Proton charge radius extraction from muon scattering at MUSE using dispersively improved chiral effective field theory

F. Gil-Domínguez,^{1, *} J. M. Alarcón,^{2, †} and C. Weiss^{3, ‡}

¹Instituto de Física Corpuscular (IFIC) (centro mixto CSIC-UV),

Institutos de Investigación de Paterna, C/Catedrático José Beltrán 2, 46980 Paterna, Valencia, Spain

² Universidad de Alcalá, Grupo de Física Nuclear y de Partículas,

Departamento de Física y Matemáticas, 28805 Alcalá de Henares (Madrid), Spain

³ Theory Center, Jefferson Lab, Newport News, VA 23606, USA

The MUSE experiment at Paul Scherrer Institute will perform the first measurement of low-energy muon-proton elastic scattering (muon lab momenta 115–210 MeV) with the aim of determining the proton charge radius. We study the prospects for the proton radius extraction using the theoretical framework of Dispersively Improved Chiral Effective Field Theory (DI χ EFT). It connects the proton radii with the finite- Q^2 behavior of the form factors through complex analyticity and enables the use of data up to $Q^2 \sim 0.1 \text{ GeV}^2$ for radius extraction. We quantify the sensitivity of the μp cross section to the proton charge radius, the theoretical uncertainty of the cross section predictions, and the size of two-photon exchange corrections. We find that the optimal kinematics for radius extraction at MUSE is at momenta 210 MeV and $Q^2 \sim 0.05-0.08 \text{ GeV}^2$. We compare the performance of electron and muon scattering in the same kinematics. As a byproduct, we obtain explicit predictions for the μp and ep cross sections at MUSE as functions of the assumed value of the proton radius.

I. INTRODUCTION

The electromagnetic size is a fundamental characteristic of the proton observed in nuclear and atomic physics. It is quantified by the root-mean-squared radii $r_E \equiv \sqrt{\langle r^2 \rangle_E}$ and $r_M \equiv \sqrt{\langle r^2 \rangle_M}$, defined by the derivatives of the electric and magnetic form factors (FFs), G_E and G_M , at momentum transfer $Q^2 = 0$, see Ref. [1] for a review. The radii can be determined experimentally either from elastic electron-proton (ep) or muon-proton (μp) scattering or from the nuclear corrections to the energy levels of electronic or muonic hydrogen atoms.

The proton charge radius has been the object of extensive studies in the last decade. The extraction from muonic hydrogen measurements in 2010, $r_E =$ 0.84184(67) fm [2], differed by 5σ from the CODATA value accepted at the time, $r_E = 0.8768(69)$ fm [3], obtained from electronic hydrogen and electron scattering data ("proton radius puzzle"). The discrepancy motivated experimental and theoretical efforts aiming to improve the extraction methods, quantify the uncertainties, and reconcile the results; see Refs. [4, 5] for reviews. The questions raised include the performance of various methods for extraction of the radius from scattering data, the comparison of scattering and atomic results, and potential differences between electron and muon interactions. The cumulative results from these studies tend to favor the "smaller" charge radius. However, one essential piece is still missing - the extraction of the radius from lowenergy μp elastic scattering.

The MUSE experiment at Paul Scherrer Institute aims to perform the first precise determination of the proton charge radius from μp elastic scattering at muon lab momenta 115–210 MeV [6]. Good understanding of the theoretical uncertainties is needed in order to optimize the extraction procedure and assess the final error in the radius. Important questions are the sensitivity of the experimental observables to the proton radius, the theoretical uncertainty in the relation between the proton radius and the finite- Q^2 FFs, the size and uncertainty of two-photon exchange (TPE) corrections, and the optimal muon energy and Q^2 range for constraining the radius.

The analytic properties of the proton FF play an essential role in the radius extraction from scattering data. As a function of complex Q^2 , the form factor has singularities at $Q^2 < 0$, resulting from the *t*-channel exchange of hadronic states (pions, resonances) between the electromagnetic current and the proton. These singularities govern the behavior of the FF at $Q^2 > 0$, where it is measured in scattering experiments. This structure implies a correlation between the derivative of the FF at $Q^2 = 0$ and its values at finite Q^2 , which is essential for the radius extraction and must be implemented in the theoretical analysis.

The recently developed method of Dispersively Improved Chiral Effective Field Theory (DI χ EFT) [7, 8] combines dispersion relations with dynamical input from chiral EFT to describe the nucleon FFs at low Q^2 from first principles. It generates FFs with correct analytic properties (position of singularities) and realistic quantitative behavior (strength of singularities), which provide an excellent description of scattering data up to $Q^2 \sim$ 1 GeV² [9]. It also quantifies the theoretical uncertainty of the FF calculations. A special feature of this method is that it generates FF predictions that depend on the assumed proton radius as a parameter. As such it explicitly realizes the correlations between the proton radius and the finite- Q^2 behavior of the FF. It permits the use of finite- Q^2 data for the radius extraction with controlled

^{*} E-mail: fernando.gil@ific.uv.es

[†] E-mail: jmanuel.alarcon@uah.es

[‡] E-mail: weiss@jlab.org

uncertainties, which has many experimental and theoretical advantages. The method has been used successfully for the extraction of the proton electric and magnic radii from electron scattering data [9, 10].

In this work we use $DI\chi EFT$ to study the prospects for proton radius determination in μp elastic scattering at MUSE and optimize the extraction procedure. We compute the μp cross section with the $DI\chi EFT$ FFs, quantify the theoretical uncertainties and TPE corrections, and evaluate the sensitivity to the proton radius. Specifically, we attempt to answer the following questions:

- 1. What is the theoretical sensitivity of the μp cross section in MUSE kinematics to the proton radius?
- 2. What are the theoretical uncertainties in the μp cross section resulting from the DI χ EFT FF predictions and from TPE corrections?
- 3. What kinematic range in beam energy and Q^2 has the most impact on the radius extraction?
- 4. What are the differences between ep and μp scattering in radius extraction in MUSE kinematics?

We demonstrate that the radius extraction is characterized by a trade-off between several effects – the sensitivity of the cross section to the radius, the theoretical uncertainty in the FF predictions for a given radius, and the size and kinematic dependence of TPE corrections [9, 10]. We determine the optimal kinematics for radius extraction at MUSE based on these considerations. In addition, we provide predictions of the expected μp and ep cross sections for the nominal value of the proton radius.

II. METHODS

A. Lepton-proton elastic scattering

The elastic lepton-proton scattering process $l(k) + p(p) \rightarrow l(k') + p(p')$, where $l = \mu^{\mp}$ or e^{\mp} , is described by the invariant variables

$$s \equiv (k+p)^2, \qquad Q^2 = -t \equiv -(k-k')^2.$$
 (1)

In the initial proton rest frame (lab frame), the initial and final muon momenta are \mathbf{k} and \mathbf{k}' , the energies are $\omega \equiv \sqrt{|\mathbf{k}|^2 + m^2}$ and $\omega' \equiv \sqrt{|\mathbf{k}'|^2 + m^2}$, and the invariants are given by

$$s = M^2 + 2M\omega + m^2, \qquad Q^2 = 2M(\omega - \omega'), \qquad (2)$$

where *m* is the lepton mass and *M* the proton mass. The scattering angle $\theta_{\text{lab}} = \text{angle}(\mathbf{k}', \mathbf{k})$ is related to the final lepton energy and momentum by

$$\cos\theta_{\rm lab} = \frac{\omega\omega' - m^2 - M(\omega - \omega')}{|\mathbf{k}||\mathbf{k}'|}.$$
 (3)

The kinematic range of the momentum transfer accessible at a given initial lepton momentum is

$$0 \le Q^2 \le \frac{4M^2 |\mathbf{k}|^2}{s} \equiv Q_{\max}^2.$$
 (4)

In the one-photon-exchange approximation, the differential cross section for unpolarized scattering is given by (see e.g. Ref. [11])

$$\frac{d\sigma_{1\gamma}}{dQ^2} = \frac{\pi\alpha^2}{2M^2|\boldsymbol{k}|^2} \frac{(\epsilon/\tau_P) G_E^2 + G_M^2}{1 - \epsilon_T}.$$
(5)

Here α is the fine structure constant, and $G_{E,M} \equiv G_{E,M}(Q^2)$ are the electric and magnetic Sachs FFs of the proton. ϵ is the virtual photon polarization parameter and given by

$$\epsilon = \frac{Q_{\max}^2 - Q^2 + \frac{m^2}{s}(4M^2 + Q^2)}{Q_{\max}^2 - Q^2 + \frac{Q^2}{2s}(4M^2 + Q^2)},$$
(6)

and $\tau_P \equiv Q^2/(4M^2)$. ϵ/τ_P is the ratio of the fluxes of longitudinal and transverse polarized photons in the one-photon-exchange approximation. ϵ_T is the degree of linear polarization of the transverse photons,

$$\epsilon_T = \frac{Q_{\max}^2 - Q^2}{Q_{\max}^2 - Q^2 + \frac{Q^2}{2s}(4M^2 + Q^2)},$$
(7)

and is bounded by $0 \le \epsilon_T < 1$. In the case of zero lepton mass (as usually assumed in electron scattering) $\epsilon = \epsilon_T$, but for non-zero lepton mass (muon scattering) there are important differences. ϵ attains values > 1 at $Q^2 = 0$, and remains nonzero at $Q^2 = Q^2_{\text{max}}$,

$$\epsilon(Q^2 = 0) = \frac{\omega^2}{|\boldsymbol{k}|^2} > 1, \qquad (8)$$

$$\epsilon(Q^2 = Q_{\max}^2) = \frac{m^2 s}{2M^2 |\mathbf{k}|^2} > 0.$$
(9)

Two-photon exchange (TPE) corrections play an important role in the analysis of low-energy lepton-proton elastic scattering, see Refs. [12, 13] for a review. At order α^3 , the correction arises from the interference between the two-photon and one-photon exchange amplitudes and is usually included through a multiplicative factor modifying the one-photon exchange cross section,

$$\frac{d\sigma}{dQ^2} \approx \frac{d\sigma_{1\gamma}}{dQ^2} \left(1 + \delta_{2\gamma}\right). \tag{10}$$

The correction $\delta_{2\gamma}$ for μp scattering has been computed in several theoretical approaches, such as dispersion theory [11, 14] and chiral effective field theory [15–17]. In this work we use the results of Ref. [14], which give corrections $\delta_{2\gamma} \leq 0.5\%$ in the kinematic range of the MUSE experiment. While in Ref. [14] the inelastic contribution to the dispersion integral for $\delta_{2\gamma}$ was computed in forward kinematics, the analysis of Ref. [11] showed that this approximation is accurate within 10%.

B. $DI\chi EFT$ representation of form factors

The foundations of the DI χ EFT method and its applications are described in detail in Refs. [7, 8, 18]. Here we provide a brief summary, emphasizing the information flow (parameters) and other features relevant to proton radius extraction.

 $\mathrm{DI}\chi\mathrm{EFT}$ is based on dispersion theory, in which the FFs $G_{E,M}(t)$ at spacelike momentum transfer t < 0 are represented as integrals over their imaginary parts $\mathrm{Im}\,G_{E,M}(t)$ on the cut at timelike t > 0, the so-called spectral functions. The main steps in the $\mathrm{DI}\chi\mathrm{EFT}$ construction of the spectral functions and the FFs are summarized in Fig. 1. In the following we describe the steps for G_E ; the ones for G_M are similar.

In the first step, one constructs the spectral function. The proton FF has an isovector and isoscalar component, $G_E \equiv G_E^V + G_E^S$. The isovector FF G_E^V has the two-pion cut at $t > t_{\rm thr} = 4M_{\pi}^2$. The isovector spectral function is represented as the sum of two parts,

$$\operatorname{Im} G_E^V(t) = \operatorname{Im} G_E^V(t)[\pi\pi] + \operatorname{Im} G_E^V(t)[\text{high-mass}].$$
(11)

The $\pi\pi$ part covers the region $4M_{\pi}^2 \leq t < t_{\max} \approx 1 \text{ GeV}^2$ and is computed theoretically, using the elastic unitarity relation in the $\pi\pi$ channel and relativistic chiral EFT for the πN amplitudes. This approach includes the ρ resonance in the $\pi\pi$ channel and delivers realistic $\pi\pi$ spectral functions, which is essential for proton radius extraction (see below). The free parameters entering in this part are the low-energy constants of the chiral EFT calculation; in the present partial N²LO implementation this is one parameter, λ [8]. The high-mass part of the spectral function covers the region $t > 1 \text{ GeV}^2$ and is parametrized by a single effective pole $\pi a_1 \delta(t-t_1)$; this form has been shown to be sufficient for the dispersive analysis of low-tFFs, which see only the overall spectral strength in the high-mass region [8]. The free parameters entering in this part are the pole strength a_1 and pole position t_1 . The isoscalar FF G_E^S has a three-pion cut, and the spectral function is represented as

$$\operatorname{Im} G_E^S(t) = \operatorname{Im} G_E^S(t)[\pi \pi \pi] + \operatorname{Im} G_E^S(t)[\text{high-mass}]. (12)$$

The $\pi\pi\pi$ part is overwhelmingly concentrated in the ω resonance and parametrized by a pole $\pi a_{\omega} \delta(t - M_{\omega}^2)$. The high-mass part is parametrized by an effective pole, whose position can be taken as the ϕ mass, $\pi a_{\phi} \delta(t - M_{\phi}^2)$. The free parameters entering in the isoscalar spectral function are the pole strengths a_{ω} and a_{ϕ} . Altogether, this step results in a theoretical parametrization of the spectral function

$$\operatorname{Im} G_E(t|\Lambda) = \operatorname{Im} G_E^V(t) + \operatorname{Im} G_E^S(t), \qquad (13)$$

where $\{\Lambda\}$ collectively denotes the free parameters; in the present analysis $\{\Lambda\} = \{\lambda, a_1, t_1; a_{\omega}, a_{\phi}\}.$

In the second step, one imposes the dispersive sum

rules for the proton charge and radius

$$\frac{1}{\pi} \int_{t_{\rm thr}}^{\infty} dt \frac{\operatorname{Im} G_E(t|\Lambda)}{t} = Q_E, \qquad (14)$$

$$\frac{1}{\pi} \int_{t_{\rm thr}}^{\infty} dt \frac{\operatorname{Im} G_E(t|\Lambda)}{t^2} = \frac{r_E^2}{6},\tag{15}$$

where $Q_E = 1$ is the proton charge and $r_E^2 > 0$ the proton charge radius squared. (The same relations are imposed for the neutron electric FF, in which case $Q_E = 0$ and $r_E^2 < 0$ is the negative neutron charge radius squared.) These relations express the FF at t = 0 and its derivative as integrals over the spectral function. One uses them to constrain the parameters in the spectral function. In particular, Eq. (15) is valid for any assumed value of the proton charge radius r_E , and one can use it to express one of (or a combination of) the original parameters in terms of the radius, i.e., to introduce the radius as a parameter:

$$\{\Lambda\} \to \{r_E, \Lambda'\}. \tag{16}$$

In this way one obtains a set of spectral functions that depend explicitly on the assumed radius, as well as on the remaining parameters Λ'

$$\operatorname{Im} G_E(t|r_E, \Lambda'). \tag{17}$$

In the present analysis we use Eqs. (14) and (15) for the proton spectral function (and the same relations for the neutron) to fix the chiral low-energy constant λ and the effective pole strengths $a_1, a_{\omega}, a_{\phi}$, retaining the isovector effective pole position t_1 as the only undetermined parameter.

In the third step, one computes the spacelike FFs $(t < 0, \text{ or } Q^2 > 0)$ as the dispersion integral with the spectral function,

$$G_E(t|r_E, \Lambda') = \frac{1}{\pi} \int_{t_{\rm thr}}^{\infty} dt' \frac{\operatorname{Im} G_E(t'|r_E, \Lambda')}{t' - t}.$$
 (18)

The FF thus obtained depends on the assumed radius r_E and the undetermined parameters Λ' . The nominal prediction for the FF with assumed radius r_E is obtained with the nominal values of Λ' ,

$$G_E(t|r_E) = G_E(t|r_E, \Lambda'_{\text{nom}}).$$
(19)

In the last step, the theoretical uncertainty of the FF with assumed radius r_E is estimated by varying Λ' over a plausible range,

$$\delta G_E(t|r_E) = \operatorname{var}_{\Lambda'} G_E(t|r_E, \Lambda'). \tag{20}$$

In this way one obtains a nominal prediction and a theoretical uncertainty estimate for the form factor with any given assumed radius. In the present analysis, the undetermined parameter is the position of the isovector highmass pole; its nominal value is $t_1 = 2.1 \text{ GeV}^2$, and the plausible range of variation for the uncertainty estimate is $t_1 = 1.4-2.8 \text{ GeV}^2$ [10].



FIG. 1. Flowchart of the DI χ EFT description of the nucleon FFs. Shown is the case of the proton electric FF G_E ; the same flow applies to the neutron electric FF. In the case of the magnetic FF G_M , the right-hand side of the dispersive sum rules is given by the magnetic moment μ and the magnetic radius r_M^2 .

The magnetic FF G_M^V and its uncertainty are constructed by an analogous procedure. The dispersive sum rules for G_M now involve the magnetic moment μ and the magnetic radius r_M^2 see Ref. [8] for details. A computer code generating the radius-dependent DI χ EFT FFs $G_{E,M}$ used in the present analysis is available in the supplemental materials of Ref. [9].

C. Proton radius extraction

The DI χ EFT representation of the FFs enables a new theory-guided method of proton radius extraction (see Fig. 2) [9, 10]. For each assumed value of the proton radius, the theory generates a spectral function whose features (height of the ρ resonance peak, strength of effective poles) quantitatively depend on the value of the radius. The dispersion integral projects these features into the spacelike region, up to spacelike momentum transfers of the order $Q^2 \sim M_{\rho}^2$ and beyond. This effectively correlates the assumed value of the radius with the behavior

of the spacelike FF at finite Q^2 of this order. The correlation described here is based on complex analyticity and the particular information flow in the DI $\chi \rm EFT$ calculation and extends far beyond what one could infer from the series expansion in Q^2 with a given first derivative. Altogether, this allows one to recruit FF data of the order $Q^2 \sim M_\rho^2$ and beyond for constraining the proton radii.

Radius extraction using $DI\chi EFT$ proceeds as follows. For a range of assumed radii, one generates the $DI\chi EFT$ FFs as functions of Q^2 , including their theoretical uncertainties resulting from the undetermined parameters, Eqs. (19) and (20). From these FFs one predicts the cross section for the given assumed radius, including its theoretical uncertainty from the FFs and two-photon exchange corrections. For a given experimental setup (kinematic coverage, statistical and systematic errors) one can then assess how data in a given range of energies and Q^2 can constrain the radii. The optimal range is determined by a trade-off between the sensitivity of the $DI\chi EFT$ FFs to the value of the radius, the theoretical uncertainty of





FIG. 2. Illustration of the correlation between the proton radius and the spacelike FF, resulting from analyticity and the information flow in $DI\chi EFT$. The assumed value of the radius constrains the spectral function through the sum rule Eq. (15). The corresponding FF at $Q^2 > 0$ is produced by the dispersion integral Eq. (18). Variation of the radius causes variation of the spectral function and the corresponding FF. (The graph shows only the isovector part of the spectral function at t > 0.)

the DI χ EFT FFs, the two-photon exchange effects, and the precision of the data [9, 10]. The actual radius can then be determined by a fit in this optimal range, taking into account all the uncertainties. In the following we apply this method to μp scattering at MUSE and discuss the prospects for proton radius extraction.

The DI χ EFT method offers several advantages compared to other methods of proton radius extraction. Compared to empirical fits (polynomials, splines), the $DI\chi EFT$ method incorporates the analytic structure of the FFs, which includes both the position of the singularities at t > 0 and the quantitative distribution of strength in the spectral function. The analytic structure governs the global behavior of the FF, which is difficult to implement in approaches based on polynomial expansions because of strong correlations between higher-order coefficients (analyticity effectively controls the "collective behavior" of higher derivatives of the FF at $Q^2 = 0$ [7]). Compared to traditional dispersion analysis [19–22], the DI χ EFT method allows the strength of the isovector spectral function in the ρ resonance region to vary with the proton radius in a theoretically controlled manner, providing critical flexibility for fitting the spacelike FF data and recruiting them for radius determination. In traditional dispersive fits the $\pi\pi$ part of the isovector spectral function is completely fixed by theory, and the spacelike FF data only constrain the high-mass part of the spectral function, which restricts the interplay of the FF data with the proton radius (the ρ resonance region of the spectral function accounts for about half the value of the proton radius in Eq. (15) [7]).

III. ANALYSIS

A. Sensitivity of μp cross section to proton radius

We now apply the DI χ EFT framework to study the prospects for proton radius extraction at MUSE. In the first step, we study the sensitivity of the μp elastic scattering cross section to the proton electric radius and compare it with the theoretical uncertainties resulting from the DI χ EFT FF predictions, from two-photon exchange corrections, and from the magnetic FF contributions.

To exhibit the various effects, we generate a set of $DI\chi EFT$ FF predictions by varying the proton electric radius over the range $r_E = 0.83-0.88$ fm in steps $\Delta r_E =$ 0.01 fm, and evaluate the μp elastic scattering cross section with each of these FFs (the magnetic radius is kept at its nominal value; the role of the magnetic FF in the cross section is discussed below). We include in the cross section the TPE correction of Ref. [14]. Figure 3 shows the predicted cross sections for various incident muon momenta $k \equiv |\mathbf{k}|$, as functions of Q^2 . The lines show the cross section obtained with the nominal $DI\chi EFT$ FF predictions for each value of the radius, Eq. (19); the associated bands show the variation due to the theoretical uncertainty of the $DI\chi EFT$ FF predictions for the given radius, Eq. (20). The bands at the bottom of the plots show the absolute size of the TPE correction in the cross section predictions (note that this is the overall size of the TPE correction, not its theoretical uncertainty). The standard dipole cross section (σ_{SD}) used for normalization is the one-photon exchange cross section evaluated assuming the standard dipole Q^2 -dependence $\propto (1 + Q^2/0.71 \,\text{GeV}^2)^{-2}$ for both FFs $G_{E,M}$.

One observes: (a) The sensitivity of the cross section to the proton radius increases with Q^2 and with the beam momentum k, because the separation of the FF predictions with different radii increases with Q^2 [9, 10]. At the highest beam momentum, k = 210 MeV, the relative variation of the cross section reaches $\Delta\sigma/\sigma \sim 1\%$ for Q^2 at the upper end of the range shown here. (b) The theoretical uncertainty of the cross section predictions for given radius also increases with Q^2 [9, 10]. Overall, the theoretical uncertainty is substantially smaller than the relative variation of the cross section for $\Delta r_E = 0.01$ fm over the kinematic range shown here. (c) The magnitude of the TPE correction does not vary strongly with Q^2 and k over the range covered here. At the upper end of the Q^2 range, the magnitude of the TPE correction is comparable to the relative variation of the cross section with $\Delta r_E = 0.01$ fm. This clearly shows the importance of the TPE correction for radius extraction.

We also need to consider the uncertainties resulting from the contribution of the magnetic FF to the μp elastic scattering cross section. This is particularly important, as with the DI χ EFT framework we can recruit data at higher Q^2 for radius extraction, comparable to Q^2_{max} at the given k. Figure 4 shows the ratio of magnetic and electric contributions to the one-photon exchange cross



FIG. 3. $DI\chi EFT$ predictions for the $\mu^- p$ differential cross section at MUSE for several assumed values of the proton radius. The cross section predictions include the TPE correction, Eq. (10), and are normalized to the standard dipole cross section without TPE effects. *Lines (solid, dashed, dotted, dahed-dotted):* Nominal $DI\chi EFT$ predictions for the assumed value of the proton radius (see legend). *Shaded bands around lines:* Intrinsic theoretical uncertainty of $DI\chi EFT$ predictions, unrelated to assumed proton radius. *Shaded band at bottom:* TPE contribution to cross section [14].

section, $(\tau_P/\epsilon)G_M^2/G_E^2$, in MUSE kinematics. One sees that the ratio depends mainly on Q^2 , having values ~0.1 at $Q^2 = 0.04 \text{ GeV}^2$ and reaching ~0.4 at $Q^2 = 0.08$ GeV². Overall, the magnetic contributions to the cross section are limited in all kinematic settings. Using the DI χ EFT framework and the results of the analysis of epscattering data of Ref. [9], we have computed the effect



FIG. 4. Ratio of magnetic and electric contributions to the μp elastic scattering cross section, $(\tau_P/\epsilon)G_M^2/G_E^2$, in MUSE kinematics. The vertical dotted lines represent the kinematic upper limits of Q^2 at the given beam momentum k, Eq. (4).

of the experimental uncertainties of G_M on the μp cross section predictions in MUSE kinematics. We observe a maximum variation in the cross sections of the order of 0.04% at the highest Q^2 , which is small compared to the variation of ~1% resulting from a change of the electric radius $\Delta r_E/r_E = 1\%$. We conclude then that the current experimental uncertainties in G_M do not limit the extraction of the proton electric radius from the μp scattering data at the accuracy considered here.

B. Optimal kinematics for proton radius extraction

In the second step, we discuss the optimal kinematic range for the radius extraction at MUSE. It is determined by the trade-off between the sensitivity of the cross section to the radius, the theoretical uncertainties of the $DI\chi EFT$ FF predictions and the TPE corrections, and the experimental errors of the cross section measurement. While the experimental errors can only be estimated at present, some interesting conclusions can already be obtained at this stage.

To make this assessment, we use the difference between the cross section predictions for different radii in Fig. 3 as an estimate of the experimental accuracy required to discriminate between these values of the radii. At each value of Q^2 in Fig. 3, we compute the minimal difference between the cross section predictions for radii differing by a given Δr_E , taking the minimum over all pairs of radii with the given Δr_E , and taking into account their theoretical uncertainties (i.e., computing the minimal gap between the theoretical uncertainty bands of the cross section predictions for a given Δr_E). The minimal cross section difference computed in this way is independent of the nominal value of r_E . Figure 5 shows the minimal cross section differences obtained in this way, for radius differences $\Delta r_E = 0.01, 0.02, 0.03$ and 0.04 fm, as functions of Q^2 . One observes: (a) The cross section



FIG. 5. Estimated accuracy of $\mu^- p$ cross section measurement required to discriminate between different values of the proton radius. *Lines:* Differences between DI_XEFT cross section predictions for proton radii differing by Δr_E (values see legend). *Shaded band at bottom:* Size of the TPE contribution [14].

differences depend strongly on Q^2 for fixed k. They depend relatively weakly on k for fixed Q^2 (when comparing them at a fixed Q^2 that is kinematically accessible at multiple values of k). The main role of k is to define the kinematically accessible range of Q^2 . (b) At low values of Q^2 , high experimental precision is needed for radius determination. At $Q^2 \leq 0.015 \text{ GeV}^2$, a relative accuracy $\leq 0.2\%$ is needed for $\Delta r_E = 0.01$ fm, independently of k. (c) The demands on the experimental accuracy decrease

at higher Q^2 . At $Q^2 \sim 0.05$ GeV², a relative accuracy $\sim 0.5\%$ is needed for $\Delta r_E = 0.01$ fm.

Another factor to consider is the uncertainty in the theoretical calculation of the TPE correction [14]. Depending on the kinematics, this contribution to the cross section can be crucial for determining the radius with the necessary precision. Figure 5 compares the value of the TPE correction with the predicted cross section differences for a given radius difference (note that the plots show the estimated total value of the TPE correction, not its uncertainty). The theoretical uncertainty of the TPE correction is not well known; however, we can assess how an assumed theoretical uncertainty of the TPE correction would impact on the overall uncertainty of the radius extraction. For the lowest beam momentum, k =115 MeV, the size of the TPE correction is larger than the variation of the cross section prediction for $\Delta r_E = 0.01$ fm. The TPE correction thus has a decisive influence on the radius extraction in this kinematics. The situation becomes more favorable at higher beam momenta, where the TPE correction is comparable or smaller than the cross section variation for $\Delta r_E = 0.01$ fm.

Overall, our analysis suggests that the optimal kinematics for proton radius determination at MUSE with the DI χ EFT method is at the highest beam momentum, k = 210 MeV, using momentum transfers $Q^2 \sim 0.05-$ 0.08 GeV², at the upper end of the experimentally accessible range. In this setting, the experimental precision required for radius determination with $\Delta r_E = 0.01$ fm is estimated at $\approx 0.8\%$. The final uncertainty of the radius extraction depends on the theoretical uncertainty of the TPE correction, which is not known at present. An alternative method for proton radius extraction with μp scattering uses the average of μ^+ and μ^- cross sections,

$$\bar{\sigma} \equiv [\sigma(\mu^+ p) + \sigma(\mu^- p)]/2, \qquad (21)$$

in which the TPE correction cancels due to its charge dependence. The same $DI\chi EFT$ analysis of radius sensitivity and optimal kinematics as above can be performed in this case. The cross section prediction is now given by the one-photon exchange cross section Eq. (5). Figure 6 shows the predicted cross section and its theoretical uncertainty. The assessment of the optimal Q^2 values is the same as for $\mu^+ p$ above.

C. Cross section prediction for nominal radius

The present study focuses on the prospects for extracting the proton radius from μp scattering experiments at MUSE. The proton radius can also be extracted from atomic spectroscopy and ep scattering experiments. In this context we can use DI χ EFT to predict the μp cross section expected for a given value of the radius and its theoretical uncertainty. For reference, we give here the prediction for the μp cross section with the proton charge radius obtained in the previous DI χ EFT analysis of ep



FIG. 6. $DI_{\chi}EFT$ predictions for the average of μ^+ and μ^- elastic scattering cross sections Eq. (21) in MUSE kinematics, normalized to the standard dipole. *Lines, shaded bands:* Same notation as in Fig. 3.

scattering results [9, 10]

$$r_E = 0.842(2) \text{ fm.}$$
 (22)

Figure 7 shows the predicted $\mu^- p$ cross section in MUSE kinematics, Eq. (10), which includes the TPE correction; and the charge-averaged cross section, Eq. (21) in which the TPE correction cancels.



FIG. 7. DI χ EFT predictions for μp elastic scattering cross at MUSE for the nominal proton charge radius $r_E = 0.842(2)$ fm. Dashed line and shaded band (blue): $\mu^- p$ cross section, including one- and two-photon exchange contributions. Solid line and shaded band (red): Average of $\mu^+ p$ and $\mu^- p$ cross sections, given by the one-photon exchange contribution.

D. Comparison of ep and μp scattering

It is interesting to compare the prospects for proton radius extraction in ep and μp scattering in the same kinematics. The MUSE experiment will measure both ep and μp scattering, and methods for proton radius extraction were studied intensively in earlier ep scattering experiments. Characteristic differences between ep and μp occur in the TPE effects [14, 23] and in the role of the magnetic FF. We exhibit them by repeating the DI χ EFT analysis for ep scattering and comparing with the μp results.

Figure 8 shows the DI χ EFT predictions for the *ep* cross section for a range of assumed values of the proton radius, in the same style as Fig. 3 for μp . One observes: (a) The





FIG. 8. DI χ EFT predictions for the differential cross section of *ep* scattering for several assumed value of the proton radius (compare with Fig. 3 for μp scattering). The cross section predictions include the TPE correction and are normalized by the standard dipole cross section without TPE effects. *Solid lines:* Nominal DI χ EFT predictions for the assumed value of the proton radius. *Bands around solid lines:* Intrinsic theoretical uncertainty of DI χ EFT prediction (unrelated to assumed proton radius). *Blue band at bottom:* TPE contribution to cross section [23].

TPE corrections have different kinematic dependence in ep than in μp scattering [14, 23]. In ep they increase strongly with Q^2 at fixed k, and decrease with k at fixed Q^2 In μp the dependencies are much weaker. (b) The size of the TPE corrections relative to the variation of the cross section with the radius is much larger in ep than in μp , especially at low beam momenta. At k = 115 MeV and $Q^2 = 0.02$ GeV², the size of TPE correction



FIG. 9. Estimated accuracy of e^-p cross section measurements required to discriminate between different values of the proton radius (compare with Fig. 5 for μp scattering). *Lines:* Differences between DI χ EFT cross section predictions for proton radii differing by Δr_E (values see legend). *Shaded band at bottom:* Size of the TPE contribution [23].

amounts to a change of the radius $\Delta r_E \approx 0.03$ fm in ep scattering, compared to $\Delta r_E \approx 0.015$ fm in μp scattering in the same kinematics. (c) Overall, the different size and kinematic dependence of the TPE corrections causes a different Q^2 -dependence of the cross section for ep and μp scattering at low Q^2 .

Figure 9 shows the estimated accuracy of the e^-p cross section measurement required for discriminating between different values of the radius, in the same style as Fig. 5



FIG. 10. DI χ EFT predictions of the ratio of μp to ep elastic scattering cross sections σ , Eq. (10), including the TPE corrections, for several assumed values of the proton radius.

for $\mu^- p$. The optimal kinematics for radius extraction in ep scattering at MUSE can be determined in the same way as for μp . The results of Figure 9 show that the experimental accuracy required for radius extraction from ep scattering is least at the highest beam momentum k = 210 MeV. The optimal Q^2 value determined by the trade-off between radius sensitivity and theoretical uncertainty is at $Q^2 \sim 0.065$ GeV², slightly below the kinematic limit.

Figure 10 shows the ratio of the μp to ep cross sections (including the TPE corrections) for the same set of assumed proton radii. The ratio directly expresses the different Q^2 -dependence of the ep and μp cross sections. Its magnitude and Q^2 -dependence are determined by the



FIG. 11. Ratio of magnetic and electric contributions to the ep elastic scattering cross section, $(\tau_P/\epsilon)G_M^2/G_E^2$, in MUSE kinematics (compare with Fig. 4 for μp scattering). The vertical dotted lines represent the kinematic upper limits of Q^2 at the given beam momentum k, Eq. (4).

kinematic factors in the one-photon-exchange cross section Eq. (5) et seq. One observes that the ratio is remarkably insensitive to the proton charge radius in this kinematic regime, especially at the lower values of k. Only at k = 210 MeV the differences between the radii become visible at the largest Q^2 values.

The results of Fig. 8 show that the TPE corrections play a much larger role in proton radius extraction from ep scattering than μp scattering, and that they limit the theoretical uncertainty of the extracted radius. With the DI χ EFT method, the influence of TPE corrections in epcan be minimized by using the data at the highest beam momentum k = 210 MeV and momentum transfers in the range $Q^2 \sim 0.03$ –0.08 GeV² for radius extraction. In this kinematics the cross section shows good sensitivity to the proton charge radius, the theoretical uncertainty of the DI χ EFT predictions is small, and the size of the TPE corrections amounts to a shift of the radius $\Delta r_E \sim 0.01$ fm (see Fig. 8). The ability to recruit higher- Q^2 data for radius extraction with DI χ EFT is thus even more advantageous in ep than in μp scattering.

An important difference between ep and μp scattering appears in the contribution of the magnetic FF at large momentum transfers, at the upper end of the allowed kinematic range. In ep scattering $\epsilon = 0$ at $Q^2 \sim Q_{\text{max}}^2$ (if one neglects the electron mass), while in μp scattering ϵ attains a finite value, see Eq. (9). In *ep* scattering the one-photon exchange cross section for $Q^2 \to Q^2_{\text{max}}$ is therefore dominated by G_M . Figure 11 shows the ratio of magnetic and electric contributions to the one-photonexchange cross section for *ep* scattering, in the same way as Fig. 4 for μp . One sees that the magnetic contribution to the cross section is substantially larger in ep than μp already for Q^2 in the middle of the kinematic range. This circumstance must be taken into account when assessing the sensitivity of the cross section to r_E in DI χ EFT, and one should remain in the region where the cross section



FIG. 12. DI χ EFT predictions for the average of e^+ and e^- elastic scattering cross sections, as in Eq. (21) for $\mu^+ p$ and $\mu^- p$, in MUSE kinematics (compare with Fig. 6 for μp scattering).

is not dominated by G_M . We have quantified the impact of the uncertainty in G_M on the proton radius extraction from ep scattering in the same way as for μp (see Sec. III A), using the DI χ EFT framework and the experimental uncertainty of G_M obtained in an earlier analysis [9]. We find that the current experimental uncertainty of G_M produces a relative variation of the cross section of at most ~ 0.05% in the range covered by Fig. 8. The uncertainty from G_M is thus not a limiting factor of the DI χ EFT-based radius extraction from ep scattering data in MUSE kinematics.

Because of the large TPE corrections in ep scattering, it would be an attractive option to perform the proton radius extraction with the average of e^-p and e^+p cross sections [see Eq. (21) for $\mu^{\pm}p$], in which the TPE effects cancel. In this case the cross section is accurately described by the one-photon-exchange formula, and the analysis greatly simplifies. Figure 12 shows the DI χ EFT predictions for the charge radius dependence of the charge-averages cross section $\bar{\sigma}$ in $e^{\pm}p$ scattering in MUSE kinematics.

IV. CONCLUSIONS

In this work we have used the DI χ EFT framework to study the prospects for the proton radius extraction from μp scattering at MUSE. The principal conclusions are:

(i) When extracting the radius from fits to cross section data at low momentum transfers $Q^2 < 0.01 \text{ GeV}^2$, the TPE corrections need to be included with high precision. At $Q^2 < 0.01 \text{ GeV}^2$, the estimated absolute size of TPE correction amounts to a shift of the extracted radius by 0.03–0.04 fm. Any theoretical uncertainty of the TPE correction will influence the extracted radius proportionally.

(ii) The DI χ EFT method allows one to extract the radius from fits to cross section data at higher momentum transfers $Q^2 \sim$ few times 0.01 GeV² in the MUSE kinematic range. This is advantageous experimentally, because the higher sensitivity of the cross section to the radius lowers the demands on the experimental precision of the cross section measurement. It is also advantageous theoretically, as it reduces the influence of the TPE correction on the radius extraction.

(iii) The optimal kinematics for the DI χ EFT-based radius extraction at MUSE is k = 210 MeV and $Q^2 \sim 0.05-$ 0.08 GeV², at the upper end of the kinematic coverage. It is determined by the trade-off between theoretical effects – the sensitivity of the cross section to the radius, the uncertainty of the DI χ EFT FF predictions, and the TPE correction. In this kinematics, even a 100% uncertainty of the TPE correction would shift the extracted radius only by 0.01 fm. An experimental precision of \leq 0.5% is required for determining the radius with 0.01 fm accuracy. Such accuracy would be sufficient for solving the proton radius puzzle.

(iv) In ep scattering in MUSE kinematics, the TPE corrections are generally larger that in μp , and the advantages of using the DI χ EFT method with higher- Q^2 data for radius extraction are even more compelling. The ratio of same-charge ep and μp cross sections is predicted to be practically independent of the proton radius and can be used for validation of the analysis.

Our findings affirm the need for accurate theoretical estimates of the TPE corrections in elastic μp and ep scattering. If the radius extraction is performed using the DI χ EFT framework and data at finite momentum transfers $Q^2 \sim 0.05$ –0.08 GeV², as recommended here, efforts should focus on improving the TPE estimates in this kinematic region. At these finite values of Q^2 the con-

straints on the TPE amplitude from the limit of forward scattering $(Q^2 = 0)$ are less restrictive, and the calculations become more dependent on dynamical assumptions [14, 23]. Methods based on the $1/N_c$ expansion of QCD could enable systematic calculations of TPE effects with controlled theoretical uncertainties; see Refs. [24, 25] for recent developments.

The DI χ EFT framework used in the present study is a general method that could be improved through further development. In particular, the theoretical uncertainties of the FF predictions could be reduced by using a more flexible parametrization of the high-mass states in the spectral function, Eq. (11), and constraining it with spacelike nucleon FF data. In the analysis here we have used the version of Ref. [8], where the high-mass states are described by a single effective pole, which permits simple uncertainty estimates and is sufficient for the MUSE Q^2 -range. A version with a more elaborate de-

- G. A. Miller, Defining the proton radius: A unified treatment, Phys. Rev. C 99, 035202 (2019), arXiv:1812.02714 [nucl-th].
- [2] R. Pohl *et al.*, The size of the proton, Nature 466, 213 (2010).
- [3] P. J. Mohr, B. N. Taylor, and D. B. Newell, CO-DATA Recommended Values of the Fundamental Physical Constants: 2006, Rev. Mod. Phys. 80, 633 (2008), arXiv:0801.0028 [physics.atom-ph].
- [4] R. Pohl, R. Gilman, G. A. Miller, and K. Pachucki, Muonic hydrogen and the proton radius puzzle, Ann. Rev. Nucl. Part. Sci. 63, 175 (2013), arXiv:1301.0905 [physics.atom-ph].
- [5] C. E. Carlson, The Proton Radius Puzzle, Prog. Part. Nucl. Phys. 82, 59 (2015), arXiv:1502.05314 [hep-ph].
- [6] R. Gilman *et al.* (MUSE), Technical Design Report for the Paul Scherrer Institute Experiment R-12-01.1: Studying the Proton "Radius" Puzzle with μp Elastic Scattering (2017), arXiv:1709.09753 [physics.ins-det].
- [7] J. M. Alarcón and C. Weiss, Nucleon form factors in dispersively improved Chiral Effective Field Theory II: Electromagnetic form factors, Phys. Rev. C 97, 055203 (2018), arXiv:1710.06430 [hep-ph].
- [8] J. M. Alarcón and C. Weiss, Accurate nucleon electromagnetic form factors from dispersively improved chiral effective field theory, Phys. Lett. B 784, 373 (2018), arXiv:1803.09748 [hep-ph].
- [9] J. M. Alarcón, D. W. Higinbotham, and C. Weiss, Precise determination of the proton magnetic radius from electron scattering data, Phys. Rev. C 102, 035203 (2020), arXiv:2002.05167 [hep-ph].
- [10] J. M. Alarcón, D. W. Higinbotham, C. Weiss, and Z. Ye, Proton charge radius extraction from electron scattering data using dispersively improved chiral effective field theory, Phys. Rev. C 99, 044303 (2019), arXiv:1809.06373 [hep-ph].
- [11] O. Tomalak and M. Vanderhaeghen, Dispersion relation formalism for the two-photon exchange correction to elas-

scription of the high-mass states, using multiple poles with randomized parameters for the uncertainty estimates, was described in Ref. [26] and could be employed for elastic scattering analysis and radius extraction at higher $Q^2 \lesssim 1 \text{ GeV}^2$.

ACKNOWLEDGMENTS

We thank A. Gramolin for making available a computer code that was used to validate the numerical calculation of the one-photon exchange cross section, and P. Blunden and W. Melnitchouk for helpful correspondence about two-photon exchange effects.

J.M.A. acknowledges support from the Spanish MICINN grant PID2019-106080GB-C21. This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics under contract DE-AC05-06OR23177.

tic muon-proton scattering: elastic intermediate state, Eur. Phys. J. C **78**, 514 (2018), arXiv:1803.05349 [hep-ph].

- [12] C. E. Carlson and M. Vanderhaeghen, Two-Photon Physics in Hadronic Processes, Ann. Rev. Nucl. Part. Sci. 57, 171 (2007), arXiv:hep-ph/0701272.
- [13] J. Arrington, P. G. Blunden, and W. Melnitchouk, Review of two-photon exchange in electron scattering, Prog. Part. Nucl. Phys. 66, 782 (2011), arXiv:1105.0951 [nuclth].
- [14] O. Tomalak and M. Vanderhaeghen, Two-photon exchange correction to muon-proton elastic scattering at low momentum transfer, Eur. Phys. J. C 76, 125 (2016), arXiv:1512.09113 [hep-ph].
- [15] P. Talukdar, V. C. Shastry, U. Raha, and F. Myhrer, Lepton-Proton Two-Photon Exchange in Chiral Perturbation Theory, Phys. Rev. D 101, 013008 (2020), arXiv:1911.06843 [nucl-th].
- [16] P. Talukdar, V. C. Shastry, U. Raha, and F. Myhrer, Radiative and chiral corrections to elastic lepton-proton scattering in chiral perturbation theory, Phys. Rev. D 104, 053001 (2021), arXiv:2010.09380 [nucl-th].
- [17] X.-H. Cao, Q.-Z. Li, and H.-Q. Zheng, Radiative correction to lepton proton scatterings in manifestly Lorentzinvariant chiral perturbation theory, Phys. Rev. D 105, 094008 (2022), arXiv:2112.06230 [hep-ph].
- [18] J. M. Alarcón and C. Weiss, Nucleon form factors in dispersively improved chiral effective field theory: Scalar form factor, Phys. Rev. C 96, 055206 (2017), arXiv:1707.07682 [hep-ph].
- [19] G. Hohler, E. Pietarinen, I. Sabba Stefanescu, F. Borkowski, G. G. Simon, V. H. Walther, and R. D. Wendling, Analysis of Electromagnetic Nucleon Form-Factors, Nucl. Phys. B **114**, 505 (1976).
- [20] M. A. Belushkin, H. W. Hammer, and U. G. Meissner, Dispersion analysis of the nucleon form-factors including meson continua, Phys. Rev. C75, 035202 (2007), arXiv:hep-ph/0608337 [hep-ph].

- [21] I. Lorenz, H.-W. Hammer, and U.-G. Meissner, The size of the proton – closing in on the radius puzzle, Eur.Phys.J.A 48, 151 (2012), arXiv:1205.6628 [hep-ph].
- [22] M. Hoferichter, B. Kubis, J. Ruiz de Elvira, H. W. Hammer, and U. G. Meißner, On the $\pi\pi$ continuum in the nucleon form factors and the proton radius puzzle, Eur. Phys. J. A **52**, 331 (2016), arXiv:1609.06722 [hep-ph].
- [23] O. Tomalak and M. Vanderhaeghen, Two-photon exchange correction in elastic unpolarized electron-proton scattering at small momentum transfer, Phys. Rev. D 93, 013023 (2016), arXiv:1508.03759 [hep-ph].
- [24] J. L. Goity, C. Weiss, and C. T. Willemyns, Target nor-

mal single-spin asymmetry in inclusive electron-nucleon scattering with two-photon exchange: Analysis using $1/N_c$ expansion, Phys. Lett. B **835**, 137580 (2022), arXiv:2207.07588 [hep-ph].

- [25] J. L. Goity, C. Weiss, and C. Willemyns, Target normal single-spin asymmetry in inclusive electron-nucleon scattering in the $1/N_c$ expansion, Phys. Rev. D **107**, 094026 (2023), arXiv:2303.12681 [hep-ph].
- [26] J. M. Alarcón and C. Weiss, Transverse charge and current densities in the nucleon from dispersively improved chiral effective field theory, Phys. Rev. D 106, 054005 (2022), arXiv:2204.11863 [hep-ph].