Inverse moment of $B$-meson quasi distribution amplitude

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We perform a study on the structure of inverse moment of quasi distributions, by taking $B$-meson quasi distribution amplitude (quasi-DA) as an example. Based on a one-loop calculation, we find that a naive perturbative matching relation for the first inverse moments of $B$-meson light-cone distribution amplitude (LCDA) and quasi-DA failed. Contrary to the naive expectation, to restore the factorization, the mixing with logarithmic moments must be included. We also derive the renormalization group equation and velocity evolution equation for the first inverse moment of quasi-DA. Our results can be useful either in understanding the patterns of perturbative matching in Large Momentum Effective Theory or evaluating inverse moment of $B$-meson LCDA on the lattice.

I. INTRODUCTION

The structure of a high energy hadron can be depicted by nonperturbative functions like parton distribution functions (PDFs), light-cone distribution amplitudes (LCDAs), etc. Because of their nonperturbative nature, parton distributions cannot be calculated with perturbation theory but should be either extracted from experimental data or evaluated with nonperturbative methods like lattice QCD. However, parton distributions are defined with matrix elements of nonlocal operators located on the light-cone, which cannot be simulated on the lattice.

In recent years, it was pointed out that the difficulties of simulating parton physics on a Euclidean lattice can be overcome by employing the Large Momentum Effective Theory (LaMET) proposed by X. Ji [1]. The idea underlying LaMET is to introduce quasi distributions, which are defined with matrix elements of equal-time nonlocal operators. The quasi distribution, when boosting to infinite momentum frame, i.e., $P_3 \to \infty$ with $P_3$ being the hadron momentum on the third direction (moving direction), can be reduced to a light-cone distribution. The quasi distribution and its light-cone counterpart are related by a matching relation and the matching coefficient can be calculated with perturbative QCD since the difference between quasi and light-cone distributions is accompanied with a hard momentum scale $P_3 \gg \Lambda_{\text{QCD}}$. In recent years, LaMET has been applied on the lattice calculation of PDFs, LCDAs, etc., for various hadrons. For recent reviews of LaMET, see e.g. [2, 3]. Other related approaches including pseudo distributions [4, 5], lattice cross sections [6], etc.

It is interesting to study the moments of a quasi distribution. The positive moments, which are related to local operators, have been discussed in Refs. [7–9]. For a quasi-PDF $\tilde{f}(x, P_3)$, the asymptotic behavior at $|x| \to \infty$ is $1/|x|$, thus the integral $\int dx x^n \tilde{f}(x, P_3)$ leads to power divergence if $n$ is a non-negative integer, hence the positive moments of quasi distributions do not exist; on the other hand, if $n$ is a negative number, such integral has no power divergence, which indicates that the inverse moment (IM) can exist. The IM is still an important nonperturbative parameter to describe a quasi distribution. Curiously, till now, there are few studies on the IM of quasi distributions, except the quasi distribution amplitude (quasi-DA) of heavy quarkonia [10].

In this work, we will study the IM of quasi distributions. As an example, we will study the IM of $B$-meson quasi-DA. The $B$-meson LCDA in LaMET is of particular interest. It is an inherent part of hard-collinear factorization theorems for many exclusive $B$ decay reactions [11–17]; moreover, it is also an essential element in the light-cone sum-rule studies of the $B$-meson decays [18–24]. The $B$-meson LCDA has been studied in the framework of quasi distribution amplitude [25, 26] and the reduced Ioffe-time distribution [27] approaches. It has been proved that the $B$-meson DA in coordinate space is multiplicatively renormalizable, and the one-loop matching which links quasi-DA to LCDA is also derived. An ultraviolet (UV) finite pseudo distribution for $B$-meson is proposed. Lattice simulations on quasi-DA or pseudo-DA will be useful to extract the $B$-meson LCDA, which is important in phenomenology.

It has been pointed out in [28] that there are no positive moments for $B$-meson LCDA, but the IM exists. Moreover, it is the first IM attending the factorization theorem of the decay fraction of $B$-meson radiative decay [29] and $B \to P, V$ form factors (see [30] for a recent review). Furthermore, the first IM is an essential parameter to build models for LCDA. The value of the first IM of $B$-meson LCDA has been estimated with various approaches, see,
Unlike the QCD case, the decay constant in HQET is scale-dependent.

It will be of phenomenological significance to study the IM of quasi-DA, which may shed light on the evaluation of LCDA on the lattice. The natural questions are how the IM of quasi-DA is related to its light-cone counterpart, and how does IM of quasi-DA evolves with the renormalization scale and velocity of \( B \)-meson.

In this paper, we will introduce the IM of \( B \)-meson quasi DA and perform a theoretical study on the first IM of \( B \)-meson quasi-DA in LaMET. We will investigate the structure of IM of \( B \)-meson quasi-DA up to one-loop level. The renormalization, as well as one-loop matching between IMs of LCDA and quasi-DA, will be investigated. Since IM is a number instead of a distribution, a naive expectation of matching relation is a multiplication form; however, we will show that the naive expectation fails. The correct form of matching will be figured out in this work.

The paper is organized as follows: in Sec. II, we will introduce the concept of IM and logarithmic moments of \( B \)-meson quasi-DA; in Sec. III, we will present the one-loop results, then discuss the renormalization and derive the renormalization group equation (RGE) in Sec. IV; the one-loop matching will be discussed in Sec. V; at last, we summarize our results and give an outlook for future researches in Sec. VI.

II. QUASI DISTRIBUTION AMPLITUDE AND INVERSE MOMENT

We follow the notations in [27]. To start with, let us consider a nonlocal heavy-light operator in heavy quark effective theory (HQET),

\[
O_\mu(z, 0; v) \equiv \bar{q}(z)\gamma_\mu\gamma_5 h_v(0),
\]

where \( h_v \) is a heavy quark field in HQET, with \( v \) denoting the velocity of \( B \)-meson. \( v \) satisfies \( v^2 = 1 \) and \( \not{h}_v = h_v \); \( \bar{q}(z) \) is a light quark field locating at \( z \); \( S(z, 0) \equiv P \exp[-ig \int_0^1 dt z_v A^\nu(tz)] \) is a Wilson line where \( P \) denotes the path ordering of operators. By analyzing the Lorentz structure of its meson-to-vacuum matrix element, we have

\[
\langle 0 | \bar{q}(z) S(z, 0) \gamma_\mu \gamma_5 h_v(0) | \mathcal{B}(v) \rangle = iF(\mu) \left[ \nu_\mu M_{B, v}(\nu, -z^2, \mu) + z_\mu M_{B, z}(\nu, -z^2, \mu) \right],
\]

where \( M_{B, v}(\nu, \mu) \) and \( M_{B, z}(\nu, \mu) \) are two scalar functions and \( \nu \equiv v \cdot z \) will be referred to as the "Ioffe-time" of the \( B \)-meson \(^1\). \( F(\mu) \) is the decay constant of \( B \)-meson defined by the matrix element of the local current

\[
\langle 0 | \bar{q}(0) \gamma_\mu \gamma_5 h_v(0) | \mathcal{B}(v) \rangle = i v_\mu F(\mu).
\]

Unlike the QCD case, the decay constant in HQET is scale-dependent.

\( M_{B, v} \) term gives the twist-2 distribution when \( z^2 \rightarrow 0 \) while \( M_{B, z} \) is a higher-twist contribution. We are only interested in the leading-twist distribution in this work so we rename \( M_{B, v} \) as \( M_B \) for convenience, and \( M_B(\nu, -z^2, \mu) \) is defined as the Ioffe-time distribution amplitude (ITDA) of the \( B \)-meson. If \( z \) is a light-like vector with minus component of \( z \) being the only nonzero component, then ITDA will reduce to the light-cone ITDA \( \mathcal{I}_B^0(\nu, \mu) \), i.e., \( M_B(\nu, 0, \mu) = \mathcal{I}_B^0(\nu, \mu) \), which is in fact the LCDA in coordinate space. The \( B \)-meson LCDA is defined by the Fourier transform of \( \mathcal{I}_B^0(\nu, \mu) \) [31]

\[
\phi_B^+(\omega, \mu) = \frac{v^+}{2\pi} \int_{-\infty}^{\infty} dz^- e^{-i\omega z^+} \mathcal{I}_B^0(v^+, z^-, \mu).
\]

It was proposed in Refs. [1, 37] that one can study equal-time separations \( z = (0, 0, 0, z_3) \) on the lattice. The same idea has also been applied for the \( B \)-meson LCDA [25–27]. In this case, \( \nu = -v_3 z_3 \) and \( z^2 = -z_3^2 \). The \( B \)-meson quasi-DA \( \phi_B^+(\omega, v_3, \mu) \) can be expressed in terms of ITDA as [25, 26]

\[
\phi_B^+(\omega, v_3, \mu) = \frac{|v_3|}{2\pi} \int_{-\infty}^{\infty} dz_3 e^{i\omega v_3 z_3} M_B(-v_3 z_3, z_3^2, \mu).
\]

The matching relations linking the LCDA and quasi-DA or ITDA were derived in Refs. [26, 27]; in the pseudo distribution-based approach, a matching relation that mapping light-cone ITDA to a UV finite reduced ITDA was also proposed [27].

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\(^1\) Note that in QCD case, Ioffe-time is the inner product of momentum \( p \) and \( z \) [35, 36].
The first IM is defined as

$$\lambda_B^{-1}(\mu) \equiv \int_{-\infty}^{\infty} d\omega \frac{\phi_B^+(\omega, \mu)}{\omega}. \quad (6)$$

Note that $\phi_B^+(\omega, \mu)$ only has nonzero support in $[0, \infty)$, hence the lower limit in Eq. (6) is effectively 0. Other important quantities in phenomenology are the logarithmic moments [28]

$$\sigma_n(\mu) = \lambda_B(\mu) \int_{-\infty}^{\infty} d\omega \ln^n \frac{\mu}{\omega} \phi_B^+(\omega, \mu). \quad (7)$$

Similarly, we introduce the IM and logarithmic moments for quasi-DA:

$$\tilde{\lambda}_B^{-1}(v_3, \mu) \equiv \int_{-\infty}^{\infty} d\omega \frac{\tilde{\phi}_B^+(\omega, v_3, \mu)}{\omega}, \quad (8)$$

$$\tilde{\sigma}_n(v_3, \mu) \equiv \tilde{\lambda}_B(\mu) \int_{-\infty}^{\infty} d\omega \ln^n \frac{\mu}{\omega} \tilde{\phi}_B^+(\omega, v_3, \mu). \quad (9)$$

In this case, however, the region of integration $(-\infty, \infty)$ is necessary because the support of quasi-DA extends to the whole axis.

In the following, we will explore the structure of the first IM of quasi-DA. In particular, we will investigate how $\tilde{\lambda}_B(\mu)$ evolve with energy scale $\mu$, and how $\tilde{\lambda}_B(\mu)$ is related to its light-cone counterpart $\lambda_B(\mu)$. One needs to calculate the quantum corrections to answer both of these two questions. We will perform an analysis on the one-loop structure of $\lambda_B$, derive the renormalization group equation, and give the matching relation between $\lambda_B$ and the light-cone quantities.

If one boosts the $B$-meson to the infinite momentum frame, i.e., $v_3 \to \infty$, then by definition, $\tilde{\lambda}_B \sim \lambda_B + O(1/v_3)$. Within the spirit of LaMET, a matching relation can be expected between $\lambda_B$ and $\tilde{\lambda}_B$. In principle, this matching relation should be a direct inference of the matching relation for DAs in [26]; however, we find that it is not straightforward. The reason is that one needs further regularization to do the $\omega$-integral; moreover, the linear divergence of quasi-DA is hidden in dimensional regularization, and this linear divergence will be weakened to a logarithmic divergence for IM. If such divergence was neglected, one cannot renormalize IM correctly. For this reason, we prefer a new calculation from the beginning, instead of using the matching coefficient reported in [26] or [27].

### III. ONE-LOOP CALCULATION

To study the renormalization and matching in perturbation theory, one can replace the hadron state with Fock state $|q\rangle$, as what we have done for DAs. Let $p$ be the momentum of the light quark, $p = \omega_0 v + p_\perp$ with $\omega_0 > 0$, where $v$ is the velocity of $B$-meson. We set $v = (v^0, 0, 0, v^4)$, or in light-cone coordinates, $v = (v^+, v^-, 0, 0)$. Assuming that the light quark is slightly offshell, i.e., $p^2 < 0$, then $-p^2$ will work as an infrared (IR) regulator. At tree level, we have $\phi_B^+(\omega, \omega_0) = \phi_B^+(\omega, \omega_0) = \delta(\omega - \omega_0)$. This leads to the tree level result for IM: $1/\tilde{\lambda}_B = 1/\lambda_B = 1/\omega_0$.

To evaluate the one-loop corrections, we work in the Feynman gauge. The Feynman diagrams are shown in Fig. 1. Dimensional regularization (DR) is applied to regularize the UV singularities, where the space-time dimension is set to $d = 4 - 2\epsilon$.

Before starting one-loop calculation, we note that the singularities at $\omega = 0$ in the integrals Eqs. (6)(8) need prescription. For future convenience, we add a small imaginary part $i\epsilon$ to the denominator, i.e., $\omega \to \omega + i\epsilon$, to ensure that the IMs are well defined. IM of LCDA is independent of prescription because $\phi_B^+(\omega) \sim \omega$ when $\omega \to 0$ [43].

In fig. 1(a), there is a hard gluon exchange between the heavy quark and Wilson line. For DAs, one has

$$\phi_B^+(\omega, \omega_0, \mu)|_{(a)} = -i g^2 \bar{\mu}^2 C_F n \cdot v \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\epsilon)(-v \cdot k + i\epsilon)(n \cdot k + i\epsilon)} \left[ \delta \left( \omega - \frac{n \cdot k}{n \cdot v} \right) - \delta \left( \omega - \frac{n \cdot p}{n \cdot v} \right) \right], \quad (10)$$

where $\bar{\mu} = \mu \sqrt{\frac{\gamma_E}{4\pi}}$, $\mu$ is an energy scale in DR and $\gamma_E$ is the Euler-Mascheroni constant; $C_F$ is a color factor with the value $4/3$. Since we have written the above expression in a Lorentz covariant form, it is applicable for both LCDA
FIG. 1. The Feynman diagrams for quasi-DA and LCDA. The horizontal double line represents the gauge link, while the vertical double line denotes the heavy-quark in HQET. The single line represents the light quark.

and quasi-DA. Then for IMs, after integrating out $\omega$, we have

$$\lambda_B^{-1}(\omega_0, \mu) = -ig^2\bar{\mu}^2 C_F n \cdot v \int_{-\infty}^{\infty} \frac{d\omega}{\omega + i\epsilon} \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + i\epsilon)(-v \cdot k + i\epsilon)(n \cdot k + i\epsilon)}$$

$$\times \left[ \delta \left( \omega - \frac{n \cdot k}{n \cdot v} - \frac{n \cdot p}{n \cdot v} \right) - \delta \left( \omega - \frac{n \cdot p}{n \cdot v} \right) \right]$$

$$= ig^2\bar{\mu}^2 C_F \frac{n \cdot v^2}{n \cdot p} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + i\epsilon)(-v \cdot k + i\epsilon)(n \cdot k + n \cdot p + i\epsilon)}.$$  \(11\)

This integral can be easily calculated by using Feynman or Schwinger parameterizations. For IM of LCDA, we let $n$ being a light-cone vector and $n \cdot a = a^+$ for arbitrary vector $a$, $n^2 = 0$. After doing the integral, one has

$$\lambda_B^{-1}(\omega_0, \mu)|_{(a)} = -\frac{\alpha_s C_F}{4\pi} \frac{1}{\omega_0} \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{\omega_0^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{\omega_0^2} + \frac{3\pi^2}{4} \right).$$  \(12\)

On the other hand, for IM of quasi-DA, we choose $n$ as a unit vector in the 3rd direction of space-time, $n \cdot a = a_3$ for an arbitrary vector $a$, and $n^2 = -1$. After integration, one has

$$\bar{\lambda}_B^{-1}(\omega_0, v_3, \mu)|_{(a)} = -\frac{\alpha_s C_F}{4\pi} \frac{1}{\omega_0} \left[ 2 \frac{\ln 2iv_3}{\epsilon} + 2 \ln 2iv_3 \left( \ln \frac{\mu^2}{\omega_0^2} - \ln 2iv_3 \right) + \frac{\pi^2}{3} \right].$$  \(13\)

In the light-cone case one can observe a double pole $1/\epsilon^2$, which is the indication of cusp singularity; in the IM of quasi-DA, there is only single pole but accompanied with a meson-velocity-dependent coefficient. There are single and double logarithmic dependence on $v_3$ in the quasi case.

In fig. 1(b), hard gluon exchanges between the light quark and Wilson line. With the same method described for
(a), one has, for IMs,

\[
iF(\omega_0, \mu) \lambda_B^{-1}(\omega_0, \mu) |(b) = ig^2 \mu^2 C_F \int \frac{d\omega}{\omega + i \epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\epsilon)(n \cdot k + i\epsilon)(-n \cdot k + i\epsilon)} \frac{\bar{v}(p)\psi(\not{p} + \not{k})\psi \gamma_5 u_v}{(k^2 + i\epsilon)(p + k)^2 + i\epsilon} (-n \cdot k + i\epsilon) \delta \left( \omega - \frac{n \cdot p}{n \cdot v} - \frac{n \cdot k}{n \cdot v} \right) \delta \left( \omega - \frac{n \cdot p}{n \cdot v} - \frac{n \cdot k}{n \cdot v} \right).
\]

On the light-cone side,

\[
\lambda_B^{-1}(\omega_0, \mu) = \frac{\alpha_s}{4\pi} C_F \frac{1}{\omega_0} \left( \frac{2}{\epsilon} + 2 \ln \frac{\mu^2}{-p^2} + 4 \right),
\]

and for the quasi side, one has

\[
\tilde{\lambda}_B^{-1}(\omega_0, \mu) = \frac{\alpha_s}{4\pi} C_F \frac{1}{\omega_0} \left( \frac{1}{\epsilon} + 2 \ln \frac{\mu^2}{-p^2} + 2 \ln 2v_3 - \ln \frac{\mu^2}{\omega_0^2} + 2 \right).
\]

Both the light-cone and quasi case have IR divergences regularized by \(\ln(-p^2)\), with the same coefficients. It indicates that \(\lambda_B\) and \(\tilde{\lambda}_B\) have the same IR behavior.

For Fig. 1(c), we have

\[
-ig^2 \mu^2 C_F n^2 \int \frac{d\omega}{\omega + i \epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\epsilon)(n \cdot k + i\epsilon)(-n \cdot k + i\epsilon)} \delta \left( \omega - \frac{n \cdot p}{n \cdot v} - \frac{n \cdot k}{n \cdot v} \right) \delta \left( \omega - \frac{n \cdot p}{n \cdot v} - \frac{n \cdot k}{n \cdot v} \right).
\]

The IM of LCDA has no such contribution because \(n^2 = 0\). For IM of quasi-DA, the result reads

\[
\tilde{\lambda}_B^{-1}(\omega_0, v_3, \mu) |(c) = \frac{\alpha_s}{2\pi} C_F \frac{1}{\omega_0} \left( \frac{1}{\epsilon} + 2 \ln 2v_3 - \ln \frac{\mu^2}{\omega_0^2} - 2 \right).
\]

In the calculation of quasi-DA, Wilson line self-energy diagram Fig. 1(c) has a linear divergence. How to deal with the linear divergence is one of the major missions in the development of quasi and pseudo distribution approaches in the last few years. Many schemes, e.g., RI/MOM [38, 39], reduced Ioffe-time distribution [40, 41], hybrid scheme [42], etc., have been proposed to renormalize the linear divergence in quasi distributions. However, the IM involves only logarithmic UV divergence and has no linear divergence problem. It allows us to renormalize IM of quasi-DA with MS scheme.

For the box diagram of Fig. 1(d), we have

\[
-ig^2 \mu^2 C_F \int \frac{d\omega}{\omega + i \epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\omega + i \epsilon} \left[ \bar{v}(p)\psi(\not{p} + \not{k})\psi \gamma_5 u_v \right] \frac{1}{(k^2 + i\epsilon)(p + k)^2 + i\epsilon} \delta \left( \omega - \frac{n \cdot p}{n \cdot v} - \frac{n \cdot k}{n \cdot v} \right) \delta \left( \omega - \frac{n \cdot p}{n \cdot v} - \frac{n \cdot k}{n \cdot v} \right)
\]

where in the second step we have performed the trace with projector \(\frac{1 + \not{v}}{2} \gamma_5\) [43]. This integral does not depend on \(n\), indicating that it contributes the same results to IMs of LCDA and quasi-DA. Furthermore, the integral is UV finite. Therefore, the box diagram can be ignored both in renormalization and perturbative matching. In fact, it has already been shown in [26, 27] that the box diagram does not contribute, both in the quasi and pseudo distribution approaches. Thus, it will not contribute to the renormalization and matching of IM as well. It can also be clarified through a \(v_3\) power-counting argument.

### IV. Renormalization Group Equation

We renormalize the IMs of LCDA and quasi-DA in MS scheme:

\[
\lambda_{B, \text{bare}}^{-1}(\omega_0) = \left[ 1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{\omega_0^2} - \frac{2}{\epsilon} \right) \right] \lambda_B^{-1}(\omega_0; \mu),
\]
\[ \tilde{\lambda}_{B, \text{bare}}^{-1}(\omega_0, v_3) = \left[ 1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left( \frac{1}{\epsilon} - \frac{2}{\epsilon} \ln 2iv_3 \right) \right] \tilde{\lambda}_{B}^{-1}(\omega_0; v_3, \mu). \] (21)

With the above renormalization equation, we obtain RGE for IM of LCDA,

\[ \frac{d}{d\mu} \tilde{\lambda}_{B}^{-1}(\mu) = - \frac{\alpha_s(\mu)}{2\pi} C_F \left[ 2\lambda_{B}^{-1}(\mu)\sigma_1(\mu) - \lambda_{B}^{-1}(\mu) \right]. \] (22)

This result repeats the RGE derived in [43, 44]; it can also be derived from the Lange-Neubert equation for LCDA [43].

On the other hand, for IM of quasi-DA, one has

\[ \frac{d}{dv_3} \tilde{\lambda}_{B}^{-1}(v_3, \mu) = - \frac{\alpha_s(\mu)}{2\pi} C_F (2\ln 2iv_3 + 1) \tilde{\lambda}_{B}^{-1}(v_3, \mu). \] (23)

One can observe that the IM of LCDA is not multiplicative renormalized, it will get mixed with logarithmic moment \( \sigma_1 \) at one-loop. However, for IM of quasi-DA, there is no mixing between 1st IM and logarithmic moments at one-loop level, which is different from IM of LCDA. If one works at next-to-leading order accuracy, one can evolve \( \tilde{\lambda}_B \) to other scales without the input of other parameters. However, we are not clear whether or not the IM of quasi-DA is multiplicatively renormalizable to all orders of perturbation theory.

V. MATCHING RELATION IN LAMET AND VELOCITY RGE

In LaMET, light-cone and equal-time (“quasi”) quantities can be linked by a matching relation, with a perturbatively calculable hard function.

When \( v_3 \to \infty \), the equal-time matrix element that defines the IM of quasi-DA will become a light-cone matrix element, it indicates that under large Lorentz boost, \( \tilde{\lambda}_B(v_3, \mu) \to \lambda_B(\mu) \). With the spirit of LaMET, one can expect the IR physics of \( \tilde{\lambda}_B(v_3, \mu) \) and \( \lambda_B(\mu) \) are the same and there is a matching formula between IMs. Since the IMs have no dependence on \( \omega \), a naive expectation for the matching relation is a multiplication instead of a convolution, i.e.,

\[ \tilde{\lambda}_B(v_3, \mu_Q) = C(v_3, \mu_Q, \mu_L) \lambda_B(\mu_L) + \mathcal{O}(1/v_3), \] (24)

where \( C(v_3, \mu_Q, \mu_L) \) is the hard coefficient, which can be expanded in series of \( \alpha_s \) as

\[ C(v_3, \mu_Q, \mu_L) = \sum_{n=0}^{\infty} \left( \frac{\alpha_s(\mu)}{4\pi} C_F \right)^n C^{(n)}(v_3, \mu_Q, \mu_L); \]

\( \mu_Q \) and \( \mu_L \) are the scales that define the IMs of quasi and light-cone DAs, respectively.

We note that, a very similar multiplication-type matching relation was introduced in [45], which connects the IMs of LCDAs defined in HQET and full QCD; it can also be reproduced from the convolution-type matching between the LCDAs defined in QCD and HQET [46, 47].

Expanding \( \lambda_B(\mu_L) \) and \( \tilde{\lambda}_B(\mu_Q) \) in series of \( \mathcal{O}(\alpha_s) \), up to one-loop level, we have \( C^{(0)} = 1 \), and

\[ C^{(1)}(v_3, \mu_Q, \mu_L) = -2 \ln^2 \frac{\mu_L}{\omega_0} + 2(2 \ln 2iv_3 + 3) \ln \frac{\mu_Q}{\omega_0} + 2(2 \ln 2iv_3 + 1) \ln \frac{\mu_Q}{\mu_L} \]

\[ - 2 \ln 2iv_3(\ln 2iv_3 + 3) - \frac{5\pi^2}{12} + 6. \] (25)

One can notice that, the IR singularities in IMs of LCDA and quasi-DA, which are regularized by \( \ln(-p^2) \), are canceled, so the matching coefficient is IR free; however, \( C^{(1)}(v_3, \mu_Q, \mu_L) \) depends on \( \omega_0 \). It means the matching coefficient depends on \( \omega_0 \), the momentum of external light quark, which indicates the failure of naive factorization relation Eq. (24); moreover, the single and double logarithmic dependence on \( \omega_0 \) should be absorbed into other nonperturbative quantities when \( \omega_0 \) is small. Thus the matching relation should not be a multiplication equation, which is different from the case of IMs in HQET and QCD [45].

Noticing that the \( \ln^2 \mu_Q/\omega_0 \) terms are related to the logarithmic moments defined in Eq. (7), we propose a modified factorization formula as

\[ \tilde{\lambda}_B(v_3, \mu_Q) = C_0 \left( v_3, \frac{\mu_Q}{\mu_L} \right) \lambda_B(\mu_L) + \sum_{n=1} C_n \left( v_3, \frac{\mu_Q}{\mu_L} \right) \sigma_n(\mu_L) + \mathcal{O}(1/v_3). \] (26)
Again, the matching coefficients \( C_n(v_3, \mu_Q/\mu_L) \) can be expanded in series of \( \alpha_s \) as

\[
C_n \left( v_3, \frac{\mu_Q}{\mu_L} \right) = \sum_{m=0} \left( \frac{\alpha_s(\mu)}{4\pi} C_F \right)^m C_n^{(m)} \left( v_3, \frac{\mu_Q}{\mu_L} \right) .
\] (27)

Performing the perturbative expansion, at the leading order, it gives \( C_0^{(0)} = 1 \) and \( C_n^{(0)} = 0 \) for \( n \geq 1 \). With one-loop results, we get the next-to-leading order corrections of the hard coefficients, which read

\[
C_0^{(1)} \left( v_3, \frac{\mu_Q}{\mu_L} \right) = 2(2\ln 2 iv_3 + 1) \ln \frac{\mu_Q}{\mu_L} - 2 \ln 2 iv_3(\ln 2 iv_3 + 3) - \frac{5\pi^2}{12} + 6 ,
\] (28)

\[
C_1^{(1)} \left( v_3, \frac{\mu_Q}{\mu_L} \right) = 2(2\ln 2 iv_3 + 3) ,
\] (29)

\[
C_2^{(1)} \left( v_3, \frac{\mu_Q}{\mu_L} \right) = -2 ,
\] (30)

\[
C_n^{(1)} \left( v_3, \frac{\mu_Q}{\mu_L} \right) = 0 \quad (n \geq 3) .
\] (31)

Coefficient \( C_0 \) has double and single logarithmic dependence on \( B \)-meson velocity \( v_3 \), while \( C_1 \) has only single logarithmic dependence on \( v_3 \), and there is no \( v_3 \) dependence in \( C_2 \) at one-loop level. This indicates that the momentum evolution of \( \tilde{\lambda}_B(v_3, \mu) \) is only related to itself and the first logarithmic moment. In fact, one can write down the velocity evolution equation for \( \tilde{\lambda}_B(v_3, \mu) \) as

\[
v_3 \frac{d}{dv_3} \tilde{\lambda}_B(v_3, \mu) = -\frac{\alpha_s}{2\pi} C_F \left( (2\ln 2 iv_3 + 3)\tilde{\lambda}_B(v_3, \mu) - 2\tilde{\sigma}_1(v_3, \mu) \right) .
\] (32)

To evolve \( \tilde{\lambda}_B(v_3, \mu) \) from one scale to another, one needs the input of the first IM and logarithmic moment.

At last, one may notice that the matching coefficients \( C_n \) are complex numbers, therefore, the IM of quasi-DA is also complex. This is due to the \( +i\epsilon \) prescription we employed at \( \omega = 0 \). One can also get real values for IM and matching coefficients with other prescritions, e.g, Cauchy principal value. Since \( P.V.(1/\omega) = \frac{1}{2} \left( \frac{1}{\omega+i\epsilon} + \frac{1}{\omega-i\epsilon} \right) \), one can get \( \tilde{\lambda}_B \) under principal value prescription by averaging the result under \( +i\epsilon \) prescription and its complex conjugate, and this is equivalent to taking the real part in our results. This is verified by a direct calculation under principal value prescription.

### VI. SUMMARY AND OUTLOOK

We have performed a detailed study on the inverse moment of a kind of quasi distributions: the \( B \)-meson quasi distribution amplitude. The non-negative moments of quasi distributions have power divergences; however, the inverse and logarithmic moments of quasi distribution are free of power divergences. The inverse moment could be an important parameter that reflects some interesting features of a quasi distribution.

In this paper, we introduce the inverse moment of \( B \)-meson quasi-DA and explore its properties. Its light-cone partner is of great significance in the phenomenology of \( B \) decays. In the factorization of many \( B \)-meson exclusive decay processes, the amplitudes at leading order are always in the form of \( 1/\omega \). So the IM of LCDA is an important nonperturbative parameter that gives the leading contribution in the factorization formula. The studies on IM of quasi-DA can shed light on the properties of IM of LCDA, and also on the lattice simulation of \( B \)-meson LCDA.

With a one-loop calculation of IMs, we derive the renormalization and RGE for the first IM of quasi-DA and figure out the correct form of matching relation in LaMET. The first IM of quasi-DA is not only factorized into the first IM but also the logarithmic moments of LCDA, accompanied with hard coefficients which are free of IR singularities and independent of external states.

The IM of quasi-DA can be simulated on a Euclidean lattice since it is defined with equal-time matrix elements. Distinguished from non-negative moments, IM is not defined by a local operator, so the determination of IM needs the information of the whole quasi-DA. One then may be worried about the significance of this work to the lattice QCD community at first glance, because if one calculated IM on the lattice, one should calculate quasi-DA first, but quasi-DA can be matched onto LCDA directly without calculating IM. In fact, the lattice simulation of nonlocal HQET matrix element is very challenging, and evaluating the continuous \( \omega \) distribution will be a very difficult task. A realistic idea will be building proper models for quasi-DA and fitting the parameters of models with a few lattice data. The IM of quasi-DA will be an essential parameter for quasi-DA models, just like the IM for the models of
LCDA [28, 31]. Then, with the fitted IM of quasi-DA from lattice data, one can map it onto the IM and logarithmic moments of LCDA. The mixing of IM and logarithmic moments of LCDA in the matching formula may cause other difficulties. Such difficulties caused by mixing may be overcome by evaluating $\tilde{\lambda}_B$ with several $v_3$ and extracting the IM and logarithmic moments by fitting the $v_3$ dependence. Further lattice simulations on IM of quasi-DA will be crucial to improve the determination of IM and other parameters of $B$-meson LCDA.

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