Ambiguities in Partial Wave Analysis of Two Spinless Meson Photoproduction

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We describe the formalism to analyze the mathematical ambiguities arising in partial-wave analysis of two spinless mesons produced with a linearly polarized photon beam. We show that partial waves are uniquely defined when all accessible observables are considered, for a wave set which includes S and D waves. The inclusion of higher partial waves does not affect our results, and we conclude that there are no mathematical ambiguities in partial-wave analysis of two mesons produced with a linearly polarized photon beam. We present Monte Carlo simulations to illustrate our results.

I. INTRODUCTION

In hadron spectroscopy, the extraction and interpretation of data from scattering experiments typically employ partial-wave analyses to isolate resonant contributions. However, these partial-wave expansions need not be unique, and, depending on the reaction, one may find multiple wave sets which produce mathematically equivalent predictions for the observables. This causes significant problems in the analysis and interpretation of data. These mathematical ambiguities have been extensively studied for various processes [1–4] and there is no generic prescription to remedy them. Hence, the issue must be addressed on a case-by-case basis (see Refs. [5–8] for some recent examples). To remedy ambiguities, typically one must generate all possible ambiguous wave sets and select one of them by enforcing additional constraints like global continuity [9] or unitarity [10]. Most previous analyses of mathematical ambiguities for partial-wave analysis examine nucleon or pion-beam production processes. In this work, we introduce the formalism for the examination of mathematical ambiguities in two pseudoscalar meson photoproduction processes with a linearly polarized photon beam, such as those present in the GlueX experiment at Jefferson Lab [11].

The physics program for the GlueX experiment focuses on the search for light exotic mesons. Some of the final states under consideration involve the two pseudoscalar mesons $\eta^{(\prime)}\pi$, for which odd waves have exotic quantum numbers incompatible with a $q\bar{q}$ assignment [12]. The dominant non-exotic signal in these final states is the $a_2(1320)$ resonance which populates the D waves [13]. It is essential to first accurately identify all relevant D-wave components before extracting the weaker exotic signal in the P-waves [5, 14–16]. In this paper, we address the issue of ambiguous solutions in partial wave analyses which are relevant to the extraction of the D-wave components, but our work is applicable to the general case of photoproproduction of any two spinless mesons. Our methods are based on the concept of Barrelet zeros, which we review in Appendix A for completeness. In Section II we introduce our notation and formalism for the photoproduction of two spinless mesons with a linearly polarized photon beam. We then demonstrate, using a wave set with two or three D-wave components accompanied by

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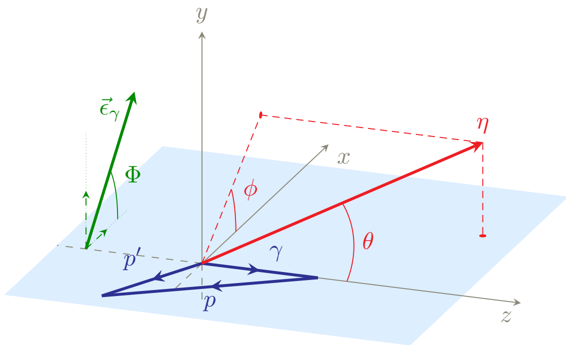

FIG. 1: Definition of the angles in the Gottfried-Jackson frame. In the two-meson rest frame, the z axis is given by the photon beam (γ), and the xz reaction plane contains also the nucleon target (p) and recoiling nucleon (p') momenta. θ and ϕ are the polar and azimuthal angles of the η. The polarization vector of the photon ($\vec{\epsilon}_\gamma$) forms an angle Φ with the reaction plane.

an S-wave, that there are no mathematical ambiguities. We also provide arguments supporting the absence of ambiguous solutions in more general cases. In Section IV we present results of numerical simulations, which show that there is indeed a unique solution with the highest likelihood. However, the likelihood function contains many local maxima that may lead to false solutions if appropriate care is not taken when performing fits. The summary and conclusions are given in Section V.

II. FORMALISM

We consider the photoproduction on a nucleon target of a meson resonance decaying into two spinless mesons, *e.g.* $\gamma p \to p\,\eta\,\pi^0$. We follow Ref. [11], writing

$$
\begin{aligned}
I(\Omega,\Phi) &= \frac{\mathrm{d}\sigma}{\mathrm{d}t\,\mathrm{d}m_{\eta\pi^0}\,\mathrm{d}\Omega\,\mathrm{d}\Phi} \\
&= \kappa \sum_{\substack{\lambda_\gamma,\lambda'_\gamma \\ \lambda_1,\lambda_2}} A_{\lambda_\gamma;\lambda_1\lambda_2}(\Omega)\rho^{\gamma}_{\lambda_\gamma\lambda'_\gamma}(\Phi)A^*_{\lambda'_\gamma;\lambda_1\lambda_2}(\Omega), \quad (1)
\end{aligned}
$$

where $\Omega = (\theta,\phi)$ are the decay angles of the resonance in the Gottfried-Jackson or helicity frame, and Φ is the polarization angle with respect to the production plane. The spin density matrix is given by $\rho_\gamma(\Phi) = (1 - P_\gamma \cos 2\Phi\,\sigma_x - P_\gamma \sin 2\Phi\,\sigma_y \cdot \sigma)/2$, and P_γ indicates the degree of polarization. Since the analysis of ambiguities is performed independently in each bin of t and $\eta\pi^0$ invariant mass, these dependences are understood. The dependence on the nucleon spin is neglected. The phase space factor κ does not depend on angular variables and will be absorbed into the amplitudes.

From the helicity amplitudes, one can construct partial waves in the reflectivity basis as linear combinations of the partial waves with different photon helicities in such a way that in the high energy limit positive (negative) reflectivity corresponds to natural (unnatural) parity exchanges [11]. The helicity amplitudes are given as:

$$
A_{\lambda_\gamma}(\Omega) = \begin{cases} \sum_{\ell m \epsilon} [\ell]^{(\epsilon)}_m Y^m_l(\Omega) & \text{for } \lambda_\gamma = +\,, \\ \sum_{\ell m \epsilon} -\epsilon(-1)^m\,[\ell]^{(\epsilon)}_{-m}\,Y^m_l(\Omega) & \text{for } \lambda_\gamma = -\,. \end{cases} \quad (2)
$$

Here, $\epsilon = \pm$ is the reflectivity and $[\ell]^{(\epsilon)}_m$ refers to the partial wave with angular momentum ℓ, spin projection m, and reflectivity ϵ. We write the intensity of the final products as:

$$
\begin{aligned}
I(\Omega,\Phi) = I^0(\Omega) &- P_\gamma I^1(\Omega)\cos(2\Phi) \\
&- P_\gamma I^2(\Omega)\sin(2\Phi), \quad (3)
\end{aligned}
$$

where I^0 is the unpolarized intensity, $I^{1,2}$ are polarized intensities.

The intensities are quadratic in the partial waves and, using the same notation as in Ref. [11], can be written in terms of the amplitudes $U^{(\epsilon)}$ and $\tilde{U}^{(\epsilon)}$ in the reflectivity basis:

$$
U^{(\epsilon)}(\Omega) = \sum_{\ell,m} [\ell]^{(\epsilon)}_m Y^m_\ell(\Omega), \quad (4a)
$$

$$
\tilde{U}^{(\epsilon)}(\Omega) = \sum_{\ell,m} [\ell]^{(\epsilon)}_m \left[Y^m_\ell(\Omega)\right]^*. \quad (4b)
$$

Then, the intensities in Eq. (3) may be expressed:

$$
I^0(\Omega) = \sum_\epsilon \left\{ |U^{(\epsilon)}(\Omega)|^2 + |\tilde{U}^{(\epsilon)}(\Omega)|^2 \right\}, \quad (5a)
$$

$$
I^1(\Omega) = -2\sum_\epsilon \epsilon\,\mathrm{Re}\left\{ U^{(\epsilon)}(\Omega)\,\left[\tilde{U}^{(\epsilon)}(\Omega)\right]^* \right\}, \quad (5b)
$$

$$
I^2(\Omega) = -2\sum_\epsilon \epsilon\,\mathrm{Im}\left\{ U^{(\epsilon)}(\Omega)\,\left[\tilde{U}^{(\epsilon)}(\Omega)\right]^* \right\}. \quad (5c)
$$

The dependence on the polar angle θ can be written explicitly by expanding the intensities in a Fourier series in the azimuthal decay angle ϕ:

$$
I^0(\Omega) = \frac{1}{2\pi}\left[\ h^0_0(\theta) + \ h^0_1(\theta)\cos(\phi) + \ \ldots\right], \quad (6a)
$$

$$
I^1(\Omega) = -\frac{1}{2\pi}\left[\ h^1_0(\theta) + \ h^1_1(\theta)\cos(\phi) + \ \ldots\right], \quad (6b)
$$

$$
I^2(\Omega) = -\frac{1}{2\pi}\left[\ 0 \ + \ h^2_1(\theta)\sin(\phi) + \ \ldots\right]. \quad (6c)
$$

Here the ellipses denote terms of higher order harmonics in ϕ.

The functions $h^\alpha_M(\theta)$, which we will refer to as (un)polarized moments, are quadratic in the partial waves and relate them to the measurable angular distribution of the two mesons in their center of mass frame.

We note that positive and negative reflectivity contributions sum up incoherently, and one can decompose $h^\alpha_M(\theta)$ into an explicit sum of reflectivity components , *i.e.* $h^\alpha_M(\theta) = {}^{(+)}h^\alpha_M(\theta) + {}^{(-)}h^\alpha_M(\theta)$. The two reflectivities can be distinguished from each other due to the dependence on the polarization angle Φ. From Eq. (5), one notes that the unpolarized intensity I^0 is a sum of the two reflectivity components, whereas the polarized intensities $I^{1,2}$ are equal to differences of the two components. We can therefore deal with each reflectivity independently, noting that the mathematical treatment of ambiguities is identical for each.

To pursue our analysis of Barrelet zeros, we need to express the observables $h^\alpha_M(\theta)$ as polynomials of $\tan\frac{\theta}{2}$, from which we can extract their roots. We first employ Eqs. (4) and (5) and rewrite Eq. (6) as:

$$I^0(\Omega) = \frac{1}{2\pi}\sum_{\epsilon m m'} f^{(\epsilon)}_m(\theta) f^{(\epsilon)*}_{m'}(\theta)\cos[(m-m')\phi], \tag{7a}$$

$$I^1(\Omega) = \frac{-1}{2\pi}\sum_{\epsilon m m'} \epsilon f^{(\epsilon)}_m(\theta) f^{(\epsilon)*}_{m'}(\theta)\cos[(m+m')\phi], \tag{7b}$$

$$I^2(\Omega) = \frac{-1}{2\pi}\sum_{\epsilon m m'} \epsilon f^{(\epsilon)}_m(\theta) f^{(\epsilon)*}_{m'}(\theta)\sin[(m+m')\phi], \tag{7c}$$

where,

$$\begin{aligned} f^{(\epsilon)}_m(\theta) &= \sum_\ell \sqrt{4\pi}\,[\ell]^{(\epsilon)}_m Y^m_\ell(\theta,0) \\ &= \sum_\ell \sqrt{2\ell+1}\,[\ell]^{(\epsilon)}_m d^\ell_{m0}(\theta). \end{aligned} \tag{8}$$

The Wigner d-function, $d^\ell_{m0}(\theta)$,[1] is a polynomial in $\cos\theta$ only for $m=0$. For $m \neq 0$ it is a polynomial of $\cos\theta$ of order $l-|m|$ multiplied by a factor $\sin^{|m|}(\theta)$. We thus represent the d-functions in terms of $u=\tan\theta/2$ by [2]:

$$d^\ell_{m0}(\theta) = \left(\frac{u}{1+u^2}\right)^\ell (-1)^m \varepsilon^\ell_m(u)\,, \tag{9}$$

with the polynomial $\varepsilon^\ell_m(u)$ defined as:

$$\varepsilon^\ell_m(u) = \sum_k (-1)^k \frac{u^{2k+m-\ell}\ell![(\ell-m)!(\ell+m)!]^{1/2}}{(\ell-m-k)!(\ell-k)!(m+k)!k!}\,. \tag{10}$$

The summation over k is restricted to the range $k \in [\max(0,-m), \min(\ell, \ell-m)]$.

By matching Eqs. (6) and (7), we obtain a relation between the observable quantities and the reflectivity partial waves:

$${}^{(\epsilon)}h^0_M = \sum_{m,m'} f^{(\epsilon)}_m f^{(\epsilon)*}_{m'} \delta_{M,|m-m'|}\,, \tag{11a}$$

$${}^{(\epsilon)}h^1_M = \epsilon\sum_{m,m'} f^{(\epsilon)}_m f^{(\epsilon)*}_{m'} \delta_{M,|m+m'|}\,, \tag{11b}$$

$${}^{(\epsilon)}h^2_M = \epsilon\sum_{m,m'} f^{(\epsilon)}_m f^{(\epsilon)*}_{m'} \delta_{M,|m+m'|}\,\mathrm{sign}(m+m')\,. \tag{11c}$$

Since each $f^{(\epsilon)}_m(\theta)$ is a complex function, and each $h^\alpha_M(\theta)$ is a real observable expressible as a sum of products of f-functions, one may simplify the problem by expressing $h^\alpha_M(\theta)$ as a sum of squares of complex functions,

$$h^\alpha_M(u) = \sum_i |g_i(u)|^2. \tag{12}$$

Here, each $g_i(u)$ is a linear combination of the $f^{(\epsilon)}_m(\theta)$, and therefore is also a rational function in u. Hence, conjugation of the roots of each $g_i(u)$ may generate ambiguities of the partial waves. We note that it is most convenient to express every moment in terms of a single basis set of g's. Eq. (11) represent bilinear matrix equations which connect the coefficients of the intensity Eq. (6) to the partial wave amplitudes, while Eq. (12) represents a diagonalization of the same equations. Since the moments can be extracted directly from experimental data, the problem of ambiguities is ultimately whether or not replacing roots of the basis functions g_i with their conjugates provide alternate solutions to this matrix equation. To address this we will consider a few examples explicitly to show that given a set of partial waves $\{[\ell]^{(\epsilon)}_m\}$, the relations Eq. (11) are uniquely determined and no ambiguities exist. In other words there is no way to construct a different set, $\{[\tilde{\ell}]^{(\epsilon)}_m\}$ which will yield the same moments.

III. CASE STUDIES

Ambiguities in partial wave analysis with a high energy pion beam were studied in Ref. [2], where several wave sets with different combinations of waves up to the G wave were considered. In all cases the spin projections were limited to $m = 0, 1$.[2] With this restriction, the intensity only includes three terms in the azimuthal expansion of Eq. (6). Since for a pion beam there is no S wave with positive reflectivity, there is one relevant g-function for the positive reflectivity components, and two for the negative reflectivity components. The polynomials which generate ambiguities in the negative reflectivity components are not independent, so the ambiguities for

[1] We use the Wigner d-function with the convention $d^j_{m'm}(\theta) = \langle jm'|\, e^{-i\theta J_y}\,|jm\rangle$

[2] For pion beams, $m \geq 0$ in the reflectivity basis. This does not hold for photon beams.

the waves in each reflectivity component are obtained using the roots of a single polynomial. That is, there are ambiguous solutions in the partial wave extraction because the observable depends on only two independent polynomials, one for each reflectivity component, and transformations built from combinations of conjugations of roots of each polynomial produce the same intensity profile.

In this section, we will argue that there are no ambiguities in the extraction of partial waves from an experiment using a linearly polarized photon beam. First, we note that any possible ambiguities arising from switching contributions between the two different reflectivity waves may be resolved by making use of the Φ dependence of the linearly polarized photon beam. We will thus only consider one reflectivity component and suppress all reflectivity superscripts for convenience.

We will consider first the simplest non-trivial case by including only the waves $\{S_0, D_0, D_1\}$. This case is analogous to Ref. [2], however, as we will see, the polarized intensity allows us to determine the partial waves without ambiguity.

We then will consider the wave set $\{S_0, D_{-1}, D_0, D_1\}$. These D waves dominate the production of the $a_2(1320)$ resonance in the $\eta\pi$ final state *via* pion exchange [17]. We will not find any ambiguous solution for the extraction of this wave set, once the polarized moments are taken into account. The $a_2(1320)$ is also produced by vector exchanges. In this case, the dominant D waves are $\{D_0, D_1, D_2\}$ [17]. We have confirmed that this wave set is also free of ambiguities, although we omit the calculation for brevity.

Our key result is that there are at least two unique g's which appear in the Fourier series of the polar angle when two or more spin projections are allowed. These polynomials are independent and have distinct roots. Consequently, these Fourier moments are enough to uniquely determine the partial waves. No transformations on the partial waves leave every observable invariant, and the observables uniquely define the partial waves for linearly polarized meson photoproduction. We illustrate this fact only with S and D waves, but the addition of other waves should not change our results. Adding more waves increases the number of roots of each g-function, and hence the number of possible ambiguities, but in general we argue that there is no relation between the roots, and therefore partial waves can be unambiguously extracted from the polarized observables.

A. S and D waves with $m = 0, 1$

We start by analyzing the wave set with S and D waves with m projections $0, 1$ and positive reflectivities, as this set has been analyzed explicitly for a pion-beam production process [2]. Suppose that we have obtained one set of partial waves, $\{S_0, D_0, D_1\}$, from an experiment. We can then attempt to generate an ambiguous set of partial waves, $\left\{\tilde{S}_0, \tilde{D}_0, \tilde{D}_1\right\}$, from the original set. We start by writing the f's from Eq. (8):

$$f_0(u) = \frac{\sqrt{5}\left(u^4 - 4u^2 + 1\right) D_0}{\left(u^2+1\right)^2} + S_0\,, \tag{13a}$$

$$f_1(u) = \frac{\sqrt{30}\, u \left(u^2 - 1\right) D_1}{\left(u^2+1\right)^2}\,. \tag{13b}$$

With this wave set, there are seven non-zero functions $h^\alpha_M(\theta)$, though they are not all linearly independent. When the wave set includes only positive m-projections, there is a simple relation between the polarized moments $h^2_M = h^1_M$ for $M > 0$ [11]. (For $M = 0$, one has $h^2_0 = 0$, see Eq. (6c)). In addition, we find the relation $h^1_1 = h^0_1$ and $h^1_2 = h^0_0 - h^1_0$ for this particular wave set. So we are left with three linearly independent $h^\alpha_M(\theta)$. We rewrite the conditions relating the h's to the f's in matrix form:

$$h^0_M(\theta) = F^\dagger H^0_M F, \qquad h^1_M(\theta) = F^\dagger H^1_M F\,. \tag{14}$$

Where $F = (f_0, f_1)^T$. The three matrices are:

$$H^0_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H^0_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H^1_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{15}$$

Since the matrices H^0_0 and H^0_1 commute, we can simultaneously diagonalize them and simplify the unpolarized moments, obtaining:

$$g_0(u) \equiv \frac{1}{\sqrt{2}}\left[f_1(u) + f_0(u)\right], \tag{16a}$$

$$g_1(u) \equiv \frac{1}{\sqrt{2}}\left[f_1(u) - f_0(u)\right]. \tag{16b}$$

Since $f_0(u)$ is even and $f_1(u)$ is odd, the new functions fulfill $g_1(-u) = -g_0(u)$. Thus, their roots and ambiguities from complex conjugation of the roots are the same. The three independent moments read:

$$h^0_0 = |g_0|^2 + |g_1|^2, \tag{17a}$$

$$h^0_1 = |g_0|^2 - |g_1|^2, \tag{17b}$$

$$h^1_0 = \frac{1}{2}|g_0 - g_1|^2. \tag{17c}$$

We note that the moments h^α_M will simply change by a sign $(-1)^M$ under the substitution $g_0 \to g_1$. It is necessary and sufficient to require that any prospective ambiguity transformation leaves invariant $|g_0|^2$ and $|g_0 - g_1|^2$ independently. In terms of the partial waves, these functions can be written:

$$g_0 = \sqrt{\frac{5}{2}}\frac{1}{(u^2+1)^2}\Big[D_0(u^4 - 4u^2 + 1) + \sqrt{6}D_1(u^3 - u)\Big] + \frac{1}{\sqrt{2}}S_0, \tag{18a}$$

$$g_0 - g_1 = \sqrt{10}\frac{(u^4 - 4u^2 + 1)D_0}{(u^2+1)^2} + \sqrt{2}S_0. \tag{18b}$$

Which can be simplified defining $v = u - 1/u = -2\cot\theta$:

$$g_0 = \sqrt{\frac{5}{2}}\frac{1}{v^2+4}\Big[Av^2 + \sqrt{6}D_1 v - 2B\Big], \tag{19a}$$

$$g_0 - g_1 = \frac{\sqrt{10}}{v^2+4}\Big[Av^2 - 2B\Big], \tag{19b}$$

where $A = D_0 + S_0/\sqrt{5}$ and $B = D_0 - 2S_0/\sqrt{5}$.

Recalling that the ambiguous waves should be generated by conjugating roots of these polynomials, we start by considering the first polynomial in Eq. (19a), and factorize it into its Barrelet zeros $r_{1,2}$:

$$g_0 \propto (v - r_1)(v - r_2), \tag{20}$$

where we have dropped the irrelevant factors. The roots read:

$$r_{1,2} = \frac{-\sqrt{3}D_1 \pm \sqrt{4AB + 3D_1^2}}{\sqrt{2}A}. \tag{21}$$

In this case, there are only two Barrelet zeros and there is thus only one non-trivial independent solution given by the substitution of one root by its complex conjugate. We invert Eq. (19a) and replace r_1 with its conjugate to obtain:

$$\tilde{S}_0 = \sqrt{5}\frac{A}{6}(2 + r_1^* r_2), \tag{22a}$$

$$\tilde{D}_0 = \frac{A}{6}(4 - r_1^* r_2), \tag{22b}$$

$$\tilde{D}_1 = -\frac{A}{\sqrt{6}}(r_1^* + r_2). \tag{22c}$$

We note that the new waves obtained by the complex conjugation of r_1 and r_2 simultaneously lead to the set $\{\tilde{S}_0^*, \tilde{D}_0^*, \tilde{D}_1^*\}$, the complex conjugate of the original wave set. For a given wave set $\{S_0, D_0, D_1\}$, the set in Eq. (22) produces the same unpolarized moments $h^0_{0,1}(\theta)$. In the absence of information on the polarized moments, the above wave set would constitute an ambiguous solution.

In this example, the use of observables only accessible via a polarized beam are essential to ensure that no mathematical ambiguities can occur. In particular, we must consider the constraints implied by the polarized moment $h^1_0 = \frac{1}{2}|g_0 - g_1|^2$. The combination $g_0 - g_1$ only has one Barrelet zero, *i.e.* $g_0 - g_1 \propto (v - r_3)(v - r_3^*)$, where $r_3 = \sqrt{2}B/A$. This is independent of $r_{1,2}$, and the only transformation that leaves $h^1_0(\theta)$ invariant is the one that replaces each wave by its complex conjugate, since all the waves are defined up to a global phase. Therefore, there is no nontrivial transformation of the partial waves which leaves both the unpolarized moments $h^0_{0,1}(\theta)$ and the polarized moment $h^1_0(\theta)$ invariant, and thus there are no ambiguous solutions for this wave set.

We illustrate this case for one single energy bin by choosing three random complex numbers for the original waves $\{S_0, D_0, D_1\}$,[3] compute the associated ambiguous solutions $\{\tilde{S}_0, \tilde{D}_0, \tilde{D}_1\}$ and display the three moments in Fig. 2. The numerical values of the waves are specified in Table I. Here again, we see the value of incorporating polarized observables. While the two wave sets produce degenerate solutions for the two unpolarized moments, the incorporation of the polarized moment h^1_0 breaks the degeneracy.

The inclusion of more waves with only the projections $m = 0, 1$ will not change our results. Adding more waves with different m projections could potentially produce ambiguous solutions, each of which leave invariant one single moment $h^\alpha_M(\theta)$, but it would also generate additional nonzero $h^\alpha_M(\theta)$ which must remain invariant under each of the ambiguity transformations. One can try to generate other prospective ambiguities, but each potentially ambiguous wave set will be subject to an increasing number of constraints. Hence, we argue that, for most sensible wave sets, the intersection between all these sets of potentially ambiguous waves will be empty.

TABLE I: Numerical values of our example wave set and the potentially ambiguous wave set generated by the unpolarized moments.

$[\ell]_m$	original	potentially ambiguous
S_0	0.229	0.630
D_0	$-0.217 + 0.310i$	$0.043 + 0.056i$
D_1	$0.770 + 0.448i$	$0.280 - 0.713i$

B. S and D waves with $m = -1, 0, 1$

We now consider the previous example in Section III A with the addition of the $m = -1$ projection. The presence of three different m projections raises the number of independent $\cos M\phi$ moments to three ($M = 0, 1, 2$ in this case), each of them being a function of the polar angle. As we will see, it is impossible to find an ambiguous set leaving all the polar angle distributions simultaneously invariant. We only consider here the $m = -1, 0, 1$ components for the D wave but our conclusions can be generalized to any wave set with three (or more) spin projections. It was already noticed by the COMPASS collaboration that no ambiguities are found in the $\eta\pi$ system once the $m = 2$ component is included in the partial wave analysis [18].

We again start with a set of partial waves, $\{S_0, D_0, D_1, D_{-1}\}$, and attempt to generate an ambigu-

[3] We choose S_0 to be real positive without loss of generality and rotate the ambiguous solution to bring $\tilde{S}_0$ also to the positive real axis, *i.e.* its phase is zero.

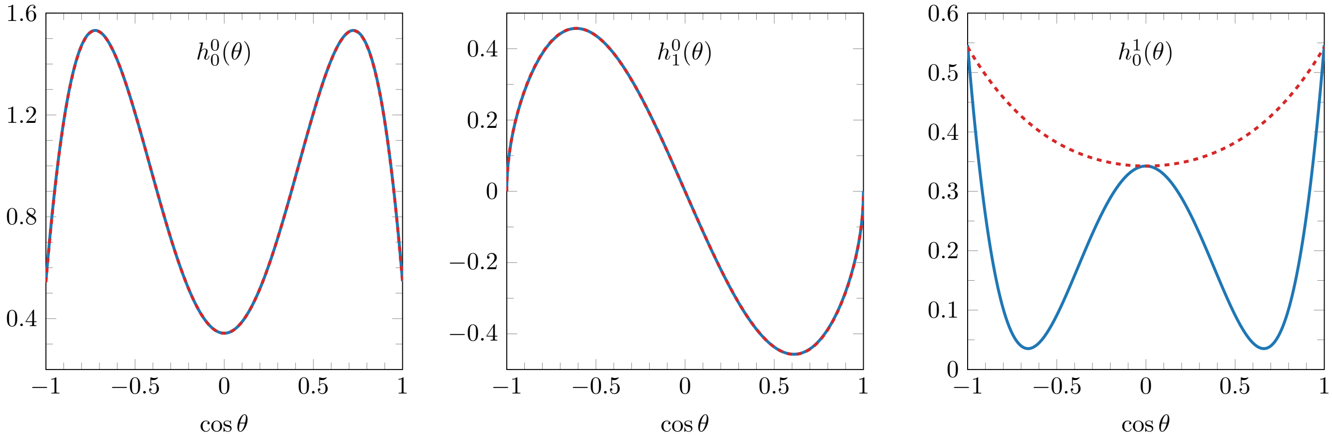

FIG. 2: *Solid blue lines*, moments obtained from the original waves of Table I; *dotted red lines*, moments obtained from the ambiguous solution of Table I. The polarized moment h_0^1 breaks the ambiguity between the two solutions.

ous set $\{\tilde{S}_0, \tilde{D}_0, \tilde{D}_1, \tilde{D}_{-1}\}$. The f's are:

$$f_0(u) = \sqrt{5}\frac{\left(u^4 - 4u^2 + 1\right)}{\left(u^2+1\right)^2} D_0 + S_0\,, \tag{23a}$$

$$f_{\pm 1}(u) = \mp\sqrt{30}\frac{u\left(1-u^2\right)}{\left(u^2+1\right)^2} D_{\pm 1}\,. \tag{23b}$$

Our wave set for this example contains all $|m| \leq 1$ but only positive reflectivity components. The structure of the moments in Eq. (11) tells us that, when only one reflectivity component is included but all m projections are allowed, the polarized moments h_M^1 are not independent of the unpolarized moments h_M^0. It suffices to study the ambiguities which leave only h_M^0 and h_M^2 invariant.

Let us first investigate only the unpolarized moments. We rewrite the conditions relating the h's to the f's in matrix form:

$$h_M^0(\theta) = F^\dagger H_M^0 F\,, \tag{24}$$

where, $F = (f_{-1}, f_0, f_1)^T$ and

$$H_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_1^0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad H_2^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{25}$$

Notice here that H_0^0, H_1^0 and H_2^0 are all *not* simultaneously diagonalizable. Nevertheless, as before, we diagonalize H_1^0, defining g_0, g_1, and g_{-1} as

$$g_{\pm 1}(u) \equiv \frac{1}{2}\left[f_1(u) \pm \sqrt{2} f_0(u) + f_{-1}(u)\right], \tag{26a}$$

$$g_0(u) \equiv \frac{-1}{\sqrt{2}}\left[f_1(u) - f_{-1}(u)\right]. \tag{26b}$$

Again, the parity of the f's functions indicates that $g_{\pm 1}(-u) = g_{\mp 1}(u)$ and $g_0(-u) = -g_0(u)$. The two functions g_1 and g_{-1} possess the same Barrelet zeros and therefore the same potential ambiguities

As in the previous example, the moments are even functions of the polar angles and read, in the g basis,

$$h_0^0 = \left(|g_{-1}|^2 + |g_0|^2 + |g_1|^2\right), \tag{27a}$$

$$h_1^0 = \sqrt{2}\left(|g_1|^2 - |g_{-1}|^2\right), \tag{27b}$$

$$h_2^0 = \frac{1}{2}|g_{-1} + g_1|^2 - |g_0|^2\,. \tag{27c}$$

Again, any transformation on the partial waves which leaves each term above independently unchanged will produce a mathematically ambiguous set of waves. Introducing the change of variables $v = u - 1/u$ as before, the relevant rational fractions are

$$g_{\pm 1} = \pm\sqrt{\frac{5}{2}}\frac{1}{v^2+4}\left[A v^2 \pm \sqrt{6} v D^- - 2B\right], \tag{28a}$$

$$g_0 = -\sqrt{30}\frac{v}{v^2+4} D^+, \tag{28b}$$

where A, B are defined as in the previous subsection and $D^\pm = (D_1 \pm D_{-1})/\sqrt{2}$. With these definitions, the roots of $g_{\pm 1}(v)$ are given by Eq. (21) with the substitution $D_1 \to \pm D^-$.

As already noted, the same ambiguous solution will simultaneously leave invariant $|g_1|^2$ and $|g_{-1}|^2$. The new wave set $\{\tilde{S}_0, \tilde{D}_0, \tilde{D}^-\}$ is easily obtained from Eq. (22) with the substitution $D_1 \to D^-$. There is, in addition, a continuous transformation $D^+ \to \exp(i\alpha^+)D^+$ leaving $|g_0|^2$ invariant. Since this transformation is independent from the set $\{\tilde{S}_0, \tilde{D}_0, \tilde{D}^-\}$, we have, so far, found an ambiguous solution, parametrized with a continuous parameter, leaving the moments h_0^0 and h_1^0 invariant. However, the invariance of h_2^0 requires a continuous transformation of the type $D^- \to \exp(i\alpha^-)D^-$, which contradicts the ambiguous solution $\{\tilde{S}_0, \tilde{D}_0, \tilde{D}^-\}$. Therefore the unpolarized moments $h_{0,1,2}^0$ are left invariant only by the 1-parameter continuous transformation

$$\{S_0, D_0, D^-, D^+\} \to \{S_0, D_0, D^-, e^{i\alpha^+} D^+\}. \tag{29}$$

Since the polarized moments $h^1_{0,1,2}$ are related to the unpolarized ones, we only need to consider the moments $h^2_{1,2}$. Their respective matrices, in the form analogous to Eq. (24), are

$$H^2_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad H^2_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{30}$$

Their expressions in the g's basis are

$$h^2_1 = 2\,\mathrm{Re}\left[(g_1 - g_{-1})g^*_0\right], \tag{31a}$$

$$h^2_2 = -\sqrt{2}\,\mathrm{Re}\left[(g_1 + g_{-1})g^*_0\right]. \tag{31b}$$

The continuous transformation in Eq. (29) changes the phase of g_0 and does not leave the polarized moments Eq. (31) invariant.

We thus conclude that there is no ambiguity associated with the extraction of partial waves with a linearly polarized beam for this wave set, other than the trivial ambiguities given by the rotation of all waves by a common phase, or by the complex conjugation of all waves.

IV. SIMULATIONS

While in the previous sections we have provided arguments that no mathematical ambiguities exist in partial-wave analysis of two mesons produced with a linearly polarized photon beam, the complicated multidimensional shape of likelihood functions or other functions used for fitting can present themselves as false solutions, which one might naively label as mathematically ambiguous. In this section, we present some Monte Carlo studies showing this effect. We wish to emphasize that here we only investigate the dependence on statistics of a perfect model. Other factors such as acceptance corrections, resolutions, and other systematic effects are experiment-dependent and may qualitatively alter the results. Studies based off pseudodata or studies involving full experiment simulations will be an important part of subsequent analyses, and might be employed to help discard false solutions or assess the impact of limited statistics.

First, pseudodata was generated following the angular intensity given by Eqs. (7) and (8). We used the wave set from Section III B, and generated the pseudodata using the fixed "true solution" wave set, with non-zero, positive reflectivity partial waves shown in Table II and a mean linear polarization degree of $P_\gamma = 0.85$.

We then performed event-by-event fits to extract these four waves. We used `MINUIT` [19] with random initial conditions to minimize the negative log likelihood. To explore the effect of differing statistical information on fit results, we examined three different cases with generated data sets of 10^2, 10^4, and 10^6 events.

In Fig. 3 we show the resulting negative log likelihood and amplitude components from 50 fits to the pseudodata with 100 events. Similar results are shown for 10^4 events in Fig. 5 and 10^6 events in Fig. 7. For clarity, in these plots we show only a single complex conjugate solution set, though the fitting procedure did also identify trivial ambiguities, i.e. the set with all phases simultaneously flipped in sign.

TABLE II: Numerical values of our "true" wave set for the simulation studies.

$[\ell]_m$	Magnitude	Phase
S_0	0.499	0°
D_{-1}	0.201	15.4°
D_0	0.567	174°
D_1	0.624	-81.6°

In Figs. 4, 6 and 8 we show the projections of the intensity onto the polarization angle Φ, and the decay angles ϕ, θ for the best five solutions, compared to the distributions generated from the true amplitudes.

We observe in the case of the second best solution for each simulation (shown in red), the $\cos\theta$ distribution is almost identical to the best fit's distribution (blue) However the second best ϕ distribution is flat (red), while the true distribution has a $\cos(2\phi)$ component. The reason for the flat distribution is that the magnitude of the D_{-1} amplitude is zero for this solution (first red triangle in the plots of Figs. 3, 5 and 7). The h^0_2 moment, which contributes to the $\cos(2\phi)$ amplitude, requires an interference between D_{-1} and D_1, which is obviously zero when either of these waves has zero magnitude. Note that this solution is found despite the fit magnitude range being $[-1, 1]$ and `MINUIT` finishing with successful status.

The other solutions do not agree well with the data and therefore clearly do not represent real solutions, but rather represent artifacts of local maxima in the likelihood. We also note that, for each level of statistics, we observe a similar behavior in the projection of the intensity onto Φ for all solutions. The best (blue) and second-best (red) solutions closely match the true solution (dashed black) for each case, while the less-favored solutions cannot be immediately discarded from this projection even at high statistics

We should emphasize here that although we have shown explicitly that there are no mathematical ambiguities present, the false solutions found in fits to data or pseudodata must still be addressed. In fits to real data one may not always be able to extract the most favored solution from a fitting procedure and claim that it is the true, mathematically unique, solution due to detector effects and other systematics. In practice, each solution could be shifted up or down in likelihood, and the 'true' solution could correspond to a local minimum rather than the global one. We do note that, in an environment with no systematics or detector effects, higher statistics allows

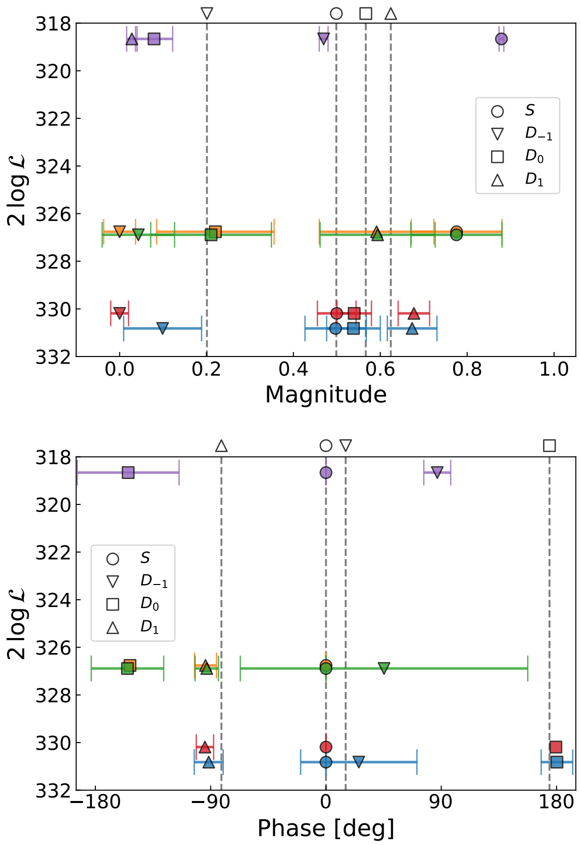

FIG. 3: Results of the 5 best (highest likelihood) fits from 50, to 100 events generated with the partial waves given in Table II showing the likelihood versus the amplitude magnitude (upper) and phase (lower), the dashed lines show the true values. The wave is indicated by the marker shape (see legend) while the color represents different solutions. The highest likelihood is at the bottom of the plots. The phase for the D_{-1} (D_1) wave is not shown for the red and orange (purple) fits, as associated magnitude is zero and, hence, the phase is undetermined. Fits and uncertainties are computed using the `HESSE` option of `MINUIT` [19].

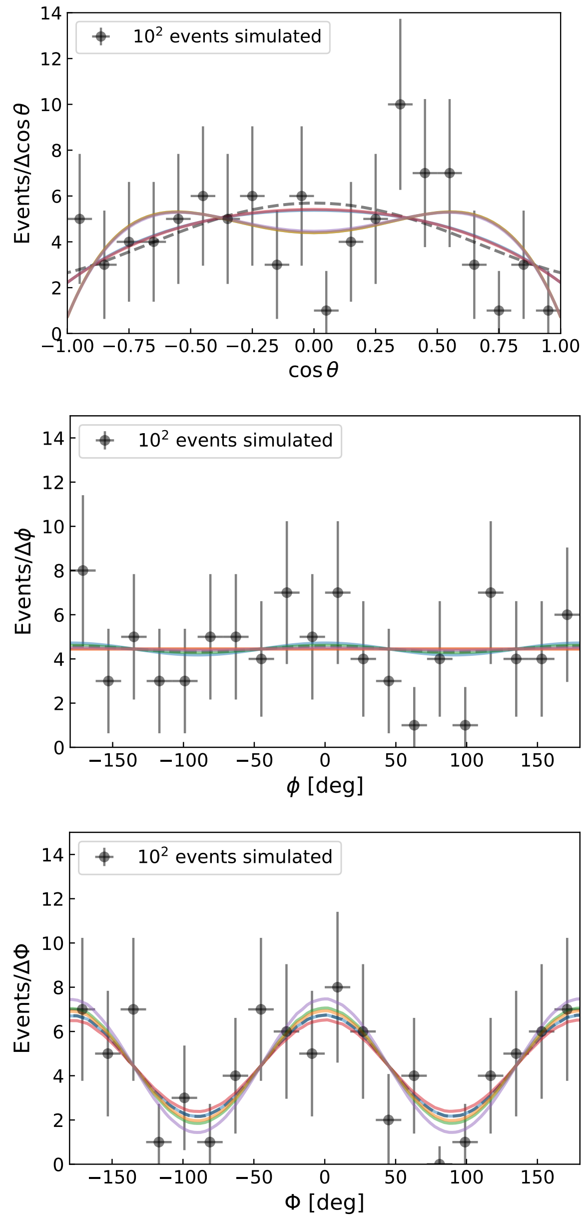

FIG. 4: Projections of the angular distributions (upper: $\cos\theta$, center: ϕ, lower: Φ) as defined in Eqs. (3) and (7). Shown are the data (black circles), the true solution (dashed black), and the different solutions (colored lines), with colors matching the plots in Fig. 3. Bin widths are $\Delta\cos\theta = 0.1$ and $\Delta\phi = \Delta\Phi = 18^{\mathrm{o}}$.

one to make qualitative judgments about which solution best fits the data by considering projections of the intensity onto the scattering angles. We also note that in these simulations we have relatively few waves. In larger wave sets, the probability of finding the global minima from fifty random starting points reduces drastically. These issues are outside the scope of this paper, and we leave methods to address them for future work.

V. SUMMARY AND CONCLUSIONS

In this work, we have presented our formalism for the analysis of mathematical ambiguities for linearly polarized photoproduction of two spinless particles. We demonstrated for two wave sets that, even with a small number of constraints on the partial waves, the partial waves are over-specified by experimental data. We illustrated our results by generating pseudodata and extracting back the partial waves. We found that the best solution matches the input waves. We do not expect larger wave sets to exhibit root-conjugation ambiguities, as the number of constraints increases rapidly with the size of the fitted wave set. Rather, we expect that false

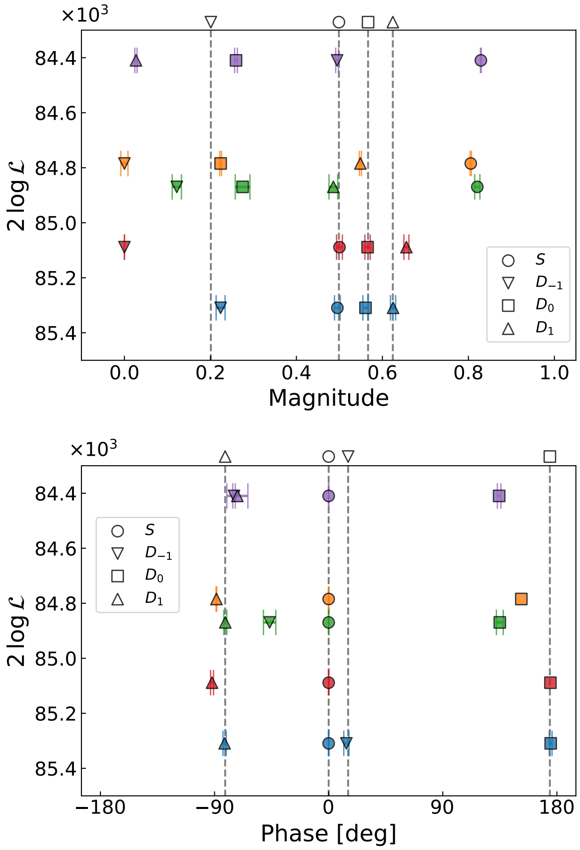

FIG. 5: As Fig. 3 for fits to 10^4 events. The phase for the D_{-1} wave is not shown for the red and orange fits, as associated magnitude is zero and, hence, the phase is undetermined.

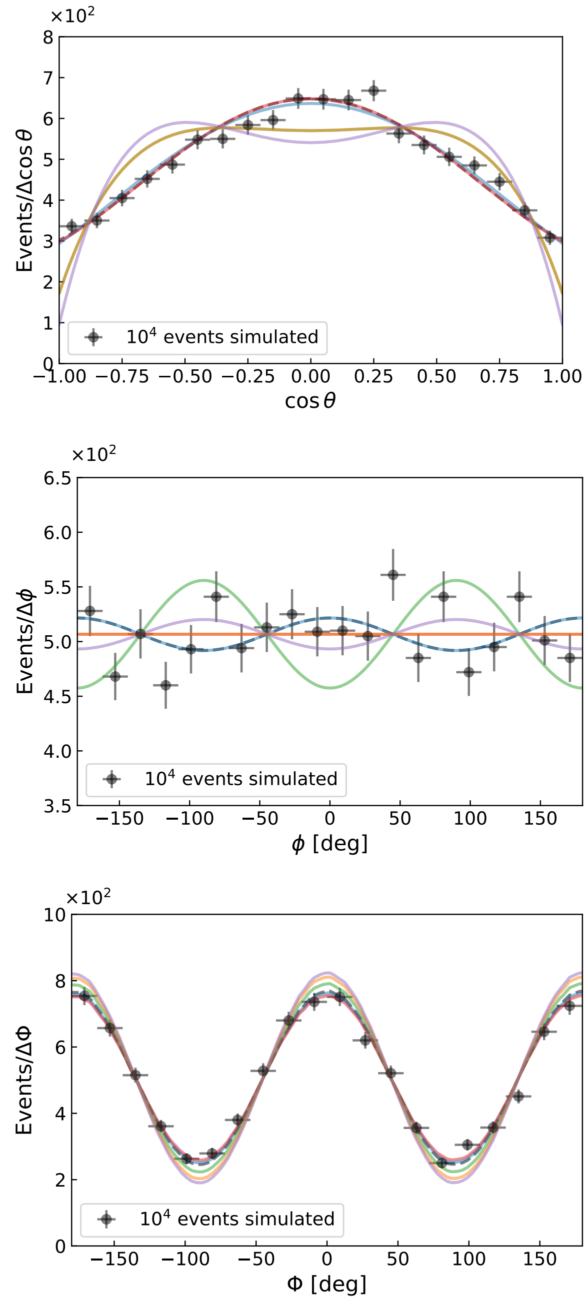

FIG. 6: As Fig. 4 for fits to 10^4 events with colors matching the plots in Fig. 5.

solutions which appear in fits to real data come about as artifacts of complicated multidimensional properties of log-likelihood functions. These may be identified through examination of the angular dependence of the polarized observables.

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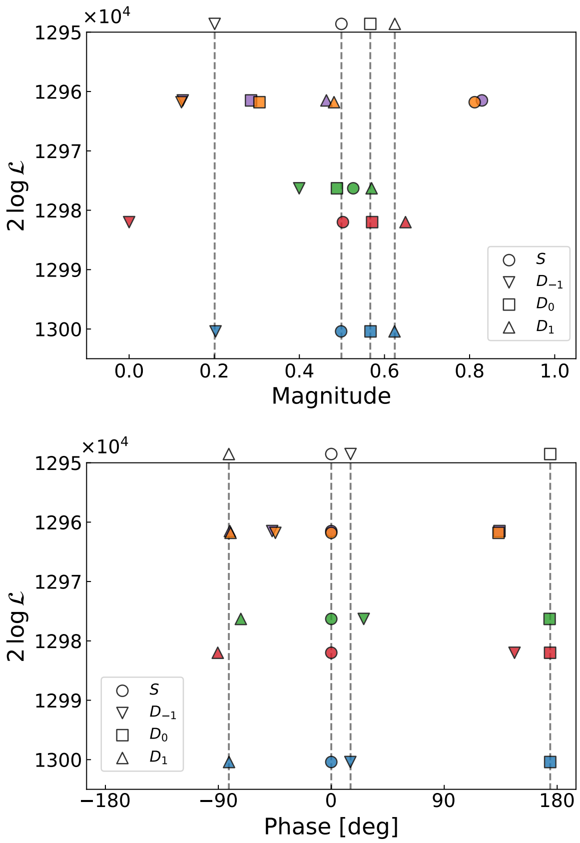

FIG. 7: As Fig. 3 for fits to 10^6 events. Uncertainties are negligible and not shown.

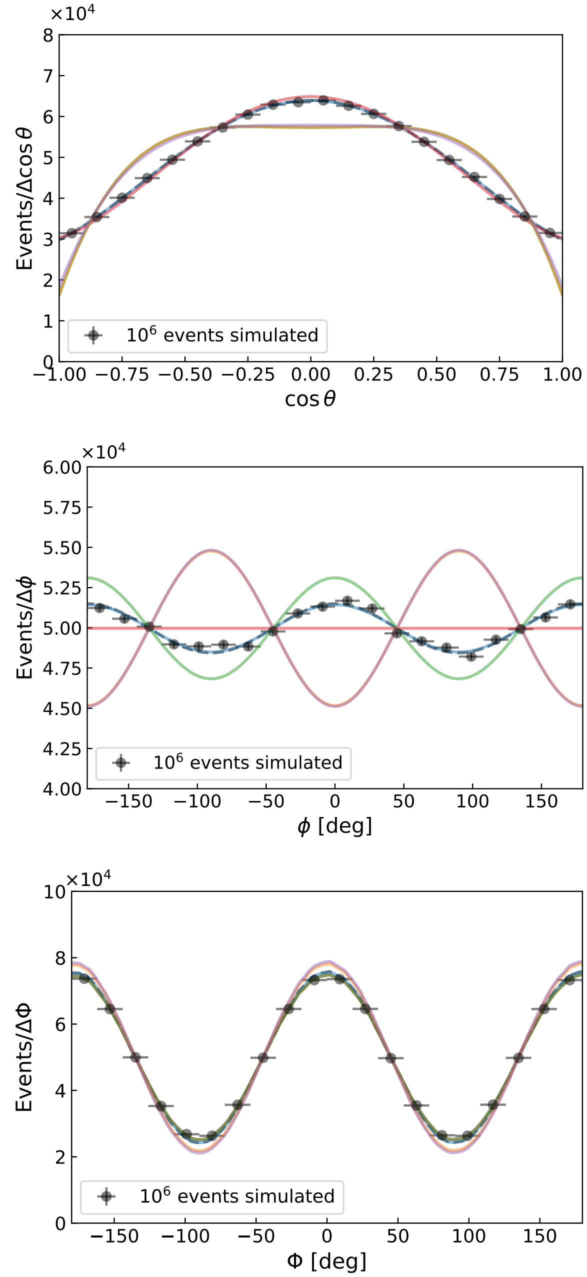

FIG. 8: As Fig. 4 for fits to 10^6 events with colors matching the plots in Fig. 7.

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Appendix A: Barrelet zeros for spinless meson scattering

We consider the elastic scattering of two spinless mesons [1, 20]. Lorentz invariance allows us to choose the scattering plane as the xz plane, and to write the intensity as a real positive function of the scattering angle $z = \cos\theta$. The differential cross section,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(s,z)|^2, \tag{A1}$$

is decomposed into partial waves of the decaying resonance, with angular momentum ℓ as

$$f(s,z) = \sum_{\ell} (2\ell+1) a_\ell(s) P_\ell(z). \tag{A2}$$

The center-of-mass energy s is a fixed variable in our treatment. In practice, for each bin in s, the sum in Eq. (A2) is truncated to ℓ_{M} and the differential cross section is thus a polynomial of order $2\ell_M$ in the cosine of the scattering angle, z. The $\ell_{\mathrm{M}}+1$ partial waves are in general complex numbers, but since the intensity is positive, the cross section can be factorized into its roots, also denoted Barrelet zeros [1, 20], in the following way

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = C \prod_{i=0}^{\ell_M} (z - z_i)(z - z_i^*), \tag{A3}$$

where the s dependence of the normalization factor C and the Barrelet zeros z_i have been omitted.

Clearly, the knowledge of a set of partial waves $\{a_\ell\}$ determines the Barrelet zeros $\{z_i\}$, and *vice versa*. However, the differential cross section includes both the roots z_i and their conjugates z_i^* while only one of $\{z_i, z_i^*\}$ is used to generate the partial waves; there is no physical distinction between a zero and its complex conjugate, which can lead to ambiguities in the values of the partial waves in Eq. (A2). To see this, suppose we know $\ell_M + 1$ Barrelet zeros $\{z_i\}$ from which we reconstruct the partial waves:

$$a_\ell = F_\ell(z_0, z_1, \ldots, z_{\ell_M - 1}, z_{\ell_M}), \tag{A4}$$

where the functions F_ℓ are known for a given ℓ_M. Alternatively one could choose to use the complex conjugate of any of the $\ell_M + 1$ Barrelet zeros. For instance by choosing

$$a'_\ell = F_\ell(z_0^*, z_1, \ldots, z_{\ell_M - 1}^*, z_{\ell_M}). \tag{A5}$$

There are $2^{\ell_M + 1}$ sets of potentially ambiguous partial waves $\{a'_\ell\}$ which lead to the same differential cross section. One can always rotate all the waves with a constant phase (in each bin of energy) such that the S-wave is real and positive. We are nevertheless left with 2^{ℓ_M} possibilities for the partial waves in the case of spinless meson scattering.

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