# The Schwinger model as an open quantum system: Dynamics of string breaking and reconnection in a medium 

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Understanding the dynamics of bound state formation is one of the key questions in confining quantum field theories such as quantum chromodynamics. In this work, we consider a key aspect of this process within the Schwinger model: the breaking of a single string into two separated strings. We study the modification to the string breaking mechanism due to the presence of an external medium. To do this, we consider the Schwinger model as an open quantum system coupled to a thermal environment, where in the weakly coupled regime the real-time evolution can be described by a Lindblad evolution equation. We observe a delay of the string breaking due to dissipation of the string tension into the environment. We discuss how this process can be simulated on quantum computers, and perform an estimate of associated Trotter errors.

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## I. INTRODUCTION

Real-time simulations of lattice field theories have recently received significant attention in fundamental nu-

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FIG. 1. Illustration of the string breaking process in the Schwinger model in a thermal background.
clear and particle physics. While these simulations pose computational challenges, especially in higher dimensions, recent advancements in quantum computing offer the potential to eventually enable large-scale simulations [1-6]. See in particular Refs. [7, 8] for recent advancements in quantum error correction. The realtime dynamics of field theories can be simulated within the Hamiltonian formulation developed by Kogut and Susskind [9]. Different than in the path integral formulation of lattice field theory that relies on a spatial and temporal lattice discretization, time is kept as a continuous variable within the Hamiltonian formulation and only the spatial directions of are discretized. The need to simulate exponentially large Hilbert spaces makes largescale classical simulations intractable. This necessitates the development of quantum algorithms to simulate for example high-energy scattering processes [10-12] or field theories at finite chemical potential, which are relevant to nuclear and particle physics [13-32]. In particular, lowerdimensional lattice field theories that share features with
quantum chromodynamics (QCD) have received an increased attention recently. An example is the Schwinger model [33, 34], which corresponds to quantum electrodynamics (QED) in $1+1$ dimensions. This $\mathrm{U}(1)$ gauge theory coupled to fermions exhibits confinement and chiral symmetry breaking. Besides the similarities with QCD, lower-dimensional field theories are an important testing ground for developing simulation protocols in order to eventually build up toward simulations of QCD. See Refs. [35-42] for recent work on quantum and tensor network simulations of the Schwinger model and Refs. [4348] for related work on simulating real-time dynamics of non-Abelian lattice field theories.

A particularly interesting aspect of the Schwinger model is the string-breaking mechanism, which is closely related to quark confinement in QCD. Consider two fermions that are connected by a string of electric flux separated by a certain distance. When the string is sufficiently long, one can observe the breaking of the string in real time and eventually two (or more) tightly-bound fermion anti-fermion pairs are formed. The energy that was initially stored in the string gets transformed to kinetic energy of the fermion pairs and they move away from each other at a certain velocity. The details of this mechanism significantly depend on the fermion mass and the value of the coupling between fermions and the gauge field. The string-breaking process is analogous to the hadronization mechanism of a quark anti-quark pair in QCD. It is are implemented as a model of hadronization for example in Monte Carlo event generators like Pythia [49] that are used to simulate high-energy particle collisions. Recent work on the string breaking mechanism within the Schwinger model can be found in Refs. [35, 37, 50]. See also Refs. [36, 40, 51-54]. In this work, we study the influence of a thermal environment on the dynamics of the string breaking mechanism. The microscopic picture of the string-breaking process and hadronization in QCD are of great relevance for ongoing and future collider experiments. This includes precision measurements of Standard Model parameters [55], studies of the quark-gluon plasma in heavy-ion collisions [56] and cold nuclear matter effects at the future Electron-Ion Collider [57]. For example, in Ref. [58] a time-dependent string tension was introduced in PYthia simulations in order to describe collective effects in high-energy collisions. Moreover, in Ref. [59] initial-state string dynamics and string junctions were found to be necessary ingredients to describe the production of particles in heavy-ion collisions within a three-dimensional dynamical initialization model.

In Refs. [60-63] the static string within the Schwinger has been studied at both finite temperature and chemical potential. It was found that the string tension decreases as the temperature and/or chemical potential are increased. In this work, we extend these studies to the dynamic case where the real-time evolution of the stringbreaking process is modified by the presence of a thermal environment. We find that the string-breaking process
is delayed and the velocity at which the fermion antifermion pair move away from each other is reduced. This can be understood in terms of a drag force at the quantum level acting on the fermion pairs, which is generally in line with the findings in the static case. In order to study the real-time dynamics, we consider the Schwinger model that interacts with a thermal scalar field via a Yukawa-type coupling. We work in the Brownian motion limit where the temperature of the environment is high compared to the typical energy levels of the system. In addition, we consider the Markovian limit where memory effects can be neglected. The evolution of the Schwinger model as an open quantum system can then be expressed in terms of a Lindblad equation [64-66]. See also Refs. [67-69] for recent work on simulations of other lattice field theories at finite temperature and/or chemical potential.

One of the key aspects of out-of-equilibrium physics are the late-time relaxation dynamics of the system toward equilibrium. These relaxation dynamics are governed by the Liouvillian gap, which is given by the smallest eigenvalue of the Liouvillian spectrum. It is a fundamental quantity of open quantum systems analogous as the energy gap of Hamiltonians describing closed quantum systems. We determine the Liouvillian gap and corresponding eigenmodes for different choices of the environmental correlator (long and short range correlations), study its dependence on the system size and compare to the free fermion model. We find that a long-range correlated environment leads to a slower thermalization of the system since the exchange between the system and environment is slowed when long range correlations are present. Moreover, we find that special care needs to be taken when constructing the Hilbert space of the Schwinger model as an open quantum system. We decompose the Hilbert space into a charge conjugation and parity (CP) even and odd sector. When the Schwinger model is considered as a closed system, the two sectors evolve independently. This is also the case for the open system when the environmental correlator that appears in the Lindblad equation is chosen as a constant (long-range correlated environment). We observe that this has an impact on the relaxation dynamics to the equilibrium state of the system, which we study in detail. As an example, we study the von Neumann entropy of the system that quantifies its decoherence due to the interaction with the environment. These results are closely related to the study and classification of field-theoretical dissipative phase transitions. Our results extend beyond the scope of this work, providing a starting point for more detailed studies in the future.

Finally, we study the resource requirements for quantum simulations of the Schwinger model as an open quantum systems. For this case study, we focus on a quantum algorithm that interleaves short time steps in the system's evolution with a time-evolution operator comprising the Lindblad operators. By using a first-order Trotter decomposition for both unitarty operators, we find
that, in practice, the Trotter errors do not necessarily increase when compared to the vacuum calculation of the Schwinger model. This is due to cancellations of errors, which is an encouraging sign for the quantum simulation of open quantum systems in the near to intermediateterm future.

The remainder of this paper is organized as follows. In section II, we introduce the lattice formulation of the Schwinger model as an open quantum system including the decomposition into separate CP sectors. In section III, we present results for the Liouvillian gap and its eigenmodes, and study its relation to the decoherence of the system and relaxation dynamics toward equilibrium. In section IV, we present numerical studies of the stringbreaking process in the vacuum and medium, and study its dependence on system parameters. We estimate Trotter errors of quantum algorithms for simulations of open quantum systems in section V and we draw conclusions in section VI.
(Cite this somewhere [70])

## II. THE SCHWINGER MODEL AS AN OPEN QUANTUM SYSTEM

The Lagrangian of the Schwinger model is given by

$$
\begin{equation*}
\mathcal{L}=\bar{\psi}(i \not D-m) \psi-\frac{1}{4} F^{\mu \nu} F_{\mu \nu} \tag{1}
\end{equation*}
$$

with a two-component fermion field $\psi$, the covariant derivative $D_{\mu}=\partial_{\mu}-i e A_{\mu}$, the $\mathrm{U}(1)$ gauge field $A_{\mu}$ and the field strength tensor $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$. The Hamiltonian of the theory can be written out on a discretized spatial lattice in the axial gauge $A_{0}=0$, by applying the stagger fermion formulation and the Jordan-Wigner transform

$$
\begin{align*}
H_{S}= & \frac{1}{2 a} \sum_{n=0}^{N_{f}-2}\left(\sigma^{+}(n) L_{n}^{-} \sigma^{-}(n+1)+\sigma^{+}(n+1) L_{n}^{+} \sigma^{-}(n)\right) \\
& +\frac{1}{2} a e^{2} \sum_{n=1}^{N_{f}-1} \ell_{n}^{2}+\frac{1}{2} m \sum_{n=0}^{N_{f}-1}(-1)^{n} \sigma_{z}(n) \tag{2}
\end{align*}
$$

where $a$ denotes the lattice spacing, $n$ indicates the index of the lattice position, $\sigma^{ \pm}=\left(\sigma_{x} \pm i \sigma_{y}\right) / 2$ and $L^{ \pm}$represent the raising and lowering operators acting on the states living on the gauge links that are labeled by their eigenvalues $\ell_{n}^{2}$ under $E^{2}(n)$. We have assumed the open boundary condition so that the upper limit of the first sum in $H_{S}$ is $N_{f}-2$, where $N_{f}$ is the number of fermion sites and twice the number of physical sites. (So $N_{f}$ is an even number.) For $N_{f}$ fermion sites, we only need $N_{f}-1$ gauge links to connect nearest neighbors in the open boundary case.

Physical states have to satisfy the Gauss' law:

$$
\begin{equation*}
E(n+1)-E(n)=-e \sigma^{+}(n) \sigma^{-}(n)-e \frac{(-1)^{n}-1}{2} \tag{3}
\end{equation*}
$$



FIG. 2. Example of how a physical state transforms under the CP transformation. Green (blue) dots are unoccupied (occupied) fermion sites. Electrons (fermions) only live on even sites while positrons (anti-fermions) only stay on odd sites. Left and right arrows on the links indicate the electric fluxes are negative and positive respectively.

For the $n=0$ and $n=N_{f}-1$ sites, imposing the Gauss' law needs information about $E(0)$ and $E\left(N_{f}\right)$, which is set by our open boundary condition:

$$
\begin{equation*}
E(0)=0, \quad E\left(N_{f}\right)=0 \tag{4}
\end{equation*}
$$

Other boundary conditions can also be studied which correspond to cases where the system has nonzero total charge.

The discretized Hamiltonian has a charge-conjugationparity $(C P)$ symmetry given by

$$
\begin{align*}
& \sigma^{ \pm}(n) \xrightarrow{C P} \sigma^{\mp}\left(N_{f}-1-n\right) \\
& \sigma_{z}(n) \xrightarrow{C P}-\sigma_{z}\left(N_{f}-1-n\right) \\
& \quad L_{n}^{ \pm} \xrightarrow{C P} L_{N_{f}-2-n}^{ \pm} \\
& \quad \ell_{n} \xrightarrow{C P} \ell_{N_{f}-2-n} . \tag{5}
\end{align*}
$$

Physical states of the theory transform as shown in Fig. 2
We consider the Schwinger model coupled to an environment, which is described by a scalar field theory at thermal equilibrium. The total Hamiltonian takes the form

$$
\begin{equation*}
H=H_{S}+H_{E}+H_{I} \tag{6}
\end{equation*}
$$

where the three terms describe the system, the environment and their interaction, respectively. The system Hamiltonian $H_{S}$ is given in Eq. (21). The environment Hamiltonian is that of a scalar field theory. Different models of the scalar field interaction terms may be considered. The interaction Hamiltonian $H_{I}$ describes the coupling between the Schwinger model and the scalar field theory. Here we consider a specific interaction

$$
\begin{equation*}
H_{I}=\lambda \int \mathrm{d} x \phi(x) \bar{\psi}(x) \psi(x) \tag{7}
\end{equation*}
$$

While the system and environment can be strongly coupled, we assume that the interaction between them is sufficiently weak such that the time evolution of the

Schwinger model itself is Markovian and a Lindblad equation can be used to describe its time evolution. In the quantum Brownian motion limit valid at high temperature, the Lindblad master equation for the density matrix of the Schwinger model can be written as [54, 71]

$$
\begin{align*}
\frac{\mathrm{d} \rho_{S}(t)}{\mathrm{d} t} & =\mathcal{L} \rho_{S}(t) \\
& =-i\left[H_{S}, \rho_{S}(t)\right]+a^{2} \sum_{x_{1}, x_{2}} D\left(x_{1}-x_{2}\right) \\
& \times\left(L\left(x_{2}\right) \rho_{S} L^{\dagger}\left(x_{1}\right)-\frac{1}{2}\left\{L^{\dagger}\left(x_{1}\right) L\left(x_{2}\right), \rho_{S}\right\}\right) \tag{8}
\end{align*}
$$

where $\mathcal{L}$ denotes the Liouvillian superoperator, $x_{1}=n_{1} a$ and $x_{2}=n_{2} a$ are discrete. The environment correlator is given by

$$
\begin{equation*}
D\left(x_{1}-x_{2}\right)=\int_{-\infty}^{+\infty} \mathrm{d}\left(t_{1}-t_{2}\right) \operatorname{Tr}\left[\phi\left(t_{1}, x_{1}\right) \phi\left(t_{2}, x_{2}\right) \rho_{E}\right] \tag{9}
\end{equation*}
$$

where $\rho_{E}=e^{-\beta H_{E}} / \operatorname{Tr}\left(e^{-\beta H_{E}}\right)$ and $\phi(t, x)$ denotes the scalar field in the interaction picture at thermal equilibrium. The Lindblad operators are

$$
\begin{align*}
L(n a) & =O(n)-\frac{1}{4 T}\left[O(n), H_{S}\right] \\
O(n) & =(-1)^{n} \frac{\sigma_{z}(n)+1}{2} \tag{10}
\end{align*}
$$

In principle, the environment correlator $D(x)$ can be calculated, which depends on the model for the scalar field theory. Here instead of calculating the correlator $D(x)$ for a specific scalar field theory model, we just use models for the correlator. In order to test the dependence on the correlation length of the environment, we use three different models for the $D$ term: (Please double check the $D$ expressions...)) Gaussian for intermediate-range correlations

$$
\begin{equation*}
D_{G}(x)=D_{0} \exp \left(-\frac{(x-\mu)^{2}}{2 \sigma^{2}}\right) \tag{11}
\end{equation*}
$$

delta function $D_{\delta}(x)=D_{G}(x=0) \delta_{0 x}$ for short-range correlations, and a constant $D_{c}(x)=D_{G}(x=0)$ for longrange correlations. With this choice, the environmental correlators all agree for $x=0$. We will study the numerical impact of different choices of the correlator below.

Under the $C P$ transformation, the $O$ operators in the Lindblad operators transform as

$$
\begin{equation*}
O(n) \xrightarrow{C P}(-1)^{n} \frac{\sigma_{z}\left(N_{f}-1-n\right)-1}{2}, \tag{12}
\end{equation*}
$$

where we have used the fact that $N_{f}$ is an even number. One can then show that if the environment correlator $D(x)$ is constant, the Lindblad equation (8) preserves the $C P$ symmetry, i.e., if an initial state $\rho_{S}(0)$ is $C P$-even (odd), the state will remain $C P$-even (odd) throughout the time evolution. In this case, one can construct two


FIG. 3. Illustration of the gap sizes for different environmental correlators.
invariant subspaces of the whole Hilbert space: one is $C P$-even and the other is $C P$-odd. The construction can be done as follows: We consider each state in the whole Hilbert space. If the state is invariant under the $C P$ transformation, then the state is $C P$-even. Otherwise, a symmetric linear combination of the original state and the state after the $C P$ transformation leads to a $C P$-even state while an antisymmetric linear combination gives a $C P$-odd state. The $C P$-even and odd sectors decouple in the time evolution and as will be shown later, thermalize independently. We want to emphasize this is not the case if the environment correlator $D$ is Gaussian or a delta function.

## III. DECOHERENCE AND RELAXATION DYNAMICS

The characterization and classification of the relaxation dynamics of open quantum systems to the steady / thermal state has received significant interest in recent years. See for example Refs. [72-82]. Typically, either short or long-time non-equilibrium dynamics are considered. At short time scales, the Lindblad evolution can be approximated by a non-hermitian Hamiltonian. Here, we focus primarily on long-time relaxation dynamics. Characterizing the relaxation dynamics of open quantum systems is relevant for studies of non-equilibrium / dissipative phase transitions. For example, the decay modes can undergo a phase transition, while the equilibrium state does not. In addition, dissipative dynamics allow for topological phases, domain walls, non-trivial boundary modes and exceptional points. While a full exploration of these aspects in the quantum field theory limit of the Schwinger model is beyond the scope of this work, this section should be considered as a starting point for more detailed future studies.


FIG. 4. Scatter plots of the Liouvillian eigenvalues for a $N=4$ lattice sites with $e=0.8, m=0.5, \beta=0.1$ for different types of environmental correlators.

## A. Liouvillian eigenmodes and relaxation dynamics

The relaxation dynamics of open quantum systems are frequently discussed in condensed matter physics and quantum optics []. Here, we study analogous nonequilibrium dynamics of field theories relevant for highenergy nuclear and particle physics. We study basic concepts of the discretized field theory, while leaving more detailed studies in the quantum field theory limit for future work.

We start by rewriting the Lindblad master equation in Eq. (8) in terms of a Liouvillian superoperator $\mathcal{L}$ that acts on the density matrix $\rho$ as

$$
\begin{equation*}
\frac{\mathrm{d} \rho}{\mathrm{~d} t} \equiv \mathcal{L}(\rho) \tag{13}
\end{equation*}
$$

In the semi-classical limit, this Liouvillian becomes equivalent to the non-hermitian Hamiltonian [83] formalism.

As expected from the open quantum system, the density matrix diagonalizes over time due to thermalization when expressed in terms of the energy eigenstate basis. With our re-expression of the Lindblad equation as a Liouvillian superoperator acting on this density matrix, we are also able to understand how we approach to the thermal state, i.e. the non-equilibrium and relaxation dynamics by carrying out a spectral analysis. That is, we expand the density matrix describing the open quantum system dynamics in terms of eigenmodes of the Liouvillian. The right and left eigenmodes $\rho_{j}^{R, L}$ are defined as

$$
\begin{equation*}
\mathcal{L}\left(\rho_{j}^{R}\right)=\lambda_{j} \rho_{j}^{R}, \mathcal{L}^{\dagger}\left(\rho_{j}^{L}\right)=\lambda_{j}^{*} \rho_{j}^{L} \tag{14}
\end{equation*}
$$

where the subscript $j=1, \ldots d^{2}$ indexes the $j$ th eigenmodes and eigenvalues $\lambda_{j}$. The left and right eigenmodes are orthogonal

$$
\begin{equation*}
\left\langle\rho_{i}^{L} \mid \rho_{j}^{R}\right\rangle \sim \delta_{i j} \tag{15}
\end{equation*}
$$

and we define the inner product as $\langle A \mid B\rangle=\operatorname{Tr}\left[A^{\dagger} B\right]$. The dimensionality of the Liouvillian is $d^{2} \times d^{2}$ acting on a vectorized density matrix of length $d^{2}$, where $d$ is the number of states in the system Hamiltonian. In Fig. 4, we plot the eigenvalues for a $N=4$ lattice sites for different types of interactions that we discussed above. Here,
in order to facilitate the visual comparison we restrict the range of real values to be $[-10,0]$, although the constant interaction can have a few eigenvalues at a much smaller value. This spectrum of eigenvalues clearly demonstrates that the non-equilibrium dynamics are nontrivially modified for different types of interactions with the medium. For example, in the case where the interaction corresponds to a delta function, we observe the emergence of a band structure. This indicates that different subspaces of the Hilbert space decay in separate stages. (Found a similar statement in [73]. Is there anything we can say about the states that the different bands correspond to?)

Assuming for now that there is no degeneracy for the stationary state, ordering the label such that eigenvalues are sorted in ascending order $0=\operatorname{Re}\left(\lambda_{0}\right)>\operatorname{Re}\left(\lambda_{1}\right) \geq$ $\ldots \geq \operatorname{Re}\left(\lambda_{d^{2}-1}\right)$, the time evolution of the general density matrix can be written, for instance with respect to the right eigenmodes, as

$$
\begin{equation*}
\rho(t)=\rho_{0}+\sum_{j=1}^{d^{2}-1} c_{j} e^{\lambda_{j} t} \rho_{j}^{R} \tag{16}
\end{equation*}
$$

where the coefficients $c_{j}$ are given by

$$
\begin{equation*}
c_{j}=\frac{\left\langle\rho_{j}^{L} \mid \rho(t=0)\right\rangle}{\left\langle\rho_{j}^{L} \mid \rho_{j}^{R}\right\rangle} \tag{17}
\end{equation*}
$$

This result is obtained by diagonalizing Eq. (13). Since the eigenvalues satisfy $\operatorname{Re}\left(\lambda_{j \geq 1}\right)<0$, the density matrix $\rho(t)$ eventually relaxes to $\rho_{0}$, which is referred to as the (non-equilibrium) steady state. This is nothing but our thermal state in the high temperature limit. (I think XJ said they are different in general. Why/how should they differ in general again?) Note that as this is the only eigenmode with the trace equal to 1 (all the others eigenmodes have vanishing trace), none of the other eigenmodes satisfy the condition to be a density matrix by themselves.

Analyzing the behavior of the open system with respect to the eigenmodes provide means to interpret the non-equilibrium and relaxation dynamics. For example, the approach of the general $\rho(t)$ to $\rho_{0}$ will be governed by the first few $\rho_{j}^{R}$. In particular, for a given observable $\hat{\mathcal{O}}$, the expectation value $\langle\hat{\mathcal{O}}\rangle_{\rho(t)}=\operatorname{Tr}[\hat{\mathcal{O}} \rho(\mathrm{t})]$ will
approach the thermal expectation given by the stationary state eigenmode $\langle\hat{\mathcal{O}}\rangle_{\rho_{0}}$ and its rate of approach will be bounded by the first non-stationary eigenmode with smallest $i$ such that $\langle\hat{\mathcal{O}}\rangle_{\rho_{i}^{R}} \neq 0$, as $e^{\lambda_{i} t} \geq e^{\lambda_{j} t}$ for $i<j$.

In general, then relaxation dynamics to the stationary state cannot last longer than the rate of the decay of $\rho_{1}^{R}$. For this reason one often defines the spectral gap $\Delta$, which describes the asymptotic decay rate, of the Liouvillian as

$$
\begin{equation*}
\Delta=\left|\operatorname{Re}\left(\lambda_{1}\right)\right| \tag{18}
\end{equation*}
$$

On the other hand, the relaxation time $\tau$ is defined as the maximum time at which the following inequality is satisfied

$$
\begin{equation*}
\max _{\tau}:\left|\langle\hat{\mathcal{O}}\rangle_{\rho(t=\tau)}-\langle\hat{\mathcal{O}}\rangle_{\rho_{0}}\right| \geq e^{-1}\left|\langle\hat{\mathcal{O}}\rangle_{\rho(t=0)}-\langle\hat{\mathcal{O}}\rangle_{\rho_{0}}\right| \tag{19}
\end{equation*}
$$

where max operation is over arbitrary density matrix state $\rho(t)$. Then from Eq. (16), one naively expects

$$
\begin{equation*}
\tau \sim \frac{1}{\Delta} \tag{20}
\end{equation*}
$$

This expectation is not always respected, and Liouvillian skin effects from the boundary condition is one potential source for the deviation from this relation []. We do not find such skin effects here, but it would be interesting to study system with skin effects in the context of quantum simulations as well.

The Liouvillian gap $\Delta$ is one of the primary features that characterizes and is used to classify the dynamics of open quantum systems. In many ways it is analogous to the spectral gap of closed system Hamiltonians. The behavior of the spectral gap $\Delta$ with the system size determines the late time dynamics and is associated with the longest-lived eigenmodes [73].(what do we mean by this? the gap value at a given system size also governs the late time dynamics, right? Why do we say that its behavior with system size "in particular" governs the late time dynamics?) In Fig. 6, we plot the Liouvillian gap for our Schwinger model with different types of interactions and show how the gap reduces as the system size increases. Mention also discussions about interesting bounds for the boundary dissipative systems

The free fermion model can be mapped onto the spin system as the Schwinger model, which gives

$$
\begin{align*}
H_{\mathrm{ff}}= & \frac{1}{2 a} \sum_{n=0}^{N_{f}-2}\left(\sigma^{+}(n) \sigma^{-}(n+1)+\sigma^{+}(n+1) \sigma^{-}(n)\right) \\
& +\frac{1}{2} m \sum_{n=0}^{N_{f}-1}(-1)^{n} \sigma_{z}(n) \tag{21}
\end{align*}
$$

In order to compare with the Schwinger model where the total net charge is fixed zero as a result of the open boundary condition with vanishing electric flux outside the lattice, we constrain the free fermion system to the
sector with zero net charge. The Liouvillian gap of the free fermion model scales with the number of lattice sites as $\Delta_{\mathrm{ff}} \sim N^{-3}$, see Refs. [73, 84].

As discussed in the section II, constant interaction with the medium exhibits CP symmetry and the existence of two CP sectors imply degeneracy in the spectrum unless we separate into distinct CP sectors. This implies that we have two stationary states, $\rho_{0}^{\text {even }}$ and $\rho_{0}^{\text {odd }}$. That is, our evolution equation in Eq. (16) is now modified as

$$
\begin{align*}
\rho(t)= & c_{0}^{e} \rho_{0}^{\text {even }}+c_{1}^{o} \rho_{0}^{\text {odd }} \\
& +\sum_{j=1}^{N_{e}-1} c_{j}^{e} e^{\lambda_{j}^{e} t} \rho_{j}^{\text {even }, \mathrm{R}}+\sum_{k=1}^{N_{o}-1} c_{k}^{o} e^{\lambda_{j}^{o} t} \rho_{j}^{\text {odd }, \mathrm{R}} \tag{22}
\end{align*}
$$

where $N_{e}$ and $N_{o}$ are dimensionality of the Hilbert space of CP even and CP odd sectors, respectively, with the condition $N_{e}+N_{o}=d^{2}$. (Should the limits for even/odd add up to the left result for high $T$ ?) Since CP is now a symmetry of the theory, initial density matrix in one CP sector will have a zero overlap with the eigenmodes in the other CP sector. In particular, as the first non-stationary eigenmode for even and odd sectors are different in general, we have different relaxation dynamics for the two sectors

$$
\begin{equation*}
\Delta_{\text {even }}=\left|\operatorname{Re}\left(\lambda_{1}^{\mathrm{e}}\right)\right|, \quad \Delta_{\text {odd }}=\left|\operatorname{Re}\left(\lambda_{1}^{\mathrm{o}}\right)\right|, \tag{23}
\end{equation*}
$$

which we plot the first several values in the Fig. ?? as a function of the number of lattice sites.

We plot the complex eigenvalues of the Liouvillian in Fig. 8. Kyle

Plots: eigenvalue plots, gap, eigenmodes, relaxation time, dependence on the number of lattice sites. Could we include a plot with only boundary dissipative terms?

## B. Decoherence and von Neumann entropy

The entropy of a quantum system is generally used to quantify its disorder or uncertainty. For example, the von Neumann entropy is given by

$$
\begin{equation*}
S=-\operatorname{tr}[\rho \log \rho] \tag{24}
\end{equation*}
$$

which is a measure of the decoherence of a quantum system. The von Neumann entropy vanishes for a pure state where $\rho^{2}=\rho$ and a finite value for $S$ measures the deviation from a pure state. This is a generalization of the Gibbs (and Shannon) entropy of thermodynamic systems to the quantum case. We note that the phenomenon of decoherence is frequently discussed in literature of highenergy heavy-ion collisions [] and the entropy were discussed in the context of parton distribution functions in Refs. [85-87] and jet physics in Ref. [88].

In the Schwinger model as an open quantum system, the pure initial state of a string decoheres due to the interaction with the thermal environment. Therefore, we obtain a finite value for the von Neumann entropy for


FIG. 5. The Liouvillian gap as a function of the number of lattice sites $N$ for the Schwinger model and a free fermion model. (Closing of the gap with the lattice size because system is less correlated and thermalizes faster. Similar to short-range correlated environment leads to a quick thermalization. (Both arguments from chatgpt)) (Add Hamiltonian of free fermion theory, add $\Delta$ on the $y$-axis) (Shouldn't the free fermion theory be $\sim 1 / N^{3}$, see [73]?)


FIG. 6. The Liouvillian gap as a function of the number of lattice sites $N$ for the Schwinger model and a free fermion model. (Closing of the gap with the lattice size because system is less correlated and thermalizes faster. Similar to short-range correlated environment leads to a quick thermalization. (Both arguments from chatgpt)) (Add Hamiltonian of free fermion theory, add $\Delta_{1}$ on the $y$-axis)
$t>0$ which increases as a function of time due to the continued interaction with the environment until the system is fully thermalized. Once the system is in a thermal state, the von Neumann entropy reaches its maximum value. The von Neumann entropy is generally bounded by

$$
\begin{equation*}
0 \leq S \leq \log d \tag{25}
\end{equation*}
$$

where $d$ is the dimension of the Hilbert space. As mentioned above, the lower limit is obtained for a pure state, whereas the upper limit is realized for a maximally mixed state proportional to the identity matrix $\rho_{\mathrm{mm}}=\frac{1}{d} \mathbb{1}$. The
thermal state which is generated at late times of the Lindblad evolution approximates the maximally mixed state in the limit $T \rightarrow \infty$. Here, we explore numerically the real-time dependence of the von Neumann entropy in the Schwinger model as an open quantum system and its temperature dependence.

In Fig. 7, we plot the von Neumann entropy of $N=4$ lattice sites with $e=0.8, m=0.5, \beta=0.1$. The relaxation dynamics is modified for different types of interactions, and we find this to be consistent with the hierarchy between the size of the gap for different interactions at $N=4$ illustrated in the Fig. ??. Furthermore, we show the plot of the von Neumann entropy of different CP sectors for the constant type interaction. We find that the relaxation rate is comparable for both sectors, which is consistent with the Fig. ?? again, and additionally find the thermalization value of the two sectors to be different. The relative size difference of the thermalization values can be understood from the fact that the dimension, which gives the maximum value of the von Neumann entropy, of the Hilbert space is larger for the CP even sector in comparison to the CP odd sector.

## IV. STRING BREAKING IN A THERMAL BACKGROUND

In this section, we study the real-time dynamics of the string-breaking process in the Schwinger model. As mentioned above, the evolution of the string in the Schwinger model can be considered as a model of hadronization in QCD where two quarks are separated by a color string, see for example the Lund string model [89]. String breaking has been studied numerically in the vacuum in several previous studies [35, 37, 52]. We will consider both the vacuum case within our setup as well as for the first time


FIG. 7. Von Neumann entropy $S_{\mathrm{vN}}$ of the Schwinger model as an open quantum system for $N=4$ lattice sites with $e=0.8$, $m=0.5$, and $\beta=0.1$. Left: $S_{\mathrm{vN}}$ for different environmental correlators starting from the trivial vacuum state (CP even). Right: $S_{\mathrm{vN}}$ for $D \sim$ const where the CP sectors are separated (starting with appropriate initial states in each sector). (Would it look better to show the $y$-axis all the way down to 0 ?)



FIG. 8. String breaking in vacuum for $N=6$ lattice sites, corresponding to 11 electric field links. Left: Numerical RK4 solution. Right: Diagram of string breaking process. In both cases, the $y$-axis shows the fermion/anti-fermion lattice sites, and the $x$-axis shows the time evolution.
the medium modification to the string breaking dynamics.

## A. Vacuum

To begin, we study the string breaking process in vacuum in our setup. We consider an initial configuration where a string of electric field links is located in the middle of the one-dimensional lattice, connecting to fermions at the string endpoints. In order to focus on the dynam-
ics of this string, we will subtract from this configuration the results when no string is initialized, i.e. the trivial (fully unoccupied) vacuum state. We choose suitable values of $m, e$ where string-breaking occurs in the vacuum; different regimes will be discussed further in Section IV B. In particular, we choose: mass $m=0.5$, electric charge $e=0.8$, and lattice spacing $a=1$. Since the numerical simulation of the medium case is computationally very expensive we limit ourselves to $N=6$ lattice sites ( $N_{f}=12$ fermion sites) corresponding to 11 electric field links throughout this section. We note that our
initial state corresponds to a bare state where effectively a string creation operator is applied to the bare vacuum. It is possible to extend this description and smear the relevant states into wave packets. In this work, we do not pursue this direction further but instead, refer the reader to Refs. [90-92].

We quantify the presence of the string by measuring the electric field content as a function of real time. The initial configuration can be seen at $t=0$ on the left side of Fig. 8. The string is shown in blue whereas green corresponds to no electric field. In our convention, we choose the electric field to be pointing in the direction where it has negative values at the beginning $\left\langle E_{i}\right\rangle=$ -1 . If it is pointing in the opposite direction it will take positive values up to $\left\langle E_{i}\right\rangle=+1$.

As time evolves, the string breaks into two spatially separated fermion anti-fermion pairs (mesons) that move away from each other with a certain velocity. These tightly bound-states can be seen in Fig. 8 as blue lines pointing toward to upper and lower edge of the figure. Eventually, there are rescattering effects at the boundary of the lattice, which is an artifact of the finite size of our setup. With tensor networks, it is possible to simulate significantly larger lattices, which we do not pursue in this work. See Refs. [37? ].

## B. Medium

Now we are going to explore how the string-breaking process described above is modified due to the presence of a thermal medium. The real-time evolution of the string is described by the Lindblad equation, see Eq. (8) above. For our numerical simulations, we choose the delta-function correlator $D=D_{0} \delta(z)$ where $z=\left|x_{1}-x_{2}\right|$, with different values of the prefactor $D_{0}$. Similar to the vacuum case, we again subtract the Lindblad evolved trivial vacuum state from the initial string configuration.

The open quantum system evolution of the string is shown for an $N=6$ lattice in Fig. 9. The constant $D_{0}=0.05-0.3$ is increased from left to right. As $D_{0}$ increases, the system is more significantly modified. Additionally, we investigate whether a delay of the string breaking mechanism is observed as $D_{0}$ is increased. In order to quantify this effect, we determine the time $t^{*}(x)$ at which each site $x=n a$ reaches the maximum electric field value,

$$
\begin{equation*}
t^{*}(x)=\underset{t}{\arg \max } E_{i}(x, t) \tag{26}
\end{equation*}
$$

Here $t$ is chosen in an interval $t \in\left[0, t_{\max }\right]$, where $t_{\max }$ is determined by the onset of boundary effects due to the finite size of the lattice. The results are shown in Fig. 9, where we plot $t^{*}$ as a function of the site index $i \in[0,1,2,3]$ (excluding the trivial sites where the string is initialized). In the limit of small $D^{0}\left(D^{0}=0.01\right)$ the open system behavior approaches the vacuum behavior. We find that $t^{*}(x)$ is larger for all $x$ in the open system,
with a larger delay observed for stronger $D_{0}$. This delay can be understood from the reduced string tension. Less energy is stored in the string, which delays the formation and the velocity at which the fermion anti-fermion pair are separated. In addition, the presence of the medium slows down the propagation of the pairs once they are formed. Actually not sure if we can say that the initial string tension is reduced since we start with the same initial state as in the vacuum. Just the time evolution is modified. So probably more of a drag force-related explanation is needed and contrasted to the static case of reduced string tension.

We also note that for a sufficiently long time, the system reaches the non-equilibrium steady state, which is beyond the effect of the delayed string breaking of the open quantum system. It will be interesting to investigate in the future if insights can be obtained from our results for the development of hadronization models in heavy-ion collisions [93, 94].

The real-time dynamics of the string breaking depend on the mass $m$ and coupling $e$. In the vacuum, there are three different regimes [] (Xiaojun, could you describe here the discussion with Magnifico in more detail?):

- In the parameter region where $m \sim e^{2} \sim 1$, the fermion mass, the electric energy stored in the electric fields and the kinetic energy are all on the same order and string breaking can happen in real time dynamics. The electric flux between an electronpositron pair will break to release enough energy to create another pair of electron and positron and form two charge-neutral "mesons" that move away from each other. This process can only happen when the electric energy stored in the electric flux is comparable to the sum of the typical kinetic energy and twice the fermion mass.
- In the region where $m \sim e^{2} \gg 1$, the string cannot break in real time dynamics since the kinetic energy released from the break of the electric flux is too big for the neutral "meson" to carry. Instead, the state oscillates, i.e. fermion pairs are created and annihilated but the string effectively stays in place and the created fermion pairs do not move away from each other. Add note about energy conservation and tradeoff of strength of terms. I think we need plots to explain this clearly.
- In the region where $m \gg e^{2}$ or $e^{2} \gg m$, the string stays intact in real time dynamic since the energy released from the breaking of one unit electric flux is too large or too small to create an electronposition pair in a long time.

Fig. 11 (left) quantifies these regimes in vacuum with a metric $\bar{E}$ defined as the average value of the electric field at the 3 central sites over a specified time window:

$$
\begin{equation*}
\bar{E} \equiv \int_{t_{1}}^{t_{2}} \sum_{i \in[4,5,6]} E_{i}(t) \mathrm{d} t \tag{27}
\end{equation*}
$$



FIG. 9. String breaking in medium for $N=6$ lattice sites for three different values of $D_{0}$ with the $\delta$-function environment correlator.


FIG. 10. The time $t^{*}(x)$ at which each site $x=n a$ reaches the maximum electric field value. We find that $t^{*}(x)$ is larger for all $x$ in the open system (excluding the trivial central three sites), with a larger delay observed for stronger $D_{0}$.

As expected, we observe that at small $m$ and $e$ the string breaks, corresponding to $\bar{E}$ large (i.e. less negative). At large $m$ and/or $e$, the string generally stays intact, corresponding to $\bar{E} \approx-1$. The "wing" structure in the plots with $m \sim e^{2}$ corresponds to the second regime described above. It appears as light-blue in our plots, corresponding to an intermediate value of $\bar{E} \approx$ due to configurations where the central site oscillates due to fermion pair creation and annihilation. The period of these oscillations depends on the value of $e, m$ which gives rise to a varying magnitude of $\bar{E}$ due to the fixed time window we examine.

The modification of this behavior is shown in Fig. 11 (right), where $\bar{E}$ is plotted as a function of $e, m$ for the open system. While we observe the same three regimes as in vacuum, their behaviors are significantly modified. At small $m$ and $e$, we observe a regime of string breaking - with slightly smaller amplitude than in vacuum, due to the delayed breaking effect discussed above. In the open system this is coupled with an approach of the system to thermalization, which reduces the magnitude of $\bar{E}$ relative to the vacuum case. As $m$ and/or $e$ are increased, the string thermalizes before it can break. At large $m$ and/or $e$, we observe thermalization of the string as op-
posed to the intact string observed in vacuum - noting that the thermalization time depends on $m, e$ such that $\bar{E}$ remains at $\bar{E} \approx-1$ in the fixed time window we examine.

Next, we consider the case where the string does not break in vacuum, i.e. it is a bound state, but the medium can induce a melting of the string. This scenario is analogous to quarkonium dissociation in heavy-ion collisions [95]. In Fig. ??, we show both the vacuum and the medium case. While the string stays intact in the vacuum, the medium effects lead to a melting such that after a sufficiently long time, the entire lattice reaches the non-equilibrium steady state. Whether the medium can melt the initial string depends on the medium properties and the parameters of the Schwinger model such as the fermion mass. (Xiaojun could you edit this and include references? Make sure arguments about string tension are correct or should be removed? Instead drag force..?)

## V. QUANTUM SIMULATION: ESTIMATION OF TROTTER ERRORS

Lindblad dynamics can be simulated with a quantum algorithm based on the Stinespring dilation theorem [96].


FIG. 11. String breaking in vacuum (left) and medium (right) for different values of the mass $m$ and coupling $e$. We show the value of the central lattice links averaged over $t=3-4$ where the string is initially located at $t=0$. We use a delta function correlator with $D_{0}=0.15$.


FIG. 12. Top: Example oscillatory behavior in the wing region. Bottom: A bound string in vacuum (left) can be melted by the medium (right). The oscillatory behavior is suppressed by larger $m$, but remains present.

The non-unitary evolution of the open quantum system can be achieved by including an ancillary register, which allows for the embedding of the evolution in an enlarged

Hilbert space. In this larger Hilbert space, the evolution is step-wise unitary and repeated reset operations of the ancillary qubit register lead to a time-irreversible


FIG. 13. Quantum algorithm to simulate Lindblad evolution based on the Stinespring dilation theorem [96]. Here $\left|\psi_{S}(0)\right\rangle$ denotes the initial state of the system.
and non-unitary evolution. Following Ref. [97], we can simulate the Lindblad evolution in terms of small time steps $\delta t=t / N_{\text {cyl }}$, where $t$ is the final time we evolve to and $N_{\text {cyl }}$ is the number of time steps or cycles. The simulation protocol illustrated in Fig. 13 proceeds by alternating between the application of the unitary evolution operator associated with the system $U_{H_{S}}=\exp \left(-i H_{S} \delta t\right)$ and the evolution operator $U_{J}=\exp (-i J \sqrt{\delta t})$, where $J$ is a block matrix that contains the Lindblad operators in the first row and column

$$
J=\left(\begin{array}{cccc}
0 & L_{1}^{\dagger} & \ldots & L_{m}^{\dagger}  \tag{28}\\
L_{1} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
L_{m} & 0 & \ldots & 0
\end{array}\right)
$$

Here we limit ourselves to the case where the environmental correlator is given by $D(z)=\delta_{0 x}$, see the discussion around Eqs. (8) and (11) above. The evolution operator $U_{J}$ acts on the system and the ancillary register of qubits. The ancillas are reset after every time step $\delta t$, which leads to a non-unitary evolution. In the limit $N_{\text {cyl }} \rightarrow \infty$, the exact Lindblad evolution is recovered. The error associated with the decomposition of the Lindblad evolution in terms of $U_{H_{S}}$ and $U_{J}$ operators scales as $\delta t^{1.5}$. To illustrate the numerical size of the error that is introduced


FIG. 14. Upper panel: Comparison of the full Lindblad evolution (RK4) and results from the (noiseless) quantum simulation using different numbers of cycles $N_{\text {cyl }}$, see Fig. 13. Lower panel: Ratio of the different approximate results to the full RK4 solution.


FIG. 15. Numerical results for the time evolution of the Schwinger as a closed system. Results with a different number of Trotter steps $r$ are shown in comparison to the full result, which is labeled as $r=\infty$.
by using a finite number of $N_{\text {cyl }}$ time steps, we show the evolution of the Schwinger model as an open quantum system for $N_{\text {cyl }}=1-4$ in Fig. 14 along with the full result based on the Runge-Kutta method. As an example, we consider the expectation value of the electric field summed over all links of the lattice, with the trivial vacuum as the initial state. All numerical results presented in this section are based on a 2-dimensional spatial lattice. As the number of steps $N_{\text {cyl }}$ is increased, the agreement with the full result improves. Here we assumed that both $U_{J}$ and $U_{H_{S}}$ can be mapped exactly to elementary quantum gates, without considering shot noise, gate errors etc. In general, the mapping of the unitary evolution operators $U_{J}$ and $U_{H_{S}}$ to elementary quantum gates requires further approximations. In Ref. [54, 98] an efficient compilation method [99] was used to approximately map the unitary operators $U_{J}$ and $U_{H_{S}}$ to elementary quantum gates. However, for unitaries acting on a larger number of qubits this compilation process can become computationally expensive. Instead, to implement the evolution operators $U_{H_{S}}$ and $U_{J}$ on a quantum computer, a Trotter-Suzuki decomposition $[100,101]$ can be employed for both operators. This decomposition will introduce additional errors, besides the errors arising due to a finite number of cycles, as shown in Fig. 13. In this section, we will quantitatively assess both types of errors.

We can write any Hamiltonian acting on $n$ qubits, in our case $H_{S}$ and $J$, as

$$
\begin{equation*}
H=\sum_{j} H_{j}=a_{j} P_{j} \tag{29}
\end{equation*}
$$

where $P_{j}:\{\mathbb{1}, X, Y, Z\}^{\otimes^{n}}$ are strings of $n$ Pauli operators (and the identity). The relevant coefficients $a_{j}$ can be obtained as

$$
\begin{equation*}
a_{j}=\frac{1}{2^{n}} \operatorname{tr}\left[P_{j} H\right] \tag{30}
\end{equation*}
$$



FIG. 16. Upper panel: Lindblad evolution of the Schwinger model using the quantum algorithm shown in Fig. 13 for $N_{\text {cyl }}=4$ as in Fig. 14. We show the result for different numbers of Trotter steps of the operators $U_{H_{S}}$ and/or $U_{J}$ indicated by $r_{H, J}$, respectively. Lower panel: Difference between the different Trotter approximations and the result without further Trotter decomposition $r_{H, J}=\infty$.

The unitary evolution with any of the terms in Eq. (29), i.e. $e^{-i H_{j} t}$, can be directly mapped to elementary quantum gates without further approximations [101]. We can implement the evolution of the full Hamiltonian $H$ in Eq. (29), using a first order Trotter decomposition

$$
\begin{equation*}
U_{1}(t)=\prod_{j} e^{-i H_{j} t} \tag{31}
\end{equation*}
$$

The upper bound for the error of this approximation, i.e. the difference between $U_{1}(t)$ and $e^{-i H t}$, is given by [102]

$$
\begin{equation*}
\left\|e^{-i H t}-U_{1}(t)\right\| \leq \frac{1}{2} \sum_{j>k}\left\|\left[H_{j}, H_{k}\right]\right\| t^{2} \tag{32}
\end{equation*}
$$

where $\|\cdot\|$ denotes the spectral norm. The error bound of the first order Trotter decomposition is proportional to the square of the time $t$ and the size of the prefactor depends on the number of non-commuting terms in Eq. (29). By decomposing the interval $t$ into $r$ time steps, the error can be reduced to

$$
\begin{equation*}
\left\|e^{-i H t}-U_{1}^{r}(t / r)\right\| \leq \frac{1}{2} \sum_{j>k}\left\|\left[H_{j}, H_{k}\right]\right\| \frac{t^{2}}{r} \tag{33}
\end{equation*}
$$

Here $U_{1}^{r}(t / r)$ denotes $r$ applications of the Trotter decomposed unitary $U_{1}$ with each time steps being $t / r$. This implies that the value that needs to be chosen for
$r$ and the computational cost to perform the simulations within spectral-norm error $\epsilon$, scales as $O\left(t^{2} / \epsilon\right)$. To further reduce the cost, higher order Trotter formulas can be used [102-104]. Here we limit ourselves to first-order Trotter decompositions since we are primarily interested in the difference between the closed and open system evolution.

In the following, we present numerical results for the Lindblad evolution of the Schwinger model using different numbers of Trotter steps. For comparison, we start by considering the error induced by the Trotter decomposition for the vacuum evolution (i.e. $N_{\text {cyl }}=1$ ) of the Schwinger model, which was also considered in Ref. [36]. The results are shown in Fig. 15, where we choose exemplary values for the number of Trotter steps $r=3,5,10$. As expected, the error increases for late times in comparison to the full result, which is labeled as RK4. Next, we consider the Trotter error for the Lindblad evolution. Our numerical results are shown in Fig. 16. For all results we choose $N_{\text {cyl }}=4$, which provides a good approximation of the full result for the time values shown here, see Fig. 14. We denote the number of Trotter steps for $U_{H_{S}}$ and $U_{J}$ by $r_{H, J}$, respectively. These Trotter steps correspond to a further decomposition of the time interval of each cycle with a time step $\delta t$. In other words, here $r_{H, J}=1$ is analogous to a Trotter decomposition of the vacuum result in Fig. 15 with $r=4$. Interestingly, we observe that the error induced by the Trotter decomposition for the open quantum system evolution is smaller compared to the time evolution of the closed system. This holds even though more qubits and gates have to be applied to realize the Lindblad evolution due to the unitaries $U_{J}$. For the closed system evolution, we use 3 qubits and for the open system, we need twice as many. From the upper bound for the Trotter error in Eq. (33) and the consecutive application of quantum gates, one might have expected an increased error for the open quantum system evolution as there are significantly more non-commuting terms that contribute to the total error when $U_{J}$ is included. While these results may not be universally applicable, they suggest the presence of interesting error cancellation effects in the Trotter decomposition of field-theoretical open quantum systems, which motivates further detailed studies in future work.

## VI. CONCLUSIONS

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