)^2 W_2 + \left(X^1\right)^2 W_3 + \left(2U^1X^1\right) W_4 && (219)\\
\left(W_s^{22}\right)_{unpol} &= W_1 + \left(U^2\right)^2 W_2 + \left(X^2\right)^2 W_3 + \left(2U^2X^2\right) W_4 && (220)\\
\left(W_s^{02}\right)_{unpol} &= \left(U^0U^2\right) W_2 + \left(X^0X^2\right) W_3 + \left(U^0X^2 + X^0U^2\right) W_4 && (221)\\
\left(W_s^{12}\right)_{unpol} &= \left(U^1U^2\right) W_2 + \left(X^1X^2\right) W_3 + \left(U^1X^2 + X^1U^2\right) W_4. && (222)
\end{aligned}$$

Note that, since the symmetric leptonic tensor in Eqs. (89-92) has no $\mu\nu = 02$ or 12 components, the last two hadronic contributions (Eqs. (221-222)) do not enter when the tensors are contracted, leaving a total of four terms, as expected for the situation where only the incident electrons may be polarized and the

ERL$_e$ is invoked [2]. Following the nomenclature in [2] we have

$$\begin{aligned}
W^L_{unpol} &\equiv \left(W_s^{00}\right)_{unpol} = -\frac{1}{\rho}W_1 + \left(U^0\right)^2 W_2 + \left(X^0\right)^2 W_3 \\
&\quad +2U^0X^0W_4 &(223)\\
W^T_{unpol} &\equiv \left(W_s^{22+11}\right)_{unpol} = 2W_1 + \left[\left(U^1\right)^2 + \left(U^2\right)^2\right] W_2 \\
&\quad + \left[\left(X^1\right)^2 + \left(X^2\right)^2\right] W_3 + 2\left[U^1X^1 + U^2X^2\right] W_4 &(224)\\
W^{TT}_{unpol} &\equiv \left(W_s^{22-11}\right)_{unpol} = \left[-\left(U^1\right)^2 + \left(U^2\right)^2\right] W_2 \\
&\quad + \left[-\left(X^1\right)^2 + \left(X^2\right)^2\right] W_3 + 2\left[-U^1X^1 + U^2X^2\right] W_4 &(225)\\
W^{TL}_{unpol} &\equiv 2\sqrt{2}\left(W_s^{01}\right)_{unpol} = 2\sqrt{2}\left[U^0U^1W_2 + X^0X^1W_3\right. \\
&\quad \left. + \left(U^0X^1 + X^0U^1\right) W_4\right]. &(226)
\end{aligned}$$

Next, for the **anti-symmetric, unpolarized** case we have the following from Eq. (189):

$$\left(W_a^{02}\right)_{unpol} = iW_5\left(U^0X^2 - X^0U^2\right) \quad (227)$$

$$\left(W_a^{12}\right)_{unpol} = iW_5\left(U^1X^2 - X^1U^2\right), \quad (228)$$

yielding

$$W^{TL'}_{unpol} \equiv 2\sqrt{2}\left(iW_a^{02}\right)_{unpol} = -2\sqrt{2}W_5\left(U^0X^2 - X^0U^2\right) \quad (229)$$

$$W^{T'}_{unpol} \equiv 2\left(iW_a^{12}\right)_{unpol} = -2W_5\left(U^1X^2 - X^1U^2\right). \quad (230)$$

These can all contribute in a situation where the incident electron is polarized. However, note the following: if mass terms in the electron tensor are retained (even in the PWBA) then one finds that the TL' and T' contributions are of leading order whereas the $\underline{TL}'$ contributions go as $1/\gamma_e$ or $1/\gamma'_e$ where $\gamma_e = \epsilon/m_e$ and $\gamma'_e = \epsilon'/m_e$ and hence may usually be neglected at high energies, leaving only the TL' and T' contributions.

Next we consider the contributions that arise from contractions of the symmetric leptonic tensor with the symmetric hadronic tensor for the case where the target is polarized – the **symmetric, polarized** case. From the developments in the last section we find that the following contributions enter in this sector:

$$\begin{aligned}
W^{L}_{pol} &\equiv \left(W^{00}_{s}\right)_{pol} \\
&= \left\{-W'_1/\rho + \left(U^0\right)^2 W'_2 + \left(X^0\right)^2 W'_3 + 2U^0X^0W'_4\right\} I_0 \\
&\quad +2\left\{U^0\bar{U}^0W'_5 + U^0\overline{X}^0W'_6 + X^0\bar{U}^0W'_7 + X^0\overline{X}^0W'_8\right\}
\end{aligned} \tag{231}$$

$$\begin{aligned}
W^{T}_{pol} &\equiv \left(W^{22}_{s} + W^{11}_{s}\right)_{pol} = \left\{2W'_1 + \left(\left(U^2\right)^2 + \left(U^1\right)^2\right) W'_2\right. \\
&\quad + \left.\left(\left(X^2\right)^2 + \left(X^1\right)^2\right) W'_3 + 2\left(U^2X^2 + U^1X^1\right) W'_4\right\} I_0 \\
&\quad +2\left\{\left(U^2\bar{U}^2 + U^1\bar{U}^1\right) W'_5 + \left(U^2\overline{X}^2 + U^1\overline{X}^1\right) W'_6\right. \\
&\quad + \left.\left(X^2\bar{U}^2 + X^1\bar{U}^1\right) W'_7 + \left(X^2\overline{X}^2 + X^1\overline{X}^1\right) W'_8\right\}
\end{aligned} \tag{232}$$

$$\begin{aligned}
W^{TT}_{pol} &\equiv \left(W^{22}_{s} - W^{11}_{s}\right)_{pol} = \left\{\left(\left(U^2\right)^2 - \left(U^1\right)^2\right) W'_2\right. \\
&\quad + \left.\left(\left(X^2\right)^2 - \left(X^1\right)^2\right) W'_3 + 2\left(U^2X^2 - U^1X^1\right) W'_4\right\} I_0 \\
&\quad +2\left\{\left(U^2\bar{U}^2 - U^1\bar{U}^1\right) W'_5 + \left(U^2\overline{X}^2 - U^1\overline{X}^1\right) W'_6\right. \\
&\quad + \left.\left(X^2\bar{U}^2 - X^1\bar{U}^1\right) W'_7 + \left(X^2\overline{X}^2 - X^1\overline{X}^1\right) W'_8\right\}
\end{aligned} \tag{233}$$

$$\begin{aligned}
W^{TL}_{pol} &\equiv 2\sqrt{2}\left(W^{01}_{s}\right)_{pol} \\
&= 2\sqrt{2}\left[\left\{U^0U^1W'_2 + X^0X^1W'_3 + \left(U^0X^1 + U^1X^0\right) W'_4\right\} I_0\right. \\
&\quad + \left(U^0\bar{U}^1 + U^1\bar{U}^0\right) W'_5 + \left(U^0\overline{X}^1 + U^1\overline{X}^0\right) W'_6 \\
&\quad + \left.\left(X^0\bar{U}^1 + X^1\bar{U}^0\right) W'_7 + \left(X^0\overline{X}^1 + X^1\overline{X}^0\right) W'_8\right]
\end{aligned} \tag{234}$$

following conventional notation.

Finally, we need to develop the **anti-symmetric, polarized** case. From

Eq. (206) we have that

$$\begin{aligned}
\left(W_a^{02}\right)_{pol} &= i\left[\frac{1}{M}W_9'\epsilon^{02\alpha\beta}\Sigma_\alpha Q_\beta\right. \\
&+W_{10}'(U^0\bar{U}^2-U^2\bar{U}^0)+W_{11}'(U^0\overline{X}^2-U^2\overline{X}^0) \\
&\left.+\,W_{12}'(X^0\bar{U}^2-X^2\bar{U}^0)+W_{13}'(X^0\overline{X}^2-X^2\overline{X}^0)\right] \qquad (235) \\
\left(W_a^{12}\right)_{pol} &= i\left[\frac{1}{M}W_9'\epsilon^{12\alpha\beta}\Sigma_\alpha Q_\beta\right. \\
&+W_{10}'(U^1\bar{U}^2-U^2\bar{U}^1)+W_{11}'(U^1\overline{X}^2-U^2\overline{X}^1) \\
&\left.+\,W_{12}'(X^1\bar{U}^2-X^2\bar{U}^1)+W_{13}'(X^1\overline{X}^2-X^2\overline{X}^1)\right], \qquad (236)
\end{aligned}$$

where no cases with components $\mu\nu = 03$, 13 or 23 are needed, since they can be eliminated using the continuity equation. These yield the three possible responses

$$\begin{aligned}
W_{pol}^{T'} &\equiv 2\left(iW_a^{12}\right)_{pol} \qquad (237) \\
&= -2\left[\frac{1}{M}W_9'\epsilon^{12\alpha\beta}\Sigma_\alpha Q_\beta\right. \\
&+W_{10}'(U^1\bar{U}^2-U^2\bar{U}^1)+W_{11}'(U^1\overline{X}^2-U^2\overline{X}^1) \\
&\left.+\,W_{12}'(X^1\bar{U}^2-X^2\bar{U}^1)+W_{13}'(X^1\overline{X}^2-X^2\overline{X}^1)\right] \qquad (238) \\
W_{pol}^{TL'} &\equiv 2\sqrt{2}\left(iW_a^{02}\right)_{pol} \qquad (239) \\
&= -2\sqrt{2}\left[\frac{1}{M}W_9'\epsilon^{02\alpha\beta}\Sigma_\alpha Q_\beta\right. \\
&+W_{10}'(U^0\bar{U}^2-U^2\bar{U}^0)+W_{11}'(U^0\overline{X}^2-U^2\overline{X}^0) \\
&\left.+\,W_{12}'(X^0\bar{U}^2-X^2\bar{U}^0)+W_{13}'(X^0\overline{X}^2-X^2\overline{X}^0)\right]. \qquad (240)
\end{aligned}$$

It can be shown that, upon knowing the responses $W^J_{unpol,pol}$ with $J = L, T, TT, TL, T', TL'$ and making use of the fact that the target polarization can be arranged to point in various directions, one can invert to obtain the invariant response functions W_i for $i = 1, 5$ and W_i' for $i = 1, 13$; see Appendix C.

This completes the general structure of both the leptonic and hadronic tensors in a general frame where the spin-1/2 target is polarized and moving with some general 4-momentum P^μ.

4. Semi-inclusive Cross Section for Electron Scattering from a Polarized Spin-1/2 Target

The full semi-inclusive electron scattering cross section in a general frame of reference may be written in terms of the Mott cross section, some kinematic factors that arise from using the Feynman rules [7], together with a general response function $\mathcal{F}^{semi}$. We begin the discussion in this section by introducing useful notation for the kinematic variables involved in semi-inclusive scattering.

4.1. Kinematics for Semi-inclusive Scattering

As discussed above, we are assuming that the initial state has two particles of masses m_e and M with 4-momenta $K^\mu = (\epsilon, \mathbf{k})$ and $P^\mu = (E, \mathbf{p})$, where $\epsilon = \sqrt{k^2 + m_e^2}$ and $E = \sqrt{p^2 + M^2}$, respectively, which collide, leaving a particle of mass m_e with 4-momentum $K'^\mu = (\epsilon', \mathbf{k}')$ where $\epsilon' = \sqrt{k'^2 + m_e^2}$ and producing a final state with 4-momentum $P'^\mu = (E', \mathbf{p}')$ and hence invariant mass $W = \sqrt{E'^2 - p'^2}$. In turn, the final state is assumed to be divided into two pieces, one the specific particle "x" that is assumed to be detected, having 4-momentum $P_x^\mu = (E_x, \mathbf{p}_x)$, where $E_x = \sqrt{p_x^2 + M_x^2}$, together with the undetected ("missing") parts of the final state having 4-momentum $P_m^\mu = (E_m^{tot}, \mathbf{p}_m)$ with missing energy E_m^{tot}, missing momentum $\mathbf{p}_m$, and invariant mass $W_m = \sqrt{\left(E_m^{tot}\right)^2 - p_m^2}$. Note: for the *total* missing energy we use E_m^{tot}, since we reserve the notation E_m to denote a different, but related quantity (see below). See Fig. 3 where conservation of 4-momentum requires that

$$Q^\mu + P^\mu = P'^\mu = P_x^\mu + P_m^\mu,$$

and thus

$$E_m^{tot} = E' - E_x \tag{241}$$

$$\mathbf{p}_m = \mathbf{p}' - \mathbf{p}_x. \tag{242}$$

From above we have that

$$P_m^\mu = Q^\mu + P^\mu - P_x^\mu \tag{243}$$

and therefore that

$$E_m^{tot} = \omega + E - E_x \tag{244}$$

$$\mathbf{p}_m = \mathbf{p}' - \mathbf{p}_x. \tag{245}$$

Following the procedures adopted in studies of scaling [8] let us employ as independent kinematic variables the missing momentum $\mathbf{p}_m$ and, rather than the missing energy E_m, the following energy

$$\mathcal{E}_m(p_m) \equiv E_m^{tot} - \left(E_m^{tot}\right)_T \geq 0 \tag{246}$$

$$= \sqrt{W_m^2 + p_m^2} - \sqrt{\left(W_m^T\right)^2 + p_m^2}, \tag{247}$$

where the threshold value of the invariant mass of the missing momentum is denoted W_m^T; examples of this are given later. This quantity has the merit of taking on the value $\mathcal{E}_m = 0$ at threshold. When used in the context of nuclear physics the missing 3-momentum is typically much smaller than the invariant masses of either the daughter threshold value (often the daughter ground-state mass) or any higher-energy daughter state and thus Eq. (247) may be written

$$\mathcal{E}_m(p_m) = W_m\sqrt{1 + \left(\frac{p_m}{W_m}\right)^2} - W_m^T\sqrt{1 + \left(\frac{p_m}{W_m^T}\right)^2} \tag{248}$$

$$= W_m\left(1 + \frac{p_m^2}{2W_m^2} + \cdots\right) - W_m^T\left(1 + \frac{p_m^2}{2\left(W_m^T\right)^2} + \cdots\right) \tag{249}$$

$$= \left(W_m - W_m^T\right)\left[1 - \delta_m + \cdots\right] \tag{250}$$

where

$$\delta_m \equiv \frac{p_m^2}{2W_m W_m^T} \ll 1 \tag{251}$$

typically. Often setting δ_m to zero is an excellent approximation; this correction involves only the difference between the kinetic energy of recoil when the daughter system is at threshold and when it is in some excited state. However, it is not necessary ever to make these approximations and the exact expressions can alway be employed.

In studies of nuclear physics it is common to define a different quantity (confusingly also called the missing energy) where kinetic energies are employed,

E_m. Defining the kinetic energies

$$T \equiv E - M \tag{252}$$

$$T_x \equiv E_x - M_x \tag{253}$$

$$T_m \equiv E_m^{tot} - W_m, \tag{254}$$

one has

$$E_m \equiv \omega - (T_x + T_m) \tag{255}$$

$$= \left(W_m - W_m^T\right) + E_s - T \tag{256}$$

$$\simeq \mathcal{E}_m(p_m) + E_s - T, \tag{257}$$

where the so-called separation energy

$$E_s \equiv M_x + W_m^T - M \geq 0 \tag{258}$$

has been introduced and the approximation in the third equation above corresponds to neglecting the correction involving δ_m discussed above.

Using the energy conservation condition in Eq. (244) we have

$$\mathcal{E}_m(p_m) = (E + \omega) - \left(E_m^{tot}\right)_T - \sqrt{M_x^2 + p'^2 + p_m^2 - 2p_m p' \cos\theta_m}, \tag{259}$$

where θ_m is the angle between $\mathbf{p}'$ and $\mathbf{p}_m$ and $p_m = |\mathbf{p}_m|$. By setting $\mathcal{E}_m$ to zero and solving the above equation for p_m under the limiting conditions where $\cos\theta_m = \pm 1$ it is straightforward to show that the above equation at $\mathcal{E}_m = 0$ has two solutions

$$p_m^+ \equiv Y = \frac{1}{W^2}\left[(E+\omega)\sqrt{\Lambda^2 - W^2\left(W_m^T\right)^2} + p'\Lambda\right] \tag{260}$$

$$-p_m^- \equiv y = \frac{1}{W^2}\left[(E+\omega)\sqrt{\Lambda^2 - W^2\left(W_m^T\right)^2} - p'\Lambda\right], \tag{261}$$

where, following the notation of [8] we have introduced the quantity

$$\Lambda \equiv \frac{1}{2}\left[W^2 + \left(W_m^T\right)^2 - M_x^2\right]. \tag{262}$$

Note that the quantity in the square root may be written

$$\Lambda^2 - W^2\left(W_m^T\right)^2 = \frac{1}{4}\left[W^2 - \left(W_m^T + M_x\right)^2\right]\left[W^2 - \left(W_m^T - M_x\right)^2\right] \tag{263}$$

and, since the argument of the square root must be non-negative, that

$$W \geq W^T = W_m^T + M_x. \tag{264}$$

Upon setting $y = 0$ one finds that

$$\omega = \omega_0 \equiv \sqrt{M_x^2 + q^2} + W_m^T - M. \tag{265}$$

Given these relationships it is then straightforward to determine the physically allowed regions in the $\mathcal{E}_m$-p_m plane: for $y \geq 0$ corresponding to $\omega \geq \omega_0$ one has

$$\begin{aligned} &\mathcal{E}_m^0(-p_m) \leq \mathcal{E}(p_m) \leq \mathcal{E}_m^0(p_m) && \text{for } 0 \leq p_m \leq y \\ &0 \leq \mathcal{E}(p_m) \leq \mathcal{E}_m^0(p_m) && \text{for } y \leq p_m \leq Y, \end{aligned} \tag{266}$$

while for $y \leq 0$ corresponding to $\omega \leq \omega_0$ one has

$$0 \leq \mathcal{E}(p_m) \leq \mathcal{E}_m^0(p_m) \quad \text{for } -y \leq p_m \leq Y, \tag{267}$$

where

$$\mathcal{E}_m^0(p_m) \equiv (E + \omega) - \left(E_m^{tot}\right)_T - \sqrt{M_x^2 + (p' - p_m)^2}, \tag{268}$$

namely, the value of $\mathcal{E}_m(p_m)$ when $\cos\theta_m = +1$. These regions are shown in Figs. 4 and 5. The region in Fig. 5 is seen to be bounded from below by the curve $\mathcal{E}_m^0(-p_m)$ which occurs when $\theta_m = \pi$ and above by the curve $\mathcal{E}_m^0(p_m)$ which occurs when $\theta_m = 0$ for $0 \leq p_m \leq y$, while the other regions are all bounded by zero from below and by the curve $\mathcal{E}_m^0(p_m)$ from above. When $\mathcal{E}_m(p_m) = 0$ one has from Eq. (259) that

$$\cos\theta_m = \frac{1}{2p_m p'} \left\{ M_x^2 + p'^2 + p_m^2 - \left[(E + \omega) - \left(E_m^{tot}\right)_T \right]^2 \right\}, \tag{269}$$

which determines θ_m for this boundary.

Thus we have the allowed regions of kinematics in the $\mathcal{E}_m$-p_m plane for given values of q and ω or, equivalently, of Q^2 and $\omega = \nu$ or q and y, where $y = y\,(q, \omega)$ given above is often used to replace ω in scaling analyses [8]. In turn these impose limits on the allowed values of the energy, 3-momentum and polar angle for the detected particle x: first, taking the scalar and cross product

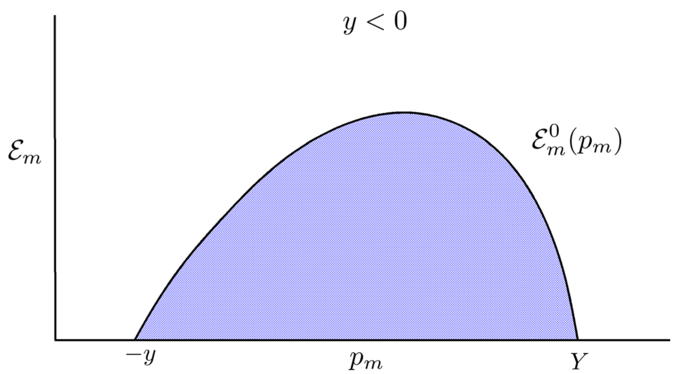

Figure 4: Physically allowed region for the situation where $y < 0$. The variables employed here are discussed in the text.

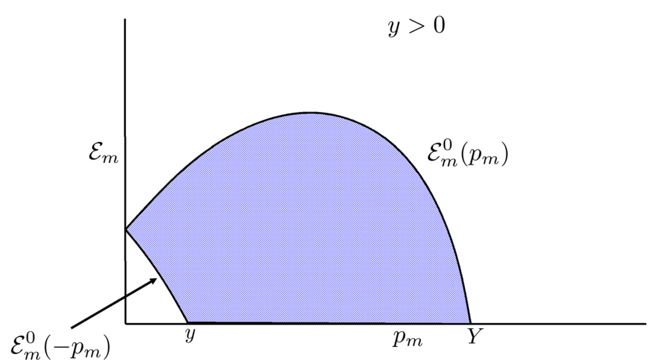

Figure 5: Physically allowed region for the situation where $y > 0$. The variables employed here are discussed in the text.

of $\mathbf{p}'$ with $\mathbf{p}_x = \mathbf{p}' - \mathbf{p}_m$ yields

$$
\begin{aligned}
p_x \cos\theta_x &= p' - p_m \cos\theta_m && (270)\\
p_x \sin\theta_x &= p_m \sin\theta_m && (271)
\end{aligned}
$$

and thus

$$
\begin{aligned}
E_x &= (E+\omega) - \left(\left(E_m^{tot}\right)_T + \mathcal{E}_m(p_m)\right) && (272)\\
p_x &= \sqrt{p'^2 + p_m^2 - 2p_m p' \cos\theta_m} && (273)\\
\tan\theta_x &= \frac{p_m \sin\theta_m}{p' - p_m \cos\theta_m}. && (274)
\end{aligned}
$$

By evaluating these expressions on the above boundaries one can then determine the physically allowed regions for P_x^μ. Let us denote the allowed region for the variables p_x (and hence E_x) and the polar angle θ_x by Γ_x. The above equations define the kinematic boundaries within which all values of (p_x, θ_x) are allowed and outside of which no physically allowed values exist. Later we discuss the roles played by the azimuthal angle ϕ_x where all values $(0, 2\pi)$ are allowed.

These results may be specialized from the general frame to the rest frame where $p = 0$ and thus $T = 0$ by making the following replacements: the energy E and the 3-momentum $\mathbf{p}'$ are replaced by M and $\mathbf{q}$, respectively, and θ_m becomes the angle between $\mathbf{q}$ and $\mathbf{p}_m$; W and Λ are Lorentz invariants and so do not change. The results one then obtains are the ones that are familiar from analyses of scaling [8].

That said, it should be noted that all of these developments are also valid for studies of particle physics at high energies.

4.2. Semi-inclusive Cross Section

Having established the allowed regions for the kinematics in semi-inclusive reactions we may now proceed to a discussion of the cross section. The Feynman rules followed in this work are those of [7]: we provide details in Appendix D of how the general expression for the six-fold semi-inclusive cross section is obtained. That general answer may be re-written in the following form to connect with the above development of the leptonic and hadronic tensors

$$\left[\frac{d^6\sigma}{d\Omega dk' dp_x d\cos\theta_x d\phi_x}\right]_x = \frac{1}{2\pi}\sigma_{\text{Mott}} f \frac{M}{E_p}\frac{p_x^2}{E_x}\left[\mathcal{F}^{semi}\right]_x \tag{275}$$

where

$$\frac{\alpha^2 v_0 k'}{Q^4 k} = \sigma_{\text{Mott}} = \left(\frac{\alpha\cos\theta_e/2}{2\epsilon\sin^2\theta_e/2}\right)^2 \tag{276}$$

is the Mott cross section and $\left[\mathcal{F}^{semi}\right]_x$ is the invariant called $\mathcal{C} = \chi_{\mu\nu}W^{\mu\nu}$ divided by the factor v_0, namely

$$\begin{aligned} \left[\mathcal{F}^{semi}\right]_x &= \chi_{\mu\nu}W_x^{\mu\nu}/v_0 & (277)\\ &= v_L\left[W_x^L\right]^{semi} + v_T\left[W_x^T\right]^{semi} + \cdots & (278) \end{aligned}$$

as discussed below and where the subscript "x" has been added to remind us that this forms the semi-inclusive cross section where particle x is assumed to be detected. The factor M/E_p arises from applying the Feynman rules in a general frame where the target is moving; this factor becomes unity in the target rest frame. Furthermore, the factor [9, 10]

$$f = \left[(\boldsymbol{\beta}_e - \boldsymbol{\beta}_p)^2 - (\boldsymbol{\beta}_e \times \boldsymbol{\beta}_p)^2\right]^{-1/2}, \tag{279}$$

with $\boldsymbol{\beta}_e = \boldsymbol{k}/\epsilon$ and $\boldsymbol{\beta}_p = \mathbf{p}/E_p$ as usual, accounts for the flux of the (in general colliding) beams. In the rest frame one has $\beta_p = 0$ and thus $f^R = 1/\beta_e$ which equals unity in the ERL$_e$.

In Eq. (275) a specific choice has been made for the normalization. In particular, while any constants or Lorentz invariants could be absorbed into the definitions of the invariant functions we choose to fix the conventions so that upon integrating the semi-inclusive cross section over the detected particle's 3-momentum and summing over all open channels, *i.e.,* all particles x while taking care not to double-count, one should recover the inclusive cross section with its conventional normalization. That is, to obtain the contribution of the channel having particle x to the inclusive cross section one should perform the integral over p_x, $\cos\theta_x$ and ϕ_x over the allowed physical region for the semi-inclusive reaction $(e, e'x)$ (see above for detailed discussion concerning the allowed region)

$$\left[\frac{d^2\sigma}{d\Omega dk'}\right]_x = \left\{\int dp_x \int d\cos\theta_x \int_0^{2\pi} d\phi_x \left[\frac{d^6\sigma}{d\Omega dk' dp_x d\cos\theta_x d\phi_x}\right]_x\right\}_{\text{allowed}} \tag{280}$$

$$= \frac{1}{2\pi}\sigma_{\text{Mott}} f \frac{M}{E_p}\left\{\int dp_x \frac{p_x^2}{E_x}\int d\cos\theta_x \left[\mathcal{G}^{semi}\right]_x\right\}_{\text{allowed}}, \tag{281}$$

where

$$\left[\mathcal{G}^{semi}\right]_x \equiv \int_0^{2\pi} d\phi_x \left[\mathcal{F}^{semi}\right]_x. \tag{282}$$

Then the full inclusive cross section is obtained by summing over all open channels, taking care not to double-count (see below for examples):

$$\frac{d^2\sigma}{d\Omega dk'} = \widehat{\sum_x}\left[\frac{d^2\sigma}{d\Omega dk'}\right]_x, \tag{283}$$

where the requirement not to double-count is indicated by the hat over the summation. In the next section the full inclusive cross section is also written in the form

$$\frac{d^2\sigma}{d\Omega dk'} = \sigma_{\text{Mott}} f \frac{M}{E_p}\mathcal{R}^{incl}, \tag{284}$$

where

$$\mathcal{R}^{incl} = R_1^{incl} + \cdots \tag{285}$$

and

$$R_1^{incl} = \left[v_L R^L_{unpol}\right]^{incl} + \cdots \tag{286}$$

Clearly the integral over ϕ_x for contributions that have no explicit ϕ_x-dependence simply accounts for the factor 2π put in the denominator above.

One may now change variables in the following ways. Beginning again in the rest frame, since $\mathbf{p}_m = \mathbf{q} - \mathbf{p}_x$ and we are keeping q constant, one has

$$p_x^2 dp_x d\cos\theta_x = p_m^2 dp_m d\cos\theta_m \tag{287}$$

and thus the semi-inclusive cross section may be written as differential in the missing-momentum plus changing p_x^2 to p_m^2. Since we have

$$\mathcal{E}_m(p_m) = (M+\omega) - \left(E_m^{tot}\right)_T - \sqrt{M_x^2 + q^2 + p_m^2 - 2p_m q\cos\theta_m} \tag{288}$$

from the discussions above, we can change variables from $\cos\theta_m$ to $\mathcal{E}$:

$$\left[\frac{\partial\mathcal{E}_m}{\partial\cos\theta_m}\right]_{p_m} = \frac{p_m q}{E_x} \tag{289}$$

and so

$$\left[\frac{d^6\sigma}{d\Omega dk' dp_m d\mathcal{E}_m d\phi_x}\right]_x = \frac{1}{2\pi q}\sigma_{\mathrm{Mott}} f \frac{M}{E_p} p_m \left[\mathcal{F}^{semi}\right]_x . \tag{290}$$

To form the inclusive cross section one may then proceed to integrate over p_m, $\mathcal{E}_m$ and ϕ_x (which is unchanged from the previous treatment), where now the physical region defining the boundaries in the $(p_m, \mathcal{E}_m)$-plane are those discussed above.

In a general frame, as above, the mass M and the 3-momentum transfer $\mathbf{q}$ are replaced by $E_p = \sqrt{M^2 + p^2}$ and $\mathbf{p}' \equiv \mathbf{q} + \mathbf{p}$, respectively, and θ_m becomes the angle between $\mathbf{p}'$and $\mathbf{p}_m$.

As discussed in detail above where the invariant response functions have been developed, the overall response can be decomposed into the four sectors that are classified by the types of polarization they involve

$$\mathcal{F}^{semi} = \mathcal{F}_1^{semi} + h\mathcal{F}_2^{semi} + h^*\mathcal{F}_3^{semi} + hh^*\mathcal{F}_4^{semi}. \tag{291}$$

In the semi-inclusive case, as we have seen earlier, the responses here depend on four scalar invariants, Q^2, $I_{1,2,3}$, together with the kinematic variables that enter through the lepton tensor. Clearly again the four sectors can be separated by flipping the electron helicity h and the direction of the target spin via the factor h^*. Explicitly we have

$$\begin{aligned}
\mathcal{F}_1^{semi} &= v_L \left[W_{unpol}^L\right]^{semi} + v_T \left[W_{unpol}^T\right]^{semi} \\
&\quad + v_{TT} \left[W_{unpol}^{TT}\right]^{semi} + v_{TL} \left[W_{unpol}^{TL}\right]^{semi} &(292)\\
h\mathcal{F}_2^{semi} &= v_{T'} \left[W_{unpol}^{T'}\right]^{semi} + v_{TL'} \left[W_{unpol}^{TL'}\right]^{semi} &(293)\\
h^*\mathcal{F}_3^{semi} &= v_L \left[W_{pol}^L\right]^{semi} + v_T \left[W_{pol}^T\right]^{semi} \\
&\quad + v_{TT} \left[W_{pol}^{TT}\right]^{semi} + v_{TL} \left[W_{pol}^{TL}\right]^{semi} &(294)\\
hh^*\mathcal{F}_4^{semi} &= v_{T'} \left[W_{pol}^{T'}\right]^{semi} + v_{TL'} \left[W_{pol}^{TL'}\right]^{semi} . &(295)
\end{aligned}$$

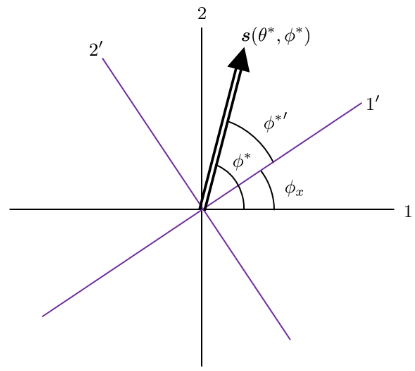

Figure 6: Two coordinate systems for the target spin. The original coordinate system is shown in Fig. 1 and here one can see how the primed system is related via a rotation around the 3-direction (the direction of the 3-momentum transfer $\mathbf{q}$) by the azimuthal angle ϕ_x. Hence in the 123-system the azimuthal angle of the target spin is ϕ^*, while in the $1'2'3'$-system it is $\phi^{*\prime} = \phi^* - \phi_x$.

Here the responses $\left[W^K_{unpol}\right]^{semi}$ and $\left[W^K_{pol}\right]^{semi}$ with $K = L$, T, TL, TT, T' and TL' are the semi-inclusive quantities developed earlier, now with the label *semi* appended to distinguish them from the inclusive responses discussed above. As we found earlier, $\mathcal{F}^{semi}_{1,4}$ are TRE while $\mathcal{F}^{semi}_{2,3}$ are TRO. In turn, the individual responses are built from the 18 invariant response functions W_m, $m = 1, \ldots, 5$ and W'_m, $m = 1, \ldots, 13$. Note: the invariant responses here are for *semi-inclusive* scattering and depend on the four chosen scalar invariants; these quantities should not be confused with the *inclusive* invariant response functions discussed below.

4.3. Two Coordinate Systems for the Target Spin

We will have occasion to use two different coordinate system to specify the axis of quantization for the target spin. In the discussions above we chose the lepton-plane oriented coordinate system where $\mathbf{q}$ is along the 3-axis and the 2-axis is normal to the electron scattering plane (see Fig. 1). It proves to be convenient to introduce a rotated (around the 3-direction) coordinate system which we denote with primes, namely one with $3'$-axis along $\mathbf{q}$ and $2'$-

axis normal to the plane formed by $\mathbf{q}$ and $\mathbf{p}_x$ (see Fig. 6). The reason for this choice of rotated system will become apparent in due course. The unit vectors in these two systems are related by

$$\mathbf{u}_{1'} = \cos\phi_x \mathbf{u}_1 + \sin\phi_x \mathbf{u}_2 \tag{296}$$

$$\mathbf{u}_{2'} = -\sin\phi_x \mathbf{u}_1 + \cos\phi_x \mathbf{u}_2 \tag{297}$$

$$\mathbf{u}_{3'} = \mathbf{u}_3 \tag{298}$$

and the inverse

$$\mathbf{u}_1 = \cos\phi_x \mathbf{u}_{1'} - \sin\phi_x \mathbf{u}_{2'} \tag{299}$$

$$\mathbf{u}_2 = \sin\phi_x \mathbf{u}_{1'} + \cos\phi_x \mathbf{u}_{2'} \tag{300}$$

$$\mathbf{u}_3 = \mathbf{u}_{3'}. \tag{301}$$

One has that

$$\mathbf{q}_R = q_R \mathbf{u}_3 = q_R \mathbf{u}_{3'} \tag{302}$$

while

$$\mathbf{p}_x = p_x \left[\sin\theta_x \mathbf{u}_{1'} + \cos\theta_x \mathbf{u}_{3'}\right] \tag{303}$$

with no $2'$ component, by construction. A simple result (which we use below) is accordingly

$$\mathbf{q} \times \mathbf{p}_x = q p_x \sin\theta_x \left(-\sin\phi_x \mathbf{u}_1 + \cos\phi_x \mathbf{u}_2\right) \tag{304}$$

$$= q p_x \sin\theta_x \mathbf{u}_{2'}, \tag{305}$$

namely having only a $2'$ component. The spin 4-vector may then be written in either the 123 system or the $1'2'3'$ system. One may define projections of the spin 3-vector in the two systems in the following way: the L, S and N directions are obtained by setting $\theta^* = 0$ (for L), $\theta^* = \pi/2$ with $\phi^* = 0$ (for S) and $\phi^* = \pi/2$ (for N), namely, making projections along the 123 system unit vectors

$$\mathcal{P}_L \equiv \mathbf{u}_3 \cdot \mathbf{s} = h^* s \cos\theta^* \tag{306}$$

$$\mathcal{P}_S \equiv \mathbf{u}_1 \cdot \mathbf{s} = h^* s \sin\theta^* \cos\phi^* \tag{307}$$

$$\mathcal{P}_N \equiv \mathbf{u}_2 \cdot \mathbf{s} = h^* s \sin\theta^* \sin\phi^* \tag{308}$$

or doing the same, but for the unit vectors in the $1'2'3'$ system

$$\begin{aligned}
\mathcal{P}_{L'} &\equiv \mathbf{u}_{3'} \cdot \mathbf{s} = h^* s \cos\theta^* && (309)\\
\mathcal{P}_{S'} &\equiv \mathbf{u}_{1'} \cdot \mathbf{s} = h^* s \sin\theta^* \cos\phi^{*'} && (310)\\
\mathcal{P}_{N'} &\equiv \mathbf{u}_{2'} \cdot \mathbf{s} = h^* s \sin\theta^* \sin\phi^{*'}. && (311)
\end{aligned}$$

The magnitude of the spin 3-vector is given in Eq. (126). Using the relationships amongst the unit vectors above one has that

$$\begin{aligned}
\mathcal{P}_L &= \mathcal{P}_{L'} && (312)\\
\mathcal{P}_S &= \cos\phi_x \mathcal{P}_{S'} - \sin\phi_x \mathcal{P}_{N'} && (313)\\
\mathcal{P}_N &= \sin\phi_x \mathcal{P}_{S'} + \cos\phi_x \mathcal{P}_{N'} && (314)\\
\mathcal{P}_{S'} &= \cos\phi_x \mathcal{P}_S + \sin\phi_x \mathcal{P}_N && (315)\\
\mathcal{P}_{N'} &= -\sin\phi_x \mathcal{P}_S + \cos\phi_x \mathcal{P}_N. && (316)
\end{aligned}$$

Note that $\mathcal{P}_{L'} = \mathcal{P}_L$ contains no dependence on ϕ_x.

5. Inclusive Cross Section

For inclusive scattering one simply needs to eliminate all contributions that contain the 4-vectors V^μ or X^μ, as well as the invariant I_0 as they involve the 4-vector P_x^μ which does not enter in the inclusive case. All invariant response functions depend only on two scalar quantities, for example, Q^2 and $Q \cdot P = Q^2 I_1$. Accordingly one obtains the following:

$$\begin{aligned}
(W_s^{\mu\nu})^{incl}_{unpol} &= -(W_1)^{incl}\left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right) + (W_2)^{incl}\, U^\mu U^\nu && (317)\\
(W_a^{\mu\nu})^{incl}_{unpol} &= 0 && (318)\\
(W_s^{\mu\nu})^{incl}_{pol} &= (W'_6)^{incl}\left(U^\mu \overline{X}^\nu + U^\nu \overline{X}^\mu\right) && (319)\\
-i\,(W_a^{\mu\nu})^{incl}_{pol} &= \frac{1}{M}(W'_9)^{incl}\,\epsilon^{\mu\nu\alpha\beta}\Sigma_\alpha Q_\beta \\
&\quad + (W'_{11})^{incl}\,(U^\mu \overline{X}^\nu - U^\nu \overline{X}^\mu) && (320)
\end{aligned}$$

with 5 inclusive invariant functions $(W_m)^{incl}$, $m = 1, 2$ and $(W'_m)^{incl}$, $m = 6, 9, 11$. Using our previous results for semi-inclusive scattering but now dropping all contributions containing V^μ or X^μ we obtain the following: for the

symmetric, unpolarized cases (now not continuing to develop the $\underline{TL}$ and $\underline{TT}$ cases)

$$
\begin{aligned}
\left[W^L_{unpol}\right]^{incl} &= -\frac{1}{\rho}\left(W_1\right)^{incl} + \left(U^0\right)^2\left(W_2\right)^{incl} && (321)\\
\left[W^T_{unpol}\right]^{incl} &= 2\left(W_1\right)^{incl} + \left[\left(U^1\right)^2 + \left(U^2\right)^2\right]\left(W_2\right)^{incl} && (322)\\
\left[W^{TT}_{unpol}\right]^{incl} &= \left[-\left(U^1\right)^2 + \left(U^2\right)^2\right]\left(W_2\right)^{incl} && (323)\\
\left[W^{TL}_{unpol}\right]^{incl} &= 2\sqrt{2}U^0U^1\left(W_2\right)^{incl}, && (324)
\end{aligned}
$$

no results for the **anti-symmetric, unpolarized** case

$$
\left(W^{\mu\nu}_a\right)^{incl}_{unpol} = 0, \qquad (325)
$$

as can be seen above in discussing the semi-inclusive responses. All contributions there contained explicit factors involving V^μ or X^μ; in fact, potential contributions of this type are parity-violating when electrons are polarized longitudinal or sideways. For the **symmetric, polarized** cases (now not continuing to develop the $\underline{TL}$ and $\underline{TT}$ cases) we have

$$
\begin{aligned}
\left[W^L_{pol}\right]^{incl} &= U^0\overline{X}^0W_6' && (326)\\
\left[W^T_{pol}\right]^{incl} &= \left(U^2\overline{X}^2 + U^1\overline{X}^1\right)W_6' && (327)\\
\left[W^{TT}_{pol}\right]^{incl} &= \left(U^2\overline{X}^2 - U^1\overline{X}^1\right)W_6' && (328)\\
\left[W^{TL}_{pol}\right]^{incl} &= 2\sqrt{2}\left(U^0\overline{X}^1 + U^1\overline{X}^0\right)W_6', && (329)
\end{aligned}
$$

all of which are proportional to the same invariant response function W_6'. And, finally, for the **anti-symmetric, polarized** situation (now not continuing to develop the $\underline{TL}'$ case, although it is very similar to the TL' case below, simply having 2 replaced by 1; as noted earlier, this term can occur when only the incident electron is polarized but when the scattered electron's polarization is not measured although the leptonic factor goes as $1/\gamma$ and hence this contribution may be safely neglected at high energies – we do so in the following) we have

$$\begin{aligned} \left[W_{pol}^{T'}\right]^{incl} &= -2\left[\frac{1}{M}\left(W_9'\right)^{incl}\epsilon^{12\alpha\beta}\Sigma_\alpha Q_\beta \right. \\ &\quad \left. +\left(W_{11}'\right)^{incl}\left(U^1\overline{X}^2-\overline{X}^1U^2\right)\right] \qquad (330) \\ \left[W_{pol}^{TL'}\right]^{incl} &= -2\sqrt{2}\left[\frac{1}{M}\left(W_9'\right)^{incl}\epsilon^{02\alpha\beta}\Sigma_\alpha Q_\beta \right. \\ &\quad \left. +\left(W_{11}'\right)^{incl}\left(U^0\overline{X}^2-\overline{X}^0U^2\right)\right]. \qquad (331) \end{aligned}$$

In total we find that 5 invariant response functions enter, $W_{1,2}$ and $W'_{9,11}$ in contributions that are TRE, plus the contributions that involve the invariant response function W'_6 and are TRO.

The general inclusive cross section may then be written in the following form:

$$\frac{d^2\sigma}{d\Omega_e dk'} \equiv \sigma_{Mott} f \frac{M}{E_p}\mathcal{R}^{incl} \qquad (332)$$

where σ_{Mott} is the Mott cross section given in Eq. (276) and the full inclusive response is given by

$$\mathcal{R}^{incl} = \mathcal{R}_1^{incl} + h\mathcal{R}_2^{incl} + h^*\mathcal{R}_3^{incl} + hh^*\mathcal{R}_4^{incl}, \qquad (333)$$

in which the four contributions correspond to completely unpolarized, electron polarization only, target polarization only, and double polarization, respectively. As above all responses here depend on two scalar invariants such as Q^2 and $Q \cdot P$ together with the electron scattering angle θ_e which enters via the leptonic factors. Clearly the four sectors can be separated by flipping the electron helicity h and the direction of the target spin via the factor h^*. Explicitly we have

$$\begin{aligned} \mathcal{R}_1^{incl} &= v_L\left[W_{unpol}^L\right]^{incl} + v_T\left[W_{unpol}^T\right]^{incl} \\ &\quad + v_{TL}\left[W_{unpol}^{TL}\right]^{incl} + v_{TT}\left[W_{unpol}^{TT}\right]^{incl} \qquad (334) \\ h\mathcal{R}_2^{incl} &= 0 \qquad (335) \\ h^*\mathcal{R}_3^{incl} &= v_L\left[W_{pol}^L\right]^{incl} + v_T\left[W_{pol}^T\right]^{incl} \\ &\quad + v_{TL}\left[W_{pol}^{TL}\right]^{incl} + v_{TT}\left[W_{pol}^{TT}\right]^{incl} \qquad (336) \\ hh^*\mathcal{R}_4^{incl} &= v_{TL'}\left[W_{pol}^{TL'}\right]^{incl} + v_{T'}\left[W_{pol}^{T'}\right]^{incl}, \qquad (337) \end{aligned}$$

where, as above, we have dropped the small $\underline{TL}'$ contribution. The leptonic factors are given in Sect. 2.2 while the inclusive hadronic response functions are given above.

5.1. The Transition from Semi-Inclusive to Inclusive Scattering

While the above developments yield the structure of the general inclusive cross section directly, it is also instructive to follow a different strategy and proceed from the semi-inclusive cross section for a given channel (*i.e.,* for a specific particle x detected in coincidence with the scattered electron), integrating over the allowed kinematics of the 4-momentum that goes with that particle, and then summing over all open channels, of course, paying close attention to issues of double-counting.

We start with the general forms for the semi-inclusive cross section for the specific channel where particle x is assumed to be detected given above in Secs. 4.2 and 4.3. The dependence on the azimuthal angle ϕ_x occurs in the explicit factors $\cos\phi_x$, $\cos 2\phi_x$ and $\sin\phi_x$ in Eqs. (395,396,397) for the cases where the target is unpolarized. Clearly, upon performing the integrals over ϕ_x over the range $(0, 2\pi)$ yields zero for the TT, TL and TL' cases, verifying the above inclusive structure (see Eqs. (412), for example). The L and T cases in Eqs. (393,394) simply pick up a factor 2π when the azimuthal integral is performed. In summary, for the target unpolarized situation one finds that each channel yields only L and T responses, as we have already seen above (see Eqs. (410,411)).

The situation where the target is polarized is a little more complicated. There one finds that as well as explicit factors $\cos\phi_x$, $\cos 2\phi_x$, $\sin\phi_x$ and $\sin 2\phi_x$ in Eqs. (403,405,409) one has implicit dependence on ϕ_x via the factors $\mathcal{P}_{S'}$ and $\mathcal{P}_{N'}$ in those equations together with Eqs. (399,401,407). In this scenario it, of course, makes no sense to use the primed spin-projection variables, since the plane in which the momentum of particle x lies is being integrated over and accordingly we must go back to the original unprimed spin projections which are referred to the electron scattering frame. Two of the symmetric, polarized cases are simple: the L and T results in Eqs. (399) and (401), respectively, depend on the azimuthal angle solely through the factor $\mathcal{P}_{N'}$, which, by Eq. (316) only has dependences $\sin\phi_x$ and $\cos\phi_x$ and accordingly upon integrations over ϕ_x yield zero, in accordance with Eqs. (414). The remaining cases require somewhat more work. The symmetric TL response in Eq. (405) has three contributions

$$\begin{aligned} x_1 &\sim \cos\phi_x \mathcal{P}_{N'} = \frac{1}{2}\left[-\sin 2\phi_x \mathcal{P}_S + (1+\cos 2\phi_x)\,\mathcal{P}_N\right] && (338)\\ x_2 &\sim \sin\phi_x \mathcal{P}_{L'} = \sin\phi_x \mathcal{P}_L && (339)\\ x_3 &\sim \sin\phi_x \mathcal{P}_{S'} = \frac{1}{2}\left[\sin 2\phi_x \mathcal{P}_S + (1-\cos 2\phi_x)\,\mathcal{P}_N\right]. && (340) \end{aligned}$$

Upon integrating over ϕ_x one then finds that the x_1 and x_3 cases yield $\pi\mathcal{P}_N$, while the x_2 case yields zero, in accord with Eq. (413), namely, a nonzero result that goes as $\mathcal{P}_N$. Similarly, the symmetric TT response in Eq. (403) also has three contributions

$$\begin{aligned} y_1 &\sim \cos 2\phi_x \mathcal{P}_{N'} = \frac{1}{2}\left[-(\sin 3\phi_x - \sin\phi_x)\,\mathcal{P}_S\right.\\ &\qquad \left. + (\cos 3\phi_x + \cos\phi_x)\,\mathcal{P}_N\right] && (341)\\ y_2 &\sim \sin 2\phi_x \mathcal{P}_{L'} = \sin 2\phi_x \mathcal{P}_L && (342)\\ y_3 &\sim \sin 2\phi_x \mathcal{P}_{S'} = \frac{1}{2}\left[(\sin 3\phi_x + \sin\phi_x)\,\mathcal{P}_S\right.\\ &\qquad \left. + (-\cos 3\phi_x + \cos\phi_x)\,\mathcal{P}_N\right], && (343) \end{aligned}$$

all of which integrate to zero and yield no contribution for the TT term, in accord with Eq. (414). Next, the anti-symmetric polarized cases are handled similarly: for the T' response in Eq. (407) the contribution that involves $\mathcal{P}_{S'}$ yields zero upon integration over ϕ_x while the contribution that involves $\mathcal{P}_{L'}$ and hence no dependence on ϕ_x yields a nonzero result arising from the factor 2π coming from the integral. Thus the T' response yields a nonzero result that is proportional to $\mathcal{P}_L$, as in Eq. (415). Finally, the TL' response in Eq. (409)

involves three contributions

$$\begin{aligned} z_1 &\sim \cos\phi_x \mathcal{P}_{L'} = \cos\phi_x \mathcal{P}_L \\ z_2 &\sim \cos\phi_x \mathcal{P}_{S'} = \frac{1}{2}\left[\left(1+\cos 2\phi_x\right)\mathcal{P}_S + \sin 2\phi_x \mathcal{P}_N\right] \\ z_3 &\sim \sin\phi_x \mathcal{P}_{N'} = \frac{1}{2}\left[-\left(1-\cos 2\phi_x\right)\mathcal{P}_S + \sin 2\phi_x \mathcal{P}_N\right]. \end{aligned}$$

As above, the term involving z_1 integrates to zero, while the z_2 and z_3 terms yields factors of π and $-\pi$, respectively, and involve the spin projection $\mathcal{P}_S$, in agreement with Eq. (416). Thus exactly the structure found above when proceeding to the inclusive cross section directly is found by integrating the semi-inclusive responses over ϕ_x.

6. Rest System Variables

One can now proceed to use the general expressions given above in any coordinate system, since everything is written in covariant form. In particular, one major goal in making the developments above is to have the semi-inclusive cross section both in the collider frame and also in the target rest frame. Typically one will develop some model for the cross section in the target rest frame and thereby identify the invariant functions this entails. This then immediately yields the cross section in the general collider frame, since these response functions are, by construction, invariant, and all of the kinematic factors discussed above are covariant.

The most straightforward way to obtain the required target rest frame variables is to use the original 123-system expressions obtained above, but to assume first that $\theta = 0$ so that $\mathbf{p}$ and $\mathbf{q}$ are collinear and second that p is set to zero. One then has

$$Q_R^\mu = (\omega_R, 0, 0, q_R) = q_R\,(\nu_R', 0, 0, 1) \tag{344}$$

$$P_R^\mu = M(1,0,0,0) \tag{345}$$

$$U_R^\mu = \frac{1}{\rho_R}\,(1,0,0,\nu_R') \tag{346}$$

with

$$\nu'_R = \frac{\omega_R}{q_R} \tag{347}$$

$$\rho_R = -\frac{Q^2}{q_R^2} = 1 - {\nu'_R}^2. \tag{348}$$

Clearly one has $Q_R \cdot U_R = 0$ as required by construction. Next, one has

$$P^\mu_{x,R} = (E_{x,R}, \mathbf{p}_{x,R}) \tag{349}$$

with

$$\mathbf{p}_{x,R} = p_{x,R} \left[\sin\theta_{x,R} \left(\cos\phi_{x,R}\mathbf{u}_1 + \sin\phi_{x,R}\mathbf{u}_2\right) + \cos\theta_{x,R}\mathbf{u}_3\right] \tag{350}$$

$$E_{x,R} = \sqrt{p^2_{x,R} + M^2_x}, \tag{351}$$

which yields

$$V^\mu_R = \left(\frac{1}{M\rho_R}\mathcal{E}_{x,R}, V^1_R, V^2_R, \frac{\nu'_R}{M\rho_R}\mathcal{E}_{x,R}\right) \tag{352}$$

with

$$\mathcal{E}_{x,R} \equiv E_{x,R} - \nu'_R p_{x,R}\cos\theta_{x,R} \tag{353}$$

$$V^1_R = \frac{P^1_{x,R}}{M} = \frac{p_{x,R}}{M}\sin\theta_{x,R}\cos\phi_{x,R} = \eta_{x,R}\cos\phi_{x,R} \tag{354}$$

$$V^2_R = \frac{P^2_{x,R}}{M} = \frac{p_{x,R}}{M}\sin\theta_{x,R}\sin\phi_{x,R} = \eta_{x,R}\sin\phi_{x,R}, \tag{355}$$

where

$$\eta_{x,R} \equiv \frac{p_{x,R}}{M}\sin\theta_{x,R}, \tag{356}$$

and again, as required by construction, one has $Q_R \cdot V_R = 0$. Upon finding that

$$U^2_R = \frac{1}{\rho_R} \tag{357}$$

$$U_R \cdot V_R = \frac{\mathcal{E}_{x,R}}{M\rho_R} \tag{358}$$

and using Eq. (138) one has that

$$X^\mu_R = V^\mu_R - \left(\frac{\mathcal{E}_{x,R}}{M}\right) U^\mu_R \tag{359}$$

$$= \left(0, V^1_R, V^2_R, 0\right) = \frac{1}{M}\left(0, P^1_{x,R}, P^2_{x,R}, 0\right) \tag{360}$$

$$= \eta_{x,R}\left(0, \cos\phi_{x,R}, \sin\phi_{x,R}, 0\right). \tag{361}$$

First, we have from above that

$$S_R^0 = 0 \tag{362}$$

and that

$$\begin{aligned} \mathbf{s}_R &= h^* \left[\sin\theta_R^* \left(\cos\phi_R^* \mathbf{u}_1 + \sin\phi_R^* \mathbf{u}_2\right) + \cos\theta_R^* \mathbf{u}_3\right] & (363) \\ &= h^* \left[\sin\theta_R^* \left(\cos\phi_R^{*'} \mathbf{u}_{1'} + \sin\phi_R^{*'} \mathbf{u}_{2'}\right) + \cos\theta_R^* \mathbf{u}_{3'}\right]. & (364) \end{aligned}$$

Clearly from Fig. C.7 one has that

$$\phi_R^* = \phi_{x,R} + \phi_R^{*'}. \tag{365}$$

One may then employ Eqs. (296-316) in the rest system (indicated by adding the label R). In particular, the target spin 4-vector becomes

$$\Sigma_R^\mu = h^* \left(-\frac{\nu_R'}{\rho_R} \cos\theta_R^*, \sin\theta_R^* \cos\phi_R^*, \sin\theta_R^* \sin\phi_R^*, -\frac{{\nu_R'}^2}{\rho_R} \cos\theta_R^* \right), \tag{366}$$

using the fact that

$$Q_R \cdot S_R = -q_R h^* \cos\theta_R^*. \tag{367}$$

One has $Q_R \cdot \Sigma_R = 0$ as required by construction and also

$$U_R \cdot \Sigma_R = -h^* \frac{\nu_R'}{\rho_R} \cos\theta_R^* \tag{368}$$

as well as

$$\begin{aligned} [I_0]_R &= \frac{1}{M^2} \left(\mathbf{q}_R \times \mathbf{p}_{x,R}\right) \cdot \mathbf{s}_R & (369) \\ &= \frac{q_R p_{x,R}}{M^2} \sin\theta_{x,R} \mathcal{P}_{N'}^R & (370) \\ &= \frac{q_R}{M} \eta_{x,R} \mathcal{P}_{N'}^R & (371) \end{aligned}$$

namely, in the $1'2'3'$ system this invariant is especially simple in that it involves only the N$'$ projection of the spin, motivating the rotation from the 123 system to the $1'2'3'$ system introduced above.

Then we can find $\overline{X}_R^\mu$ in terms of these 4-vectors. We have from Eq. (166) that

$$\overline{X}^\mu = \frac{1}{M^2} \epsilon^{\mu\alpha\beta\gamma} S_\alpha Q_\beta P_\gamma, \tag{372}$$

where, using the above expressions for Q_β and P_γ, one must have $\gamma = 0$ and hence $\beta = 3$; accordingly only the cases with $\mu\alpha = 12$ and 21 occur. Using Eqs. (168) and (169), in the target rest system we then have specifically that

$$\begin{aligned}
\overline{X}^0_R &= \overline{X}^3_R = 0 && (373)\\
\overline{X}^1_R &= -\frac{1}{M}\left(\mathbf{q}_R \times \mathbf{s}_R\right)^1 = \frac{q_R}{M}\mathcal{P}^R_N = h^* \frac{q_R}{M} \sin\theta^*_R \sin\phi^*_R && (374)\\
\overline{X}^2_R &= -\frac{1}{M}\left(\mathbf{q}_R \times \mathbf{s}_R\right)^2 = -\frac{q_R}{M}\mathcal{P}^R_S = -h^* \frac{q_R}{M} \sin\theta^*_R \cos\phi^*_R, && (375)
\end{aligned}$$

namely

$$\begin{aligned}
\overline{X}^\mu_R &= h^* \frac{q_R}{M} \sin\theta^*_R \left(0, \sin\phi^*_R, -\cos\phi^*_R, 0\right) && (376)\\
&= \frac{q_R}{M}\left(0, \mathcal{P}_N, -\mathcal{P}_S, 0\right). && (377)
\end{aligned}$$

From these results in the 123 system the corresponding results in the $1'2'3'$ system are immediate:

$$\begin{aligned}
\overline{X}^{1'}_R &= h^* \frac{q_R}{M} \sin\theta^*_R \sin\phi^{*\prime}_R && (378)\\
\overline{X}^{2'}_R &= -h^* \frac{q_R}{M} \sin\theta^*_R \cos\phi^{*\prime}_R, && (379)
\end{aligned}$$

where here we use primes on the transverse Lorentz components to indicate that they are in the rotated coordinate system. Finally, we have the last remaining 4-vector $\overline{U}^\mu_R$ which is given in terms of $\overline{T}^\mu_R$ and $\overline{V}^\mu_R$:

$$\overline{U}^\mu_R = \overline{T}^\mu_R - \left(\frac{\mathcal{E}_{x,R}}{M}\right)\overline{X}^\mu_R, \qquad (380)$$

where

$$\begin{aligned}
\overline{T}^0_R &= -h^* \frac{q_R p_{x,R}}{M^2} \sin\theta_{x,R} \sin\theta^*_R \sin\left(\phi_{x,R} - \phi^*_R\right) && (381)\\
\overline{T}^3_R &= \nu'_R \overline{T}^0_R && (382)\\
\overline{T}^1_R &= h^* \frac{q_R}{M^2}\left[\mathcal{E}_{x,R} \sin\theta^*_R \sin\phi^*_R\right.\\
&\quad \left.+\nu'_R p_{x,R} \sin\theta_{x,R} \sin\phi_{x,R} \cos\theta^*_R\right] && (383)\\
\overline{T}^2_R &= -h^* \frac{q_R}{M^2}\left[\mathcal{E}_{x,R} \sin\theta^*_R \cos\phi^*_R\right.\\
&\quad \left.+\nu'_R p_{x,R} \sin\theta_{x,R} \cos\phi_{x,R} \cos\theta^*_R\right]. && (384)
\end{aligned}$$

Then, assembling all of the developments in the above section, using in particular Eq. (304), in the target rest system we find that

$$\begin{aligned}
\overline{U}^0_R &= \frac{1}{\nu'_R}\overline{U}^3_R = \frac{1}{M^2}\left(\mathbf{q}_R \times \mathbf{p}_{x,R}\right)\cdot \mathbf{s}_R && (385)\\
&= \frac{q_R}{M}\eta_{x,R}\mathcal{P}^R_{N'} && (386)\\
\overline{U}^1_R &= -\frac{\omega_R}{M^2 q_R^2}\left(\mathbf{q}_R \times \mathbf{p}_{x,R}\right)^1\left(\mathbf{q}_R\cdot \mathbf{s}_R\right) && (387)\\
&= \frac{\omega_R}{M}\eta_{x,R}\sin\phi_{x,R}\mathcal{P}^R_{L'} && (388)\\
\overline{U}^2_R &= -\frac{\omega_R}{M^2 q_R^2}\left(\mathbf{q}_R \times \mathbf{p}_{x,R}\right)^2\left(\mathbf{q}_R\cdot \mathbf{s}_R\right) && (389)\\
&= -\frac{\omega_R}{M}\eta_{x,R}\cos\phi_{x,R}\mathcal{P}^R_{L'}, && (390)
\end{aligned}$$

that is,

$$\overline{U}^\mu_R = \frac{q_R}{M}\eta_{x,R}\left(\mathcal{P}^R_{N'}, \nu'_R\sin\phi_{x,R}\mathcal{P}^R_{L'}, -\nu'_R\cos\phi_{x,R}\mathcal{P}^R_{L'}, \nu'_R\mathcal{P}^R_{N'}\right). \tag{391}$$

Note that

$$\overline{U}^0_R = [I_0]_R\,. \tag{392}$$

We are now in a position to write explicit expressions for the hadronic tensors in the rest system.

6.1. Semi-inclusive Tensors in the Rest System

For the **symmetric, unpolarized** case we immediately have

$$\begin{aligned}
\left[W^L_{unpol}\right]_R &= \frac{1}{\rho_R^2}\left(-\rho_R W_1 + W_2\right) && (393)\\
\left[W^T_{unpol}\right]_R &= 2W_1 + \eta^2_{x,R}W_3 && (394)\\
\left[W^{TT}_{unpol}\right]_R &= -\eta^2_{x,R}\cos 2\phi_{x,R}W_3 && (395)\\
\left[W^{TL}_{unpol}\right]_R &= 2\sqrt{2}\frac{1}{\rho_R}\eta_{x,R}\cos\phi_{x,R}W_4, && (396)
\end{aligned}$$

all of which are TRE. For the **anti-symmetric, unpolarized** case one has

$$\left[W^{TL'}_{unpol}\right]_R = -2\sqrt{2}\frac{1}{\rho_R}\eta_{x,R}\sin\phi_{x,R}W_5 \tag{397}$$

namely, the usual TL' so-called TRO 5th response function [4] which goes as $\sin\phi_{x,R}$, as expected; there is no $\mu\nu = 12$, T' response in the rest frame. Each of these has explicit dependence on the azimuthal angle $\phi_{x,R}$ and consequently when one wishes to relate any specific model for the semi-inclusive reaction to the invariant response functions the procedures here are clear. For the anti-symmetric, unpolarized case only a single invariant response enters and thus W_5 may immediately be extracted. In the symmetric, unpolarized case the $\phi_{x,R}$ dependences allow the W_3 and W_4 invariant responses to be isolated using Eqs. (395) and (396), respectively. Then knowing W_3 one can deduce W_1 from Eq. (394) which contains a linear combination of W_1 and W_3. Finally, using Eq. (393) which involves a linear combination of W_1 and W_2, the latter can also be extracted. This procedure can, in principle, be used with experimental data; however, frequently one or more of the responses may be so small that the extractions become very difficult. In contrast, when the goal is to relate some model to the invariant responses, these procedures can always be followed.

For the cases where the target is polarized it is helpful now to label the response functions by the polarizations. Each of the six types of response (L, T, TT, TL, T', TL'), in addition to depending on the kinematic variables in the problem (Q^2, x, *etc.;* see the previous discussions), also depends on the polar and azimuthal angles that specify the direction in which the polarization axis of quantization points, *i.e.,* the angles θ_R^* and ϕ_R^*. Or, equivalently, one may use the three particular directions given in Eqs. (296–311); here we do the latter and write the responses in the form $\left[W_{pol}^K\right]_R^{\Lambda'}$, where $K = L, T, TT, TL, T'$ or TL' and $\Lambda' = L', S'$ or N'.

For the **symmetric, polarized** case one has

$$\begin{aligned}
\left[W^{L}_{pol}\right]^{\Lambda'}_{R} &= \frac{1}{\rho_R^2}\left(-\rho_R W_1' + W_2'\right)\left[I_0\right]_R + \frac{2}{\rho_R}\overline{U}^0_R W_5' && (398)\\
&= \frac{1}{\rho_R^2}\frac{q_R}{M}\eta_{x,R}\left(\rho_R\left(2W_5' - W_1'\right) + W_2'\right)\mathcal{P}^R_{N'} && (399)\\
\left[W^{T}_{pol}\right]^{\Lambda'}_{R} &= \left(2W_1' + \eta^2_{x,R}W_3'\right)\left[I_0\right]_R && (400)\\
&\quad +2\left\{\left(X^2_R\overline{U}^2_R + X^1_R\overline{U}^1_R\right)W_7' + \left(X^2_R\overline{X}^2_R + X^1_R\overline{X}^1_R\right)W_8'\right\}\\
&= \frac{q_R}{M}\eta_{x,R}\left(2W_1' + 2W_8' + \eta^2_{x,R}W_3'\right)\mathcal{P}^R_{N'} && (401)\\
\left[W^{TT}_{pol}\right]^{\Lambda'}_{R} &= \left(-\eta^2_{x,R}\cos 2\phi_{x,R}W_3'\right)\left[I_0\right]_R && (402)\\
&\quad +2\left\{\left(X^2_R\overline{U}^2_R - X^1_R\overline{U}^1_R\right)W_7' + \left(X^2_R\overline{X}^2_R - X^1_R\overline{X}^1_R\right)W_8'\right\}\\
&= -\frac{q_R}{M}\eta_{x,R}\left[\left(2W_8' + \eta^2_{x,R}W_3'\right)\cos 2\phi_{x,R}\mathcal{P}^R_{N'}\right.\\
&\quad \left.+2\sin 2\phi_{x,R}\left(\nu_R'\eta_{x,R}\mathcal{P}_{L'}W_7' + \mathcal{P}_{S'}W_8'\right)\right] && (403)\\
\left[W^{TL}_{pol}\right]^{\Lambda'}_{R} &= 2\sqrt{2}\left[\left(\frac{1}{\rho_R}\eta_{x,R}\cos\phi_{x,R}W_4'\right)\left[I_0\right]_R\right.\\
&\quad +\frac{1}{\rho_R}\left(\overline{U}^1_R W_5' + \overline{X}^1_R W_6'\right)\\
&\quad \left.+\eta_{x,R}\cos\phi_{x,R}\left(\overline{U}^0_R W_7'\right)\right] && (404)\\
&= 2\sqrt{2}\frac{1}{\rho_R}\frac{q_R}{M}\left[\cos\phi_{x,R}\mathcal{P}^R_{N'}\left(W_6' + \eta^2_{x,R}\left(W_4' + \rho_R W_7'\right)\right)\right.\\
&\quad \left.+\nu_R'\eta_{x,R}\sin\phi_{x,R}\mathcal{P}^R_{L'}W_5' + \sin\phi_{x,R}\mathcal{P}^R_{S'}W_6'\right], && (405)
\end{aligned}$$

all of which are TRO. Clearly $W'_{5,6,7,8}$ may immediately be isolated by choosing $\Lambda' = L'$ and S' in Eqs. (403) and (405). Then, choosing $\Lambda' = N'$ in Eq. (403), W_3' can be determined. Following this, W_1' may be deduced from Eq. (401) and W_1' may be deduced from Eq. (399), thereby yielding the full set of symmetric, polarized invariant response functions.

Finally, for the **anti-symmetric, polarized** case the required results are the following:

$$
\begin{aligned}
\left[W^{T'}_{pol}\right]^{\Lambda'}_{R} &= -2\left[\frac{1}{M}W'_9\epsilon^{12\alpha\beta}\Sigma_\alpha Q_\beta\right. \\
&\quad \left.+W'_{12}(X^1_R\bar{U}^2_R - X^2_R\bar{U}^1_R) + W'_{13}(X^1_R\overline{X}^2_R - X^2_R\overline{X}^1_R)\right] \qquad (406) \\
&= 2\frac{q_R}{M}\left[\nu'_R\left(W'_9 + \eta^2_{x,R}W'_{12}\right)\mathcal{P}^R_{L'} + \eta_{x,R}W'_{13}\mathcal{P}^R_{S'}\right] \qquad (407) \\
\left[W^{TL'}_{pol}\right]^{\Lambda'}_{R} &= -2\sqrt{2}\left[\frac{1}{M}W'_9\epsilon^{02\alpha\beta}\Sigma_\alpha Q_\beta\right. \\
&\quad \left.+U^0\left(\bar{U}^2_R W'_{10} + \overline{X}^2_R W'_{11}\right) - X^2_R\bar{U}^0_R W'_{12}\right] \qquad (408) \\
&= 2\sqrt{2}\frac{q_R}{M}\left[\left(\frac{1}{\rho_R}\nu'_R\eta_{x,R}W'_{10}\right)\cos\phi_{x,R}\mathcal{P}^R_{L'}\right. \\
&\quad -\left(W'_9 - \frac{1}{\rho_R}W'_{11}\right)\cos\phi_{x,R}\mathcal{P}^R_{S'} \\
&\quad \left.+\left(W'_9 - \frac{1}{\rho_R}W'_{11} + \eta^2_{x,R}W'_{12}\right)\sin\phi_{x,R}\mathcal{P}^R_{N'}\right], \qquad (409)
\end{aligned}
$$

all of which are TRE. By choosing $\Lambda' = S'$ in Eq. (407) and $\Lambda' = L'$ in Eq. (409) $W'_{10,13}$ may both be isolated. Then by choosing $\Lambda' = S'$ in Eq. (409) the combination $W'_9 - \frac{1}{\rho_R}W'_{11}$ may be extracted, and choosing $\Lambda' = N'$ in Eq. (409) the response W'_{12} determined. Finally, using Eq. (407) and knowing W'_{12} the response W'_9 may be determined from the $\Lambda' = L'$ there, and hence the W'_{11} response, since the combination $W'_9 - \frac{1}{\rho_R}W'_{11}$ has been determined above. Thus, when the goal is to provide relationships between any model responses for the semi-inclusive cross section and the full set of invariant response functions, these procedures provide the proof that such can be accomplished.

Note the behavior of the 18 types of contributions when the results are written in terms of the $1'2'3'$ system with polarizations $\mathcal{P}_{L'}$, $\mathcal{P}_{S'}$ and $\mathcal{P}_{N'}$ times explicit dependence on the angle $\phi_{x,R}$ are summarized in Table 2.

For the L, T, TT and TL cases, the unpolarized responses are TRE and the polarized responses are TRO, while for T' and TL' cases the reverse is true with the unpolarized case being TRO and the polarized cases being TRE.

Using a very different procedure where the goal was to develop semi-inclusive electron scattering with polarizations for situations where the target could have any spin in [5], the case of pion electroproduction was developed as an example

	unpol	$\mathcal{P}^R_{L'}$	$\mathcal{P}^R_{S'}$	$\mathcal{P}^R_{N'}$
L	1	-	-	1
T	1	-	-	1
TL	$\cos\phi_{x,R}$	$\sin\phi_{x,R}$	$\sin\phi_{x,R}$	$\cos\phi_{x,R}$
TT	$\cos 2\phi_{x,R}$	$\sin 2\phi_{x,R}$	$\sin 2\phi_{x,R}$	$\cos 2\phi_{x,R}$
T'	-	1	1	-
TL'	$\sin\phi_{x,R}$	$\cos\phi_{x,R}$	$\cos\phi_{x,R}$	$\sin\phi_{x,R}$

Table 2: This table summarizes the dependence of the response functions on the angle $\phi_{x,R}$

.

of applying that approach. Clearly this is an alternative to the present approach for the case where the target spin is 1/2. The results in that cited work were vetted against much earlier studies specifically of pion electroproduction (see the references in [5]). The behavior summarized in the above table was exactly what was found in the earlier studies.

Again, the strategy in the present work is the following: given some model for the polarized semi-inclusive cross section in the rest system one can deduce what are the invariant response functions for that model. With these the expressions in a general system immediately yield results for any choice of kinematics. The key feature is having everything written in terms of kinematic factors and invariant responses, since the latter are independent of the choice of frame. So, for example, while the earlier studies referred to above are completely general, they must be re-cast in terms of invariant response functions if one wishes to relate the results in different frames of reference.

In the rest frame the results are relatively simple: there one obtains the following for the **symmetric cases without and with spin**:

$$\left[W^{L}_{unpol}\right]^{incl}_{R} = \frac{1}{\rho_R^2}\left(-\rho_R\,(W_1)^{incl} + (W_2)^{incl}\right) \tag{410}$$

$$\left[W^{T}_{unpol}\right]^{incl}_{R} = 2\,(W_1)^{incl} \tag{411}$$

$$\left[W^{TT}_{unpol}\right]^{incl}_{R} = \left[W^{TL}_{unpol}\right]^{incl}_{R} = 0. \tag{412}$$

$$\left[W^{TL}_{pol}\right]^{incl}_{R} = 2\sqrt{2}\frac{q_R h^*}{M\rho_R}\,(W_6')^{incl}\sin\theta_R^*\sin\phi_R^* \tag{413}$$

$$\left[W^{L}_{pol}\right]^{incl}_{R} = \left[W^{T}_{pol}\right]^{incl}_{R} = \left[W^{TT}_{pol}\right]^{incl}_{R} = 0; \tag{414}$$

here the only non-zero contribution goes as $\mathcal{P}_N = h^*\sin\theta_R^*\sin\phi_R^*$. Note that one cannot in general assume that $\left[W^{TL}_{pol}\right]^{incl}_{R}$ is zero. Indeed, the final states reached via inelastic scattering in general contain interfering channels with complex amplitudes. An example of how this can occur is, for instance, in the region where the Δ is important and one might model the final states as containing a resonant Δ and non-resonant pion production with different phase shifts. Or, at high energies one might go beyond the lowest-order approximation for the inelastic processes involved and incorporate higher-order loop diagrams, which are in general complex. As discussed in [5] and references therein, in such cases the TRO response $\left[W^{TL}_{pol}\right]^{incl}_{R}$ is found to be non-zero.

As above no **anti-symmetric unpolarized** case survives and finally for the **anti-symmetric, polarized** situation the required results are the following:

$$\left[W^{T'}_{pol}\right]^{incl}_{R} = 2h^*\frac{\omega_R}{M}\,(W_9')^{incl}\cos\theta_R^* \tag{415}$$

$$\left[W^{TL'}_{pol}\right]^{incl}_{R} = -2\sqrt{2}h^*\frac{q_R}{M}\left[(W_9')^{incl} - \frac{1}{\rho_R}\,(W_{11}')^{incl}\right]\sin\theta_R^*\cos\phi_R^*; \tag{416}$$

in this sector the TL' contribution goes as $\mathcal{P}_S = h^*\sin\theta_R^*\cos\phi_R^*$ while the T' contribution goes as $\mathcal{P}_L = h^*\cos\theta_R^*$.

Let us assemble these results into the cross section for inclusive scattering in the target rest frame. First note that for the completely unpolarized contributions we have

$$\mathcal{R}_{1,R}^{incl} \equiv v_L^R \left[W_{unpol}^L\right]_R^{incl} + v_T^R \left[W_{unpol}^T\right]_R^{incl} = (W_2)^{incl} + 2\,(W_1)^{incl} \tan^2 \theta_e^R/2 \tag{417}$$

which, upon implementing the Feynman rules in the standard way, yields the following familiar form for the unpolarized inclusive cross section:

$$\left[\frac{d^2\sigma}{d\Omega_e dk'}\right]_R^{unpol} = \sigma_{Mott}^R \left[(W_2)^{incl} + 2\,(W_1)^{incl} \tan^2 \theta_e^R/2\right] = \sigma_{Mott}^R \mathcal{R}_{1,R}^{incl}, \tag{418}$$

where the Mott cross section in the rest frame is given by

$$\sigma_{Mott}^R = \left(\frac{\alpha \cos \theta_e^R/2}{2\epsilon_R \sin^2 \theta_e^R/2}\right)^2. \tag{419}$$

Here the invariant response functions $(W_{1,2})^{incl}$ have dimensions of GeV^{-1}. In Appendix F we develop the inclusive cross section in more detail as this may help the reader by making contact with more familiar expressions.

7. Summary

The present study has focused on the scattering of polarized electrons from polarized spin-1/2 targets in situations where the scattered electron and some (unpolarized) particle x are detected in coincidence, *viz.,* semi-inclusive scattering. Together with the well-known leptonic tensor that arises from products of the electron EM current matrix elements the EM hadronic tensor has been constructed using specific general basis sets of 4-vectors. When the target is unpolarized, following standard procedures these are taken to be the mutually orthogonal set Q^μ, U^μ and X^μ given in Eqs. (85), (130) and (136), respectively. When the target is polarized and the target spin 4-vector S^μ is involved (see Eqs. (128) and (154)) it proves to be convenient to employ the set $\bar{X}^\mu$, $\bar{U}^\mu$, given in Eqs. (157) and (158), respectively, together with U^μ and X^μ along with an invariant I_0 (Eq. (162)) and a special tensor obtained using the Levi-Civita symbol (Eq. (206)). In total one finds that there are 18 basis tensors, four symmetric ones when both the electron and target are unpolarized, a single anti-symmetric one when the electron is longitudinally polarized while the target is unpolarized, eight symmetric ones when the electron is unpolarized but the target is polarized, and five anti-symmetric ones when the electron and the target are both polarized.

The contraction of the leptonic and hadronic tensors that enters when applying the Feynman rules, which is a Lorentz invariant, is then formed as a linear combination involving these 18 hadronic tensors weighted with 18 invariant response functions, W_i, $i = 1, 5$ when the target is unpolarized and W_i', $i = 1, 13$ when the target is polarized. Each of these invariant responses is a function of four Lorentz scalars $(Q^2, I_{1,2,3})$ (see Eqs. (141–143)). Thus one has the kinematics of the reaction and the target spin dependence expressed in terms of the basis 4-vectors while the dynamics are contained in the 18 invariant response functions. Clearly the former are frame-dependent while the latter are not.

Given the Lorentz invariant contraction of the leptonic and hadronic tensors one can proceed using the Feynman rules to obtain the semi-inclusive cross section in a general frame where both the incident electron and the target are assumed to be moving, the latter with momentum $\mathbf{p}$. All of the kinematic factors summarized above must then be evaluated in this specific frame. One may obtain the corresponding results in a different frame where the target has a different value for its momentum simply by choosing a different value for $\mathbf{p}$; all other kinematic variables are then to be evaluated in that different frame. Specifically, one can express the semi-inclusive cross section in the target rest frame by setting $p = 0$ and the results of doing so are detailed in the paper. Importantly, the dynamical content in the problem, which is encapsulated in the invariant response functions summarized above does not change when changing frames. Also, the 18 invariant response functions are functions only of the four Lorentz scalars listed above; these are also invariant.

The semi-inclusive cross section separates into four sectors according to the electron and target polarizations, namely, (I) both unpolarized, (II) electron polarized, target unpolarized, (III) target polarized, electron unpolarized, and (IV) both polarized. Having control of these polarizations then immediately allows the four sectors to be isolated. Furthermore, the cross section has explicit dependence on several kinematic variables that may be evaluated in principle to obtain enough linear equations in the 18 unknowns — the 18 invariant response functions — to invert and thereby determine those response functions. Specifically, the dependences on the electron scattering angle θ_e, on the azimuthal angle for the 3-momentum of the detected particle, ϕ_x, and on the angles θ^* and ϕ^* that specify the axis of quantization of the target spin can be used to isolate the required linear equations (an appendix is provided with the details).

Hence several strategies are available. In one approach where measurements are made in two different types of experiments the experimental results could be used in principle to isolate the 18 invariant response functions for the kinematical situation involved in the two experiments. Specifically, one could envision one experiment being performed in the target rest frame (fixed-target experiments) and from those measurements the 18 invariant response functions or some subset thereof being determined. One might then have a different experiment where the electron and target are both in motion (collider experiments): nevertheless, the same strategy could be followed and the 18 invariant response functions determined, albeit, perhaps for non-overlapping kinematics. The two sets of invariant responses could then be analyzed in a universal way.

A similar strategy occurs when using theory to make predictions of the semi-inclusive cross section. For instance, one may be forced to work in the target rest frame when modeling the dynamics using ingredients that are not “boostable”, which is almost always the case in nuclear physics for nuclei other than the deuteron. However, one could deduce the corresponding invariant response functions working in the target rest frame and then employ them in, say, the collider frame. Specific modeling of this sort will be undertaken by the authors in the future.

To make contact with other approaches, in the process of developing the semi-inclusive cross section we have chosen to express the results in terms of specific Lorentz components of the general hadronic tensor which are governed by the helicity projections of the exchanged virtual photon. We have included an appendix where this step is skipped and the contraction of leptonic and hadronic tensors is expressed directly in terms of invariant quantities. The two approaches are completely equivalent, but each may have advantages in particular applications.

Finally, we have shown how the inclusive scattering of polarized electrons from polarized spin-1/2 targets is related to integrations of the semi-inclusive cross sections plus sums over all open channels. We have included another appendix containing a few more details on inclusive scattering to help the reader find more familiar ground to aid in navigating the much more intricate problem of semi-inclusive scattering.

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Appendix A. Conventions

In this work we employ the following conventions: 4-vectors are written $A^\mu = (A^0, A^1, A^2, A^3) = (A^0, \mathbf{a})$ with capital letters for the 4-vectors and lower-case letters for 3-vectors. The magnitude of a 3-vector is written as $a = |\mathbf{a}|$. One also has $A_\mu = g_{\mu\nu} A^\mu = (A^0, -A^1, -A^2, -A^3)$ with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

The scalar product of two 4-vectors is given by $A \cdot B = A_\mu B^\mu = (A^0)^2 - a^2$, following the conventions of [7]. For instance, for the 4-momentum of an on-shell particle of mass M, energy E and 3-momentum p we have $P^\mu = (E, \mathbf{p})$ and hence $P^2 = P_\mu P^\mu = E^2 - p^2 = M^2$. One problem occurs with these conventions, *viz.* for the momentum transfer 4-vector we have $Q^2 = (Q^0)^2 - q^2$ which, for electron scattering is spacelike, and accordingly $Q^2 < 0$. One should be careful not to confuse our sign convention for this quantity with the so-called SLAC convention which has the opposite sign. The totally anti-symmetric Levi-Civita symbol follows the conventions of [7] where

$$\epsilon_{0123} = -\epsilon^{0123} = +1. \tag{A.2}$$

When applying the Feynman rules we also employ the conventions of [7].

Appendix B. Contracted Tensors

The contraction of the electron and hadron tensors can be written as

$$\eta_{\mu\nu}\chi^{\mu\nu} = \sum_{i=1}^{5} C_i W_i + \sum_{i=1}^{13} C_i' W_i'. \tag{B.1}$$

Since this is a Lorentz scalar, as are the W_i and W_i', the coefficients C_i and C_i' are also Lorentz invariants. From Eqs. (80,83,184,189,196,206) these coefficients can be written in terms of inner products of Lorentz 4-vectors as:

$$C_1 = Q^2 \tag{B.2}$$

$$C_2 = \frac{-4K \cdot PP \cdot Q + 4K \cdot P^2 + M^2 Q^2}{2M^2} \tag{B.3}$$

$$
\begin{aligned}
C_3 =& \frac{1}{2M^2\left(P\cdot Q^2 - M^2Q^2\right)^2}(-4K\cdot P_x\left(M^2Q^2 - P\cdot Q^2\right) \\
&\times\left(P_x\cdot Q\left(M^2Q^2 - 2K\cdot PP\cdot Q\right) + P\cdot P_xQ^2(2K\cdot P - P\cdot Q)\right) \\
&+ 2P\cdot P_xP_x\cdot QQ^2\left(2K\cdot P\left(M^2Q^2 + P\cdot Q^2\right) - 4K\cdot P^2P\cdot Q - P\cdot Q^3\right) \\
&+ P\cdot P_x^2Q^4\left(-4K\cdot PP\cdot Q + 4K\cdot P^2 - M^2Q^2 + 2P\cdot Q^2\right) \\
&+ P\cdot QP_x\cdot Q^2\left(P\cdot Q\left(4K\cdot P^2 + M^2Q^2\right) - 4K\cdot PM^2Q^2\right) \\
&+ 4K\cdot P_x^2\left(P\cdot Q^2 - M^2Q^2\right)^2 + M_x^2Q^2\left(P\cdot Q^2 - M^2Q^2\right)^2) \qquad \text{(B.4)}
\end{aligned}
$$

$$
\begin{aligned}
C_4 =& \frac{(2K\cdot P - P\cdot Q)}{M^4Q^2 - M^2P\cdot Q^2}\left[K\cdot P\left(2P\cdot QP_x\cdot Q - 2P\cdot P_xQ^2\right)\right. \\
&\left.+2K\cdot P_x\left(M^2Q^2 - P\cdot Q^2\right) + Q^2\left(P\cdot P_xP\cdot Q - M^2P_x\cdot Q\right)\right] \qquad \text{(B.5)}
\end{aligned}
$$

$$
C_5 = \frac{2h\epsilon_{\alpha\beta\gamma\delta}K^\alpha P^\beta P_x^\gamma Q^\delta}{M^2} \qquad \text{(B.6)}
$$

$$
C_1' = -\frac{h^*Q^2\epsilon_{\alpha\beta\gamma\delta}P^\alpha P_x^\beta Q^\gamma S^\delta}{M^3} \qquad \text{(B.7)}
$$

$$
C_2' = \frac{h^*\epsilon_{\alpha\beta\gamma\delta}P^\alpha P_x^\beta Q^\gamma S^\delta\left(-4K\cdot PP\cdot Q + 4K\cdot P^2 + M^2Q^2\right)}{2M^5} \qquad \text{(B.8)}
$$

$$
\begin{aligned}
C_3' =& \frac{h^*\epsilon_{\alpha\beta\gamma\delta}P^\alpha P_x^\beta Q^\gamma S^\delta}{2M^5\left(P\cdot Q^2 - M^2Q^2\right)^2}\left(-4K\cdot P_x\left(M^2Q^2 - P\cdot Q^2\right)\right. \\
&\times\left(P_x\cdot Q\left(M^2Q^2 - 2K\cdot PP\cdot Q\right) + P\cdot P_xQ^2(2K\cdot P - P\cdot Q)\right) \\
&+2P\cdot P_xP_x\cdot QQ^2\left(2K\cdot P\left(M^2Q^2 + P\cdot Q^2\right) - 4K\cdot P^2P\cdot Q - P\cdot Q^3\right) \\
&+P\cdot P_x^2Q^4\left(-4K\cdot PP\cdot Q + 4K\cdot P^2 - M^2Q^2 + 2P\cdot Q^2\right) \\
&+P\cdot QP_x\cdot Q^2\left(P\cdot Q\left(4K\cdot P^2 + M^2Q^2\right) - 4K\cdot PM^2Q^2\right) \\
&\left.+4K\cdot P_x^2\left(P\cdot Q^2 - M^2Q^2\right)^2 + M_x^2Q^2\left(P\cdot Q^2 - M^2Q^2\right)^2\right) \qquad \text{(B.9)}
\end{aligned}
$$

$$
\begin{aligned}
C_4' =& h^*\frac{\epsilon_{\alpha\beta\gamma\delta}P^\alpha P_x^\beta Q^\gamma S^\delta(2K\cdot P - P\cdot Q)}{M^7Q^2 - M^5P\cdot Q^2}\left[P_x\cdot Q\left(2K\cdot PP\cdot Q - M^2Q^2\right)\right. \\
&\left.+P\cdot P_xQ^2(P\cdot Q - 2K\cdot P) + 2K\cdot P_x\left(M^2Q^2 - P\cdot Q^2\right)\right] \qquad \text{(B.10)}
\end{aligned}
$$

$$
\begin{aligned}
C_5' =& \frac{h^*}{M^5 Q^2 - M^3 P\cdot Q^2}\left[2(2K\cdot P - P\cdot Q)\left(\epsilon_{\alpha\beta\gamma\delta}K^\alpha P^\beta Q^\gamma S^\delta\right.\right.\\
&\times\left(P\cdot P_x Q^2 - P\cdot Q P_x\cdot Q\right) + \epsilon_{\alpha\beta\gamma\delta}K^\alpha P_x^\beta Q^\gamma S^\delta\left.\left(P\cdot Q^2 - M^2 Q^2\right)\right)\\
&\left.+\epsilon_{\alpha\beta\gamma\delta}P^\alpha P_x^\beta Q^\gamma S^\delta Q^2\left(P\cdot Q^2 - M^2 Q^2\right)\right]
\end{aligned} \tag{B.11}
$$

$$
C_6' = \frac{2h^*\epsilon_{\alpha\beta\gamma\delta}K^\alpha P^\beta Q^\gamma S^\delta(P\cdot Q - 2K\cdot P)}{M^3} \tag{B.12}
$$

$$
\begin{aligned}
C_7' =& \frac{2h^*}{M^3\left(P\cdot Q^2 - M^2 Q^2\right)^2}\left(\epsilon_{\alpha\beta\gamma\delta}K^\alpha P^\beta Q^\gamma S^\delta\left(P\cdot Q P_x\cdot Q - P\cdot P_x Q^2\right)\right.\\
&\left.+\epsilon_{\alpha\beta\gamma\delta}K^\alpha P_x^\beta Q^\gamma S^\delta\left(M^2 Q^2 - P\cdot Q^2\right)\right)\left(P_x\cdot Q\left(M^2 Q^2 - 2K\cdot P P\cdot Q\right)\right.\\
&\left.+P\cdot P_x Q^2(2K\cdot P - P\cdot Q) + 2K\cdot P_x\left(P\cdot Q^2 - M^2 Q^2\right)\right)
\end{aligned} \tag{B.13}
$$

$$
\begin{aligned}
C_8' =& \frac{h^*}{M^5 Q^2 - M^3 P\cdot Q^2}\left[2\epsilon_{\alpha\beta\gamma\delta}K^\alpha P^\beta Q^\gamma S^\delta\left(P_x\cdot Q\left(M^2 Q^2 - 2K\cdot P P\cdot Q\right)\right.\right.\\
&\left.+P\cdot P_x Q^2(2K\cdot P - P\cdot Q) + 2K\cdot P_x\left(P\cdot Q^2 - M^2 Q^2\right)\right)\\
&\left.+\epsilon_{\alpha\beta\gamma\delta}P^\alpha P_x^\beta Q^\gamma S^\delta\left(M^2 Q^4 - P\cdot Q^2 Q^2\right)\right]
\end{aligned} \tag{B.14}
$$

$$
C_9' = \frac{hh^* Q^2(Q\cdot S - 2K\cdot S)}{M} \tag{B.15}
$$

$$
\begin{aligned}
C_{10}' =& \frac{hh^* P\cdot Q Q\cdot S}{M^3\left(P\cdot Q^2 - M^2 Q^2\right)}\left[P_x\cdot Q\left(M^2 Q^2 - 2K\cdot P P\cdot Q\right)\right.\\
&\left.+P\cdot P_x Q^2(2K\cdot P - P\cdot Q) + 2K\cdot P_x\left(P\cdot Q^2 - M^2 Q^2\right)\right]
\end{aligned} \tag{B.16}
$$

$$
C_{11}' = \frac{hh^*\left(Q\cdot S\left(2K\cdot P P\cdot Q - M^2 Q^2\right) + 2K\cdot S\left(M^2 Q^2 - P\cdot Q^2\right)\right)}{M^3} \tag{B.17}
$$

$$
\begin{aligned}
C'_{12} =& \frac{hh^* Q^2}{M^3 \left(P\cdot Q^2 - M^2 Q^2\right)^2} \Big\{ -Q\cdot S \Big[2P\cdot P_x \big(P_x\cdot Q \big(K\cdot P \left(M^2Q^2 + P\cdot Q^2 \right) \\
& - P\cdot Q^3 \big) + K\cdot P_x P\cdot Q (M^2Q^2 - P\cdot Q^2) \big) \\
& - P\cdot P_x^2 Q^2 (2P\cdot Q(K\cdot P - P\cdot Q) + M^2Q^2) \\
& + M^2 P\cdot Q P_x\cdot Q^2 (P\cdot Q - 2K\cdot P) + 2K\cdot P_x M^2 P_x\cdot Q \left(P\cdot Q^2 - M^2Q^2 \right) \\
& + M_x^2 \left(P\cdot Q^2 - M^2Q^2 \right)^2 \Big] - P_x\cdot S \left(M^2Q^2 - P\cdot Q^2 \right) \\
& \times \big(P_x\cdot Q \left(2K\cdot P P\cdot Q - M^2Q^2 \right) + P\cdot P_x Q^2 (P\cdot Q - 2K\cdot P) \\
& + 2K\cdot P_x \left(M^2Q^2 - P\cdot Q^2 \right) \big) + 2K\cdot S \left(M^2Q^2 - P\cdot Q^2 \right) \\
& \times \left(M_x^2 \left(M^2Q^2 - P\cdot Q^2 \right) - M^2 P_x\cdot Q^2 + 2P\cdot P_x P\cdot Q P_x\cdot Q - P\cdot P_x^2 Q^2 \right) \Big\}
\end{aligned}
\tag{B.18}
$$

$$
\begin{aligned}
C'_{13} =& \frac{hh^* Q^2 (2K\cdot P - P\cdot Q)}{M^5 Q^2 - M^3 P\cdot Q^2} \left[Q\cdot S \left(M^2 P_x\cdot Q - P\cdot P_x P\cdot Q \right) \right. \\
& \left. + P_x\cdot S \left(P\cdot Q^2 - M^2Q^2 \right) \right]
\end{aligned}
\tag{B.19}
$$

Appendix C. Invariant Functions

Appendix C.1. Semi-inclusive

Using Eqs. (309–311), Eqs. (393–409) can be inverted to give the invariant functions in terms of the response functions as

$$
W_1 = \frac{1}{2}(\left[W^{TT}_{unpol}\right]\sec 2\phi_x + \left[W^{T}_{unpol}\right])
\tag{C.1}
$$

$$
W_2 = \frac{1}{2}\rho(\left[W^{TT}_{unpol}\right]\sec 2\phi_x + 2\rho\left[W^{L}_{unpol}\right] + \left[W^{T}_{unpol}\right])
\tag{C.2}
$$

$$
W_3 = -\frac{\left[W^{TT}_{unpol}\right]\sec 2\phi_x}{\eta_x^2}
\tag{C.3}
$$

$$
W_4 = \frac{\rho\left[W^{TL}_{unpol}\right]\sec \phi_x}{2\sqrt{2}\eta_x}
\tag{C.4}
$$

$$W_5 = -\frac{\rho \left[W^{TL'}_{unpol}\right] \csc\phi_x}{2\sqrt{2}\eta_x} \tag{C.5}$$

$$W'_1 = \frac{M(\left[W^{TT}_{pol}\right]^{N'} \sec 2\phi_x + \left[W^{TT}_{pol}\right]^{N'})}{2\eta_x q} \tag{C.6}$$

$$\begin{aligned} W'_2 =& \frac{M\rho}{2\eta_x \nu' q} \left(\nu' \left[W^{TT}_{pol}\right]^{N'} \sec 2\phi_x + 2\nu'\rho \left[W^{L}_{pol}\right]^{N'} + \nu' \left[W^{TT}_{pol}\right]^{N'} \right. \\ & \left. -\sqrt{2}\rho \left[W^{TL}_{pol}\right]^{L'} \csc\phi_x\right) \end{aligned} \tag{C.7}$$

$$W'_3 = -\frac{M\sec(2\phi_x)(\left[W^{TT}_{pol}\right]^{N'} - \left[W^{TT}_{pol}\right]^{S'} \cot 2\phi_x)}{\eta_x^3 q} \tag{C.8}$$

$$\begin{aligned} W'_4 =& \frac{M\rho\sec\phi_x}{2\sqrt{2}\eta_x^2 \nu' q} \left(-\nu' \left[W^{TL}_{pol}\right]^{S} \cot\phi_x + \nu' \left[W^{TL}_{pol}\right]^{N'} \right. \\ & \left. +\sqrt{2} \left[W^{TT}_{pol}\right]^{L'} \cos\phi_x \csc 2\phi_x\right) \end{aligned} \tag{C.9}$$

$$W'_5 = \frac{M\rho \left[W^{TL}_{pol}\right]^{L'} \csc\phi_x}{2\sqrt{2}\eta_x \nu' q} \tag{C.10}$$

$$W'_6 = -\frac{M\rho \left[W^{TL}_{pol}\right]^{S'} \csc\phi_x}{2\sqrt{2} q} \tag{C.11}$$

$$W'_7 = -\frac{M \left[W^{TT}_{pol}\right]^{L'} \csc 2\phi_x}{2\eta_x^2 \nu' q} \tag{C.12}$$

$$W'_8 = -\frac{M \left[W^{TT}_{pol}\right]^{S'} \csc 2\phi_x}{2\eta_x q} \tag{C.13}$$

$$W'_9 = -\frac{M\left(\sqrt{2}\nu' \left[W^{TL'}_{pol}\right]^{N'} \csc\phi_x + \sqrt{2}\nu' \left[W^{TL}_{pol}\right]^{S'} \sec\phi_x - 2\left[W^{T'}_{pol}\right]^{L'}\right)}{4\nu' q} \tag{C.14}$$

$$W'_{10} = \frac{M\rho \left[W_{pol}^{TL'}\right]^{N'} \sec\phi_x}{2\sqrt{2}\eta_x \nu' q} \tag{C.15}$$

$$W'_{11} = -\frac{M\rho \left(\sqrt{2}\nu' \left[W_{pol}^{TL'}\right]^{N'} \csc\phi_x - 2\left[W_{pol}^{T'}\right]^{L'}\right)}{4\nu' q} \tag{C.16}$$

$$W'_{12} = \frac{M(\left[W_{pol}^{TL'}\right]^{N'} \csc\phi_x + \left[W_{pol}^{TL}\right]^{S'} \sec\phi_x)}{2\sqrt{2}\eta_x^2 q} \tag{C.17}$$

$$W'_{13} = \frac{M\left[W_{pol}^{T'}\right]^{S'}}{2\eta_x q} \tag{C.18}$$

Note that although Eqs. (393–409) are derived in the rest frame, the expressions a valid in all frames where $\boldsymbol{q}$ is defined to be parallel to the z-axis. Therefore, we have dropped the subscript R in the expressions given above.

855 *Appendix C.2. Inclusive*

Inverting Eqs. (410–416)) gives the inclusive invariant functions in terms of the response functions

$$(W_1)^{incl} = \frac{\left[W_{unpol}^{T}\right]^{incl}}{2} \tag{C.19}$$

$$(W_2)^{incl} = \frac{1}{2}\rho\left(2\rho\left[W_{unpol}^{L}\right]^{incl} + \left[W_{unpol}^{T}\right]^{incl}\right) \tag{C.20}$$

$$(W'_6)^{incl} = \frac{M\rho\left[W_{pol}^{TL}\right]^{incl} \csc\theta^* \csc\phi^*}{2\sqrt{2}q} \tag{C.21}$$

$$(W'_9)^{incl} = \frac{M\left[W_{pol}^{T'}\right]^{incl} \sec\theta^*}{2\nu' q} \tag{C.22}$$

$$(W'_{11})^{incl} = \frac{M\rho\left(\nu'\left[W_{pol}^{TL'}\right]^{incl} \csc\theta^* \sec\phi^* + \sqrt{2}\left[W_{pol}^{T'}\right]^{incl} \sec\theta^*\right)}{2\sqrt{2}\nu q} \tag{C.23}$$

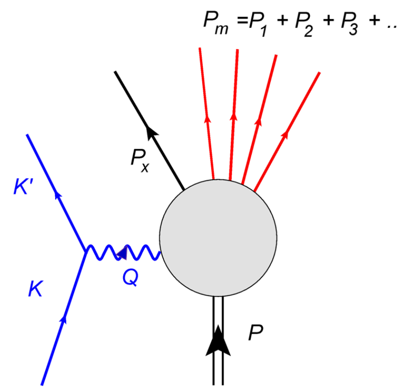

Figure C.7: Feynman diagram for semi-inclusive electron scattering. The 4-momenta here are discussed in the text. In particular, particle x is assumed to be detected in coincidence with the scattered electron and thus P_x^μ is assumed to be known. Since the total final-state momentum P'^μ is known (see Fig. 2 for inclusive scattering) this implies that the missing 4-momentum is also known via the relationship $P_m^\mu = P'^\mu - P_x^\mu$ (see Fig. 3). Furthermore, for given kinematics the missing momentum is the sum of a set of momenta for the individual particles that constitute that unobserved part of the final state.

Appendix D. General Semi-Inclusive Cross Sections

Consider the case of electron scattering from a hadronic target with 4-momentum P^μ producing $\mathcal{N}+1$ hadrons in the final state. For semi-inclusive scattering, the hadron P_x^μ is detected while the remaining $\mathcal{N}$ hadrons are not detected. This process is represented by the diagram in Fig. C.7.

We will use the conventions of [7] giving the differential cross section as

$$d\sigma_{\mathcal{N}} = f\frac{m_e}{\epsilon}\frac{M}{E_p}\frac{d^3k' m_e}{(2\pi)^3\epsilon'}\frac{d^3p_x\zeta_x}{(2\pi)^3\sqrt{p_x^2+M_x^2}}\left(\prod_{i=1}^{\mathcal{N}}\int\frac{d^3p_i\zeta_i}{(2\pi)^3\sqrt{p_i^2+m_i^2}}\right)$$
$$\times|\mathcal{M}|^2(2\pi)^4\delta^4(K+P-K'-P_x-\sum_{j=1}^{\mathcal{N}}P_j) \tag{D.1}$$

where $\zeta_{i(x)} = m_{i(x)}$ for Fermions, $\zeta_{i(x)} = 1/2$ for Bosons. The flux factor is given by [9, 10]

$$f = \frac{1}{\sqrt{(\boldsymbol{\beta_e}-\boldsymbol{\beta_P})^2-(\boldsymbol{\beta_e}\times\boldsymbol{\beta_P})^2}}\,, \tag{D.2}$$

where

$$\boldsymbol{\beta_e} = \frac{\boldsymbol{k}}{\epsilon} \tag{D.3}$$

and

$$\boldsymbol{\beta_P} = \frac{\boldsymbol{p}}{E_P}\,. \tag{D.4}$$

This is the general form of this factor whereas Bjorken and Drell omit the cross product constraining the electron and target velocities to be collinear. Equation (D.1) is correct in all Lorentz frames [7].

It is convenient to use

$$\int \frac{d^3p_i}{(2\pi)^3\sqrt{p_i^2+m_i^2}} = 2\int \frac{d^4P_i}{(2\pi)^3}\delta(P_i^2-m_i^2)\theta(P_i^0) \tag{D.5}$$

$$\begin{aligned} d\sigma_{\mathcal{N}} =& f\frac{m_e}{\epsilon}\frac{M}{E_p}\frac{d^3k'm_e}{(2\pi)^3\epsilon'}\frac{d^3p_x\zeta_x}{(2\pi)^3\sqrt{p_x^2+M_x^2}}\left(\prod_{i=1}^{\mathcal{N}}\frac{2\zeta_i}{(2\pi)^3}\int d^4P_i\delta(P_i^2-m_i^2)\theta(P_i^0)\right) \\ &\times |\mathcal{M}|^2(2\pi)^4\delta^4(K+P-K'-P_x-\sum_{j=1}^{\mathcal{N}}P_j) \end{aligned} \tag{D.6}$$

Now define the missing 4-momentum as

$$P_m = \sum_{n}^{\mathcal{N}} P_n \tag{D.7}$$

$$\begin{aligned} d\sigma_{\mathcal{N}} =& f\frac{m_e}{\epsilon}\frac{M}{E_p}\frac{d^3k'm_e}{(2\pi)^3\epsilon'}\frac{d^3p_x\zeta_x}{(2\pi)^3\sqrt{p_x^2+M_x^2}} \\ &\times\int d^4P_m(2\pi)^4\delta^4(K+P-K'-P_x-P_m) \\ &\times\left(\prod_{i=1}^{\mathcal{N}}\frac{2\zeta_i}{(2\pi)^3}\int d^4P_i\delta(P_i^2-m_i^2)\theta(P_i^0)\right)\delta(P_m-\sum_{i=1}^{\mathcal{N}}P_n)|\mathcal{M}|^2 \end{aligned} \tag{D.8}$$

Writing

$$P_n = \frac{P_m}{\mathcal{N}} + \mathcal{L}_n\,, \tag{D.9}$$

then

$$P_m = \sum_{n=1}^{\mathcal{N}} P_n = \sum_{n=1}^{\mathcal{N}}\left(\frac{P_m}{\mathcal{N}}+\mathcal{L}_n\right) = P_m + \sum_{n=1}^{\mathcal{N}}\mathcal{L}_n\,. \tag{D.10}$$

This then implies that

$$\sum_{n=1}^{\mathcal{N}}\mathcal{L}_n = 0\,. \tag{D.11}$$

The differential cross section then becomes

$$
\begin{aligned}
d\sigma_{\mathcal{N}} =& f\frac{m_e}{\epsilon}\frac{M}{E_p}\frac{d^3k'm_e}{(2\pi)^3\epsilon'}\frac{d^3p_x\zeta_x}{(2\pi)^3\sqrt{p_x^2+M_x^2}} \\
&\times\int d^4P_m(2\pi)^4\delta^4(K+P-K'-P_x-P_m) \\
&\times\left(\prod_{i=1}^{\mathcal{N}}\frac{2\zeta_i}{(2\pi)^3}\int d^4\mathcal{L}_i\delta((\frac{P_m}{\mathcal{N}}+\mathcal{L}_i)^2-m_i^2)\theta(\frac{P_m^0}{\mathcal{N}}+\mathcal{L}_i^0)\right) \\
&\times\delta(P_m-\sum_{n=1}^{\mathcal{N}}(\frac{P_m}{\mathcal{N}}+\mathcal{L}_n))|\mathcal{M}|^2 \\
=& f\frac{m_e}{\epsilon}\frac{M}{E_p}\frac{d^3k'm_e}{(2\pi)^3\epsilon'}\frac{d^3p_x\zeta_x}{(2\pi)^3\sqrt{p_x^2+M_x^2}} \\
&\times\int d^4P_m(2\pi)^4\delta^4(K+P-K'-P_x-P_m) \\
&\times\left(\prod_{i=1}^{\mathcal{N}}\frac{2\zeta_i}{(2\pi)^3}\int d^4\mathcal{L}_i\delta((\frac{P_m}{\mathcal{N}}+\mathcal{L}_i)^2-m_i^2)\theta(\frac{P_m^0}{\mathcal{N}}+\mathcal{L}_i^0)\right) \\
&\times\delta(\sum_{n=1}^{\mathcal{N}}\mathcal{L}_n)|\mathcal{M}|^2\,.
\end{aligned}
\tag{D.12}
$$

The minimum value of the invariant mass of the undetected particles is

$$
W_m^T=\sum_{j=1}^{\mathcal{N}}m_j>0 \tag{D.13}
$$

Using

$$
1=\int_{W_m^{T\,2}}^{\infty}dW_m^2\delta(P_m^2-W_m^2)\theta(P_m^0)=2\int_{W_m^T}^{\infty}dW_mW_m\delta(W_m^2-P_m^2)\theta(P_m^0) \tag{D.14}
$$

and

$$
\int d^4P_m\delta(p_m^2-W_m^2)\theta(P_m^0)=\frac{1}{2}\int\frac{d^3p_m}{\sqrt{p_m^2+W_m^2}}\,, \tag{D.15}
$$

the differential cross section becomes

$$
\begin{aligned}
d\sigma_{\mathcal{N}} =& f \frac{m_e}{\epsilon} \frac{M}{E_p} \frac{d^3k' m_e}{(2\pi)^3 \epsilon'} \frac{d^3 p_x \zeta_x}{(2\pi)^3 \sqrt{p_x^2 + M_x^2}} \\
&\times \int_{W_m^T}^{\infty} dW_m W_m \int \frac{d^3 p_m}{\sqrt{p_m^2 + W_m^2}} (2\pi)^4 \delta^4(K + P - K' - P_x - P_m) \\
&\times \left(\prod_{i=1}^{\mathcal{N}} \frac{2\zeta_i}{(2\pi)^3} \int d^4\mathcal{L}_i \delta((\frac{P_m}{\mathcal{N}} + \mathcal{L}_i)^2 - m_i^2) \theta(\frac{P_m^0}{\mathcal{N}} + \mathcal{L}_i^0) \right) \\
&\times \delta(\sum_{n=1}^{\mathcal{N}} \mathcal{L}_n) |\mathcal{M}|^2 \, .
\end{aligned}
\tag{D.16}
$$

The absolute square of the reduced scattering matrix is given by

$$
m_e^2 |\mathcal{M}|^2 = \frac{4\pi^2\alpha^2}{Q^4} \chi_{\mu\nu} W_{\mathcal{N}}^{\mu\nu} \tag{D.17}
$$

where the hadronic tensor is

$$
\begin{aligned}
W_{\mathcal{N}}^{\mu\nu} =& \sum_{s_x} \sum_{s_1} \cdots \sum_{s_{\mathcal{N}}} \langle P, s_R | \, J^\mu(Q) \, | P_x, s_x; P_1, s_1; \ldots; P_{\mathcal{N}}, s_{\mathcal{N}}; (-) \rangle^* \\
&\times \langle P_x, s_x; P_1, s_1; \ldots; P_{\mathcal{N}}, s_{\mathcal{N}}; (-) | \, J^\nu(Q) \, | P, s_R \rangle \, ,
\end{aligned}
\tag{D.18}
$$

where $(-)$ indicates that the many-particle final state must be constructed with incoming scatting boundary conditions. The final state must have the complete symmetry associated with the combination of Fermions and Bosons contributing to this state. Note that the current operator $J(Q)$ appearing in the matrix element may consist of a complete set of one-body and many-body contributions appropriate for any particular system.

Now define

$$
\begin{aligned}
W^{\mu\nu} =& (2\pi)^3 \left(\prod_{i=1}^{\mathcal{N}} \frac{2\zeta_i}{(2\pi)^3} \int d^4\mathcal{L}_i \delta((\frac{P_m}{\mathcal{N}} + \mathcal{L}_i)^2 - m_i^2) \theta(\frac{P_m^0}{\mathcal{N}} + \mathcal{L}_i^0) \right) \\
&\times \delta(\sum_{n=1}^{\mathcal{N}} \mathcal{L}_n) W_{\mathcal{N}}^{\mu\nu} \, .
\end{aligned}
\tag{D.19}
$$

The differential cross section can then be written as

$$
\begin{aligned}
d\sigma_{\mathcal{N}} =& f\frac{1}{\epsilon}\frac{M}{E_p}\frac{4\pi^2\alpha^2}{Q^4}\frac{d^3k'}{(2\pi)^3\epsilon'}\frac{d^3p_x\zeta_x}{(2\pi)^3\sqrt{p_x^2+M_x^2}}\int_{W_m^T}^{\infty}dW_mW_m\int\frac{d^3p_m}{(2\pi)^3\sqrt{p_m^2+W_m^2}} \\
&\times(2\pi)^4\delta^4(K+P-K'-P_x-P_m)\chi_{\mu\nu}W^{\mu\nu} \\
=& f\frac{1}{(2\pi)^3}\frac{1}{\epsilon}\frac{M}{E_p}\frac{\alpha^2 v_0}{Q^4}\frac{d^3k'}{\epsilon'}\frac{d^3p_x\zeta_x}{\sqrt{p_x^2+M_x^2}}\int_{W_m^T}^{\infty}dW_mW_m\int\frac{d^3p_m}{\sqrt{p_m^2+W_m^2}} \\
&\times\delta^4(K+P-K'-P_x-P_m)\frac{\chi_{\mu\nu}W^{\mu\nu}}{v_0}
\end{aligned} \tag{D.20}
$$

In the extreme relativistic limit let

$$
v_0 = 4kk'\cos^2\frac{\theta_e}{2}\,, \tag{D.21}
$$

and

$$
\begin{aligned}
Q^2 \cong& (k-k')^2-q^2 = k^2-2kk'+k'^2-k^2+2kk'\cos\theta_e+k'^2 \\
=& 2kk'(-1+\cos\theta_e) = -4kk'\sin^2\frac{\theta_e}{2}\,.
\end{aligned} \tag{D.22}
$$

Using combination of constants

$$
\frac{1}{k}\frac{\alpha^2 v_0}{Q^4} \cong \frac{1}{k'}\frac{\alpha^2\cos^2\frac{\theta_e}{2}}{4k^2\sin^4\frac{\theta_e}{2}} = \frac{1}{k'}\sigma_{Mott}\,, \tag{D.23}
$$

the differential cross section becomes

$$
\begin{aligned}
d\sigma_{\mathcal{N}} =& \frac{f}{(2\pi)^3}\sigma_{Mott}\frac{M}{E_p}\frac{d^3k'}{k'^2}\frac{d^3p_x\zeta_x}{\sqrt{p_x^2+M_x^2}}\int_{W_m^T}^{\infty}dW_mW_m\int\frac{d^3p_m}{\sqrt{p_m^2+W_m^2}} \\
&\times\delta^4(K+P-K'-P_x-P_m)\frac{\chi_{\mu\nu}W^{\mu\nu}}{v_0}\,.
\end{aligned} \tag{D.24}
$$

The six-fold differential cross section is then

$$
\begin{aligned}
\frac{d^6\sigma_{\mathcal{N}}}{dk'd\Omega'dp_xd\Omega_x} =& \frac{f}{(2\pi)^3}\sigma_{Mott}\frac{M}{E_p}\frac{p_x^2\zeta_x}{\sqrt{p_x^2+M_x^2}}\int_{W_m^T}^{\infty}dW_mW_m\int\frac{d^3p_m}{\sqrt{p_m^2+W_m^2}} \\
&\times\delta^4(K+P-K'-P_x-P_m)\frac{\chi_{\mu\nu}W^{\mu\nu}}{v_0}\,.
\end{aligned} \tag{D.25}
$$

For some reactions it is possible that the residual system contains only one particle. A particular example of this is the case of semi-inclusive scattering from nuclei where the residual system may consist of one or more stable states of the daughter nucleus with masses M_i. Using Eq. (D.1) for $\mathcal{N} = 1$ with the unmeasured particle with mass M_i

$$
\begin{aligned}
d\sigma_i =& f \frac{m_e}{\epsilon} \frac{M}{E_p} \frac{d^3k' m_e}{(2\pi)^3 \epsilon'} \frac{d^3 p_x \zeta_x}{(2\pi)^3 \sqrt{p_x^2 + M_x^2}} \int \frac{d^3 p_m \zeta_m}{(2\pi)^3 \sqrt{p_m^2 + M_i^2}} \\
& \times |\mathcal{M}|^2 (2\pi)^4 \delta^4 (K + P - K' - P_x - P_m) \\
\cong & \frac{f}{(2\pi)^3} \sigma_{Mott} \frac{M}{E_p} \frac{d^3 k'}{k'^2} \frac{d^3 p_x \zeta_x}{\sqrt{p_x^2 + M_x^2}} \int \frac{d^3 p_m \zeta_m}{\sqrt{p_m^2 + M_i^2}} \\
& \times \frac{\chi_{\mu\nu} W^{\mu\nu}}{v_0} \delta^4 (K + P - K' - P_x - P_m)
\end{aligned} \tag{D.26}
$$

The six-fold differential cross section is then

$$
\begin{aligned}
\frac{d^6 \sigma_i}{dk' d\Omega' dp_x d\Omega_x} =& \frac{f}{(2\pi)^3} \sigma_{Mott} \frac{M}{E_p} \frac{p_x^2 \zeta_x}{\sqrt{p_x^2 + M_x^2}} \int \frac{d^3 p_m \zeta_m}{\sqrt{p_m^2 + M_i^2}} \\
& \times \frac{\chi_{\mu\nu} W^{\mu\nu}}{v_0} \delta^4 (K + P - K' - P_x - P_m)
\end{aligned} \tag{D.27}
$$

Appendix E. Kinematic Variables

Here we have collected some useful kinematical variables. From the energy and momentum transfer variables we can define the following dimensionless quantities [11]:

$$\lambda \equiv \frac{\omega}{2M} \tag{E.1}$$

$$\kappa \equiv \frac{q}{2M} \tag{E.2}$$

$$\tau \equiv \frac{-Q^2}{4M^2} \tag{E.3}$$

where then

$$\tau = \kappa^2 - \lambda^2. \tag{E.4}$$

In the rest system we have

$$\lambda_R \equiv \frac{\omega_R}{2M} \tag{E.5}$$

$$\kappa_R \equiv \frac{q_R}{2M} \tag{E.6}$$

$$\tau = \kappa_R^2 - \lambda_R^2, \tag{E.7}$$

where, of course, τ is an invariant. In the target rest frame the x-variable is given by (see the following appendix)

$$x_R = \frac{-Q^2}{2M\omega_R} = \frac{\tau}{\lambda_R}. \tag{E.8}$$

It is often convenient to use τ and x_R as two independent variables; Eq. (E.8) then yields

$$\lambda_R = \frac{\tau}{x_R} \tag{E.9}$$

and using Eq. (E.7) one has

$$\kappa_R = \frac{\tau}{x_R}\sqrt{1 + \frac{x_R^2}{\tau}}. \tag{E.10}$$

This results in the following:

$$\frac{\lambda_R}{\kappa_R} = \frac{\omega_R}{q_R} = \frac{\nu_R}{q_R} = \nu' = \frac{1}{\sqrt{1 + \frac{x_R^2}{\tau}}} \tag{E.11}$$

$$\rho_R \equiv \frac{-Q^2}{q_R^2} = 1 - {\nu'_R}^2 = \frac{x_R^2}{\tau + x_R^2}. \tag{E.12}$$

One has that

$$0 < x_R < 1, \tag{E.13}$$

as discussed in the following appendix. Also one can define the "high-energy regime (HER)" as being where

$$\tau \gg 1. \tag{E.14}$$

Accordingly, from the above identities, we find that in this regime

$$\lambda_R \simeq \kappa_R, \tag{E.15}$$

implying that

$$\omega_R \simeq q_R \tag{E.16}$$

and that

$$\rho_R \simeq \frac{x_R^2}{\tau} \ll 1. \tag{E.17}$$

Appendix F. Inclusive Scattering

We continue with some developments of the inclusive cross section: following standard practice, the expressions in Sec. 6.2 can be related to dimensionless invariant functions via

$$F_1^{incl}\left(x_R, Q^2\right) \equiv M\left(W_1\left(x_R, Q^2\right)\right)^{incl} \tag{F.1}$$

$$F_2^{incl}\left(x_R, Q^2\right) \equiv \omega_R\left(W_2\left(x_R, Q^2\right)\right)^{incl} = \nu_R\left(W_2\left(x_R, Q^2\right)\right)^{incl}. \tag{F.2}$$

Note that these definitions are specific to the rest frame. To make the expressions invariant one should use $x \equiv |Q^2|/2P \cdot Q$ and instead of $\omega_R = \nu_R$ use $P \cdot Q/M$. At very high momentum transfers one finds reasonable (Bjorken) scaling:

$$F_1^{incl}\left(x_R, Q^2\right) \underset{Bj}{\longrightarrow} F_1^{incl}\left(x_R\right) \tag{F.3}$$

$$F_2^{incl}\left(x_R, Q^2\right) \underset{Bj}{\longrightarrow} F_2^{incl}\left(x_R\right), \tag{F.4}$$

namely, these two responses become functions only of x_R. Moreover, let us define

$$\mathcal{R}_L^R \equiv v_L^R\left[W_{unpol}^L\right]_R^{incl} \tag{F.5}$$

$$\mathcal{R}_T^R \equiv v_T^R\left[W_{unpol}^T\right]_R^{incl} \tag{F.6}$$

so that

$$\mathcal{R}_{1,R}^{incl} = \mathcal{R}_L^R + \mathcal{R}_T^R \tag{F.7}$$

$$= \mathcal{R}_T^R\left(1 + \delta_R\right), \tag{F.8}$$

where

$$\delta_R \equiv \frac{\mathcal{R}_L^R}{\mathcal{R}_T^R}. \tag{F.9}$$

In principle $\mathcal{R}_L^R$ and $\mathcal{R}_T^R$ can be separated by making a Rosenbluth plot of the unpolarized cross section versus $\tan^2\theta_e^R/2$ which occurs in v_T^R but not in v_L^R.

Substituting from above one then finds that

$$\delta_R = \left(\frac{q_R}{\omega_R}\right)^2 \left[\frac{\rho_R F_1^{incl} + \frac{1}{2x_R}\left(F_2^{incl} - 2x_R F_1^{incl}\right)}{F_1^{incl}\left(1 + \frac{2}{\rho_R}\tan^2\theta_e^R/2\right)}\right] \tag{F.10}$$

$$= \frac{1}{{\nu'_R}^2}\mathcal{E}_R\left[\rho_R + \frac{1}{2x_R}\left(F_2^{incl} - 2x_R F_1^{incl}\right)/F_1^{incl}\right], \tag{F.11}$$

where the kinematical variables here are discussed in Appendix E and $\mathcal{E}_R$ is the so-called longitudinal photon polarization given in Eq. (103). In the very high-energy regime (HER) one finds that

$$\mathcal{R}_L \ll \mathcal{R}_T, \tag{F.12}$$

namely, given that the usual conditions obtain where $\mathcal{E}_R$ is not especially small, then

$$\delta_R \ll 1. \tag{F.13}$$

In this regime one has from the developments in Appendix E that

$$\frac{q_R}{\omega_R} \simeq 1 \tag{F.14}$$

and that

$$\rho_R \ll 1; \tag{F.15}$$

accordingly one has that

$$\frac{1}{2x_R}\left(F_2^{incl} - 2x_R F_1^{incl}\right) \ll 1, \tag{F.16}$$

namely, the Callan-Gross relationship

$$F_2^{incl} \simeq 2x_R F_1^{incl}. \tag{F.17}$$

However, if extreme conditions obtain where $\mathcal{E}_R \ll 1$ then δ_R may also be small even when Eq. (F.17) is not satisfied.

To the above unpolarized results we now add the contributions that involve the target polarization. We can define

$$\mathcal{R}_{TL}^R \equiv v_{TL}^R \left[W_{pol}^{TL}\right]_R^{incl} \tag{F.18}$$

$$\mathcal{R}_{T'}^R \equiv v_{T'}^R \left[W_{pol}^{T'}\right]_R^{incl} \tag{F.19}$$

$$\mathcal{R}_{TL'}^R \equiv v_{TL'}^R \left[W_{pol}^{TL'}\right]_R^{incl}, \tag{F.20}$$

where the first does not involve polarized electrons, whereas the second and third do and one has

$$
\begin{aligned}
h^* \left[\mathcal{R}_3^{incl}\right]_R &= \mathcal{R}_{TL}^R && \text{(F.21)}\\
hh^* \left[\mathcal{R}_4^{incl}\right]_R &= v_{TL'}^R \left[W_{pol}^{TL'}\right]_R^{incl} + v_{T'}^R \left[W_{pol}^{T'}\right]_R^{incl} . && \text{(F.22)}
\end{aligned}
$$

From the identities above together with identities involving the leptonic factors introduced in Sect. 2.2 one can show that the above parts of the response involve the following:

$$
\begin{aligned}
\mathcal{R}_{TL}^R &\equiv -2\left(\frac{\epsilon_R+\epsilon'_R}{M}\right)\tan\theta_e^R/2h^*\sin\theta_R^*\sin\phi_R^*\,(W'_6)^{incl} && \text{(F.23)}\\
\mathcal{R}_{T'}^R &\equiv 2\left(\frac{\omega_R}{q_R}\right)\left(\frac{\epsilon_R+\epsilon'_R}{M}\right)\tan^2\theta_e^R/2h^*\cos\theta_R^*\,(W'_9)^{incl} && \text{(F.24)}\\
\mathcal{R}_{TL'}^R &\equiv 2\left(\frac{q_R}{M}\right)\tan\theta_e^R/2h^*\sin\theta_R^*\cos\phi_R^*\left[\rho_R\,(W'_9)^{incl}-(W'_{11})^{incl}\right] . && \\
& && \text{(F.25)}
\end{aligned}
$$

As noted above, clearly the three sectors ($\mathcal{R}_{1,R}^{incl}$, $\mathcal{R}_{3,R}^{incl}$ and $\mathcal{R}_{4,R}^{incl}$) can in principle be separated by flipping the electron helicity h and the direction of the target spin via h^*. Then $\mathcal{R}_{TL'}^R$ and $\mathcal{R}_{T'}^R$ can be separated by pointing the target spin in different directions as seen from Eqs. (F.24–F.25). Accordingly, all five invariant response functions $(W_{1,2})^{incl}$ and $\left(W'_{6,9,11}\right)^{incl}$ may be determined separately either experimentally or via specific modeling in the rest frame.We end this section by rewriting the single and double-polarized results in a form that is closer to that in Eq. (417):

$$
\begin{aligned}
\left[\mathcal{R}_3^{incl}\right]_R &= -2h^*\frac{1}{\rho'_R}\left(\frac{q_R}{M}\right)\tan\theta_e^R/2\,(W'_6)^{incl}\sin\theta_R^*\sin\phi_R^* && \text{(F.26)}\\
\left[\mathcal{R}_4^{incl}\right]_R &= 2hh^*\left(\frac{q_R}{M}\right)\tan\theta_e^R/2\left[\nu'_R\eta_R\tan\theta_e^R\,(W'_9)^{incl}\cos\theta_R^*\right. && \\
&\quad \left.+\left(\rho_R\,(W'_9)^{incl}-(W'_{11})^{incl}\right)\sin\theta_R^*\cos\phi_R^*\right] , && \text{(F.27)}
\end{aligned}
$$

where as earlier we have

$$
\rho'_R = \frac{q_R}{\epsilon_R+\epsilon'_R} . \qquad \text{(F.28)}
$$

In the high-energy regime, as discussed above one has $\rho_R \ll 1$ and accordingly the term above involving $(W'_9)^{incl}$ in that regime becomes negligible if $(W'_9)^{incl}$ and $(W'_{11})^{incl}$ are comparable in size.

As for the symmetric case, the anti-symmetric (double-polarized) case may be written in terms of other conventionally-defined invariant response functions. From [12] and [13]

$$\left[W_{pol}^{TL'}\right]_R^{incl} \equiv -\frac{2\sqrt{2}}{M\nu'_R}\left(g_1+g_2\right)\cdot\mathcal{P}_S \tag{F.29}$$

$$\left[W_{pol}^{T'}\right]_R^{incl} \equiv -\frac{2}{M}\left(g_1-\frac{\rho_R}{{\nu'_R}^2}g_2\right)\cdot\mathcal{P}_L \tag{F.30}$$

and hence, using Eqs. (415,416)

$$g_1+g_2 = \omega_R\left[\left(W'_9\right)^{incl}-\frac{1}{\rho_R}\left(W'_{11}\right)^{incl}\right] \tag{F.31}$$

$$g_1-\frac{\rho_R}{{\nu'_R}^2}g_2 = -\omega_R\left(W'_9\right)^{incl}. \tag{F.32}$$

For reference recall that

$$\nu'_R \equiv \frac{\omega_R}{q_R}=\frac{\nu_R}{q_R} \tag{F.33}$$

$$\rho_R = \left|\frac{Q^2}{q_R^2}\right| = 1-{\nu'_R}^2. \tag{F.34}$$

This yields the following identities

$$g_1 = \omega_R\left[\left(2\rho_R-1\right)\left(W'_9\right)^{incl}-\left(W'_{11}\right)^{incl}\right] \tag{F.35}$$

$$g_2 = \omega_R\frac{1-\rho_R}{\rho_R}\left[2\rho_R\left(W'_9\right)^{incl}-\left(W'_{11}\right)^{incl}\right] \tag{F.36}$$

and their inverses

$$\left(W'_9\right)^{incl} = -\frac{1}{\omega_R}\left[g_1-\frac{\rho_R}{1-\rho_R}g_2\right] \tag{F.37}$$

$$\left(W'_{11}\right)^{incl} = -\frac{1}{\omega_R}\rho_R\left[2g_1+\frac{1-2\rho_R}{1-\rho_R}g_2\right]. \tag{F.38}$$

Note that if $\left(W'_9\right)^{incl}$ and $\left(W'_{11}\right)^{incl}$ are similar in magnitude and one is in the HER where $\rho_R \ll 1$ then one finds that

$$\left|\frac{g_1}{g_2}\right| \ll 1. \tag{F.39}$$

Conversely, if g_1 and g_2 are similar in magnitude and one is in the HER then one finds that

$$\left|\frac{\left(W'_{11}\right)^{incl}}{\left(W'_9\right)^{incl}}\right| \ll 1. \tag{F.40}$$

We note that all of the developments in this study are for completely general kinematics, aside from the fact that the ERL$_e$ has been evoked, and even that can easily be extended to inclusion of corrections arising from keeping the electron mass finite (see [2]). Thus, for example if the polarized target is assumed to be a proton and one is studying charged-pion electroproduction, in the resonance region one type of behavior may be observed while at very high energies a different type may pertain.

Finally, we note that these developments are easily inter-related to the treatment of the special case of elastic scattering of polarized electrons from polarized protons given in [14].

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