Double Distributions and Pseudo-Distributions

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We describe an approach of lattice extraction of Generalized Parton Distributions that is based on use of the double distributions (DDs) formalism within the pseudo-distribution approach. The advantage of using DDs is hat GPDs obtained in this way have the mandatory polynomiality property, a non-trivial correlation between x- and ξ-dependences of GPDs. Another advantage of using DDs is that the D-term appears as an independent quantity in the DD formalism rather than part of GPDs H and E. We relate the ξ-dependence of GPDs to the width of the α-profiles of the corresponding DDs, and discuss strategies for fitting lattice-extracted pseudo-distributions by DDs. The approach described in the present paper may be used in ongoing and future lattice extractions of GPDs.

I. INTRODUCTION

Generalized Parton Distributions (GPDs) [1–6] (for reviews see [7–9]) are a major object of study at future Electron-Ion Collider and existing facilities at Jefferson Lab and CERN. They provide a detailed information about hadronic structure. Being functions of 3 kinematic variables, e.g. $H(x, \xi, t)$ (while there are other GPDs: $E, \tilde{H}, \tilde{E}$, etc., for brevity we will use $H(x, \xi, t)$ as a generic notation), they combine properties of usual parton distributions $f(x)$, hadronic form factors $F(t)$ and, in the region $|x| < \xi$, of the distribution amplitudes $\varphi(x/\xi)$.

However, this multi-dimensional nature of GPDs highly complicates their extraction from experimental data. In particular, deeply virtual Compton scattering (DVCS), which is the most popular tool for obtaining information about GPDs, gives information about GPDs on the diagonal $x = \pm\xi$ or through the Compton form factors that are x-integrals of GPDs with the $1/(x - \xi)$ weight.

More complicated processes like double DVCS or recently proposed single diffractive hard exclusive photoproduction [10] may provide information about GPDs off the $x = \pm\xi$ diagonals. The study of such processes is in its early stage.

During the last decade, starting with the pioneering paper of X. Ji [11] that introduced the quasi-distribution approach (see also Ref. [12] for "lattice cross sections" approach), strong efforts have been made to calculate parton distributions on the lattice (for reviews see Refs. [13–16]). In particular, matching conditions for GPDs in the quasi-distribution approach were discussed in Refs. [17–19]. For a review of recent lattice calculations of GPDs see Refs. [20, 21].

In our paper [22], general aspects of lattice QCD extraction of GPDs have been discussed in the framework of the pseudo-distribution approach [23, 24]. The advantage of lattice calculations is that matrix elements $M(\nu, \xi, t)$ ("Ioffe-time" distributions) of nonlocal operators measured on the lattice are related to Fourier transforms of GPDs like $H(x, \xi, t)$, etc., which may be inverted using various technics to produce GPDs as functions of x for fixed values of skewness ξ and invariant momentum transfer t.

An important property of GPDs is *polynomiality* [7], which states that x^N moment of $H(x, \xi, t)$ must be a polynomial of ξ of not larger than $(N + 1)$th power. This nontrivial correlation between x- and ξ- dependences of $H(x, \xi, t)$ is automatically satisfied when GPDs are obtained from double distributions (DDs) $F(\beta, \alpha, t)$ [1, 3, 4, 25, 26].

The goal of the present work is to describe an approach of lattice extraction of double distributions from lattice calculations. The paper organized as follows. To make it self-contained, in Sec. II we formulate the definitions of usual (light-cone) GPDs, DDs and discuss their relationship. Some basic properties of GPDs are discussed in Sec III. There we also introduce Ioffe-time distributions (ITDs). Pseudo-distributions, as generalizations of the ITDs onto correlators off the light cone are introduced in Sec. IV. Some strategies for fitting lattice-extracted pseudo-distributions by DDs are discussed in Sec. V. Finally, in Section VI, we summarize our results.

II. GPDS AND DDS

A. Definition of GPD

In the GPD description of a nonforward kinematics proposed by X. Ji [2], the plus-components of the initial p_1 and final p_2 hadron momenta are given by $(1 + \xi)\mathcal{P}^+$ and $(1-\xi)\mathcal{P}^+$, respectively, with $\mathcal{P}$ being the average momentum $\mathcal{P} = (p+p')/2$, while the partons have $(x+\xi)\mathcal{P}^+$ and $(x-\xi)\mathcal{P}^+$ as the plus-components of their momenta, see Fig. 1.

For the pion, one may define the light-cone GPDs $H(x, \xi, t; \mu^2)$ [1, 2, 6] by

$$\langle p_2|\mathcal{O}^\lambda(z, A)|p_1\rangle = 2\mathcal{P}^\lambda \int_{-1}^{1} dx\, e^{-ix(\mathcal{P}z)}\, H(x, \xi, t; \mu^2) \;, \tag{2.1}$$

where $\mathcal{O}^\lambda(z, A) = \bar{\psi}(-z/2)\gamma^\lambda \hat{W}(-z/2, z/2; A)\psi(z/2)$ is the quark bilocal operator with $\hat{W}(-z/2, z/2; A)$ being

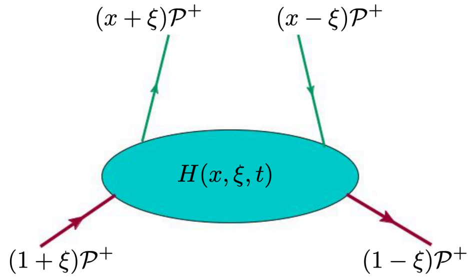

FIG. 1. Flux of the momentum plus-components in terms of GPD variables.

Wilson line in the fundamental representation, the coordinate z has only the z_- light-cone component and $\gamma^\lambda = \gamma^+$. The matrix element is singular on the light cone, so one should use some regularization for it specified by a scale μ. For brevity, we will skip reference to μ^2 in what follows.

The invariant momentum transfer is given by $t = (p_1 - p_2)^2$. In principle, the r.h.s. of Eq. (2.1) has also the r^λ term, where $r = p_1 - p_2$ is the momentum transfer. However, the GPD convention is to write $r^+ = 2\xi\mathcal{P}^+$, where ξ is the skewness variable, and the two terms are combined in one GPD $H(x,\xi,t)$.

A similar definition holds for the nucleon,

$$\langle p_2, s'|\mathcal{O}^+(z,A)|p_1,s\rangle = \int_{-1}^{1} dx\, e^{-ix\mathcal{P}^+z_-} \times \left[(\bar{u}'\gamma^+u)H(x,\xi,t) - \frac{1}{2M}(\bar{u}'i\sigma^{+\mu}r_\mu u)E(x,\xi,t)\right], \tag{2.2}$$

where $\bar{u}' \equiv \bar{u}(p_2,s')$ and $u \equiv u(p_1,s)$ are the nucleon spinors, while $H(x,\xi,t)$ and $E(x,\xi,t)$ are the nucleon GPDs.

One may re-write these definitions in a more covariant form that uses Lorentz invariants $(\mathcal{P}z)$ and (rz) only. For pion, we have

$$\langle p_2|z_\lambda\mathcal{O}^\lambda(z,A)|p_1\rangle\big|_{z^2=0} = 2(\mathcal{P}z)\int_{-1}^{1} dx\, e^{-ix(\mathcal{P}z)}\, H(x,\xi,t;\mu^2)\Big|_{z^2=0}\ . \tag{2.3}$$

In the case of the nucleon, we have two GPDs

$$\langle p_2,s'|z_\lambda\mathcal{O}^\lambda(z,A)|p_1,s\rangle\big|_{z^2=0} = \int_{-1}^{1} dx\, e^{-ix(\mathcal{P}z)}\Big\{(\bar{u}'\not{z}u)H(x,\xi,t) - \frac{1}{2M}(\bar{u}'i\sigma^{zr}u)E(x,\xi,t)\Big\}_{z^2=0}, \tag{2.4}$$

B. Double distribution description

An alternative approach to describe nonforward matrix elements is based on double distributions formalism [1, 3, 4, 25, 26]. Its guiding idea is to treat P^+ and r^+ as independent variables and organize the plus-momentum flow as a "superposition" of P^+ and r^+ momentum flows.

The parton momentum in this picture is written as $k^+ = \beta\mathcal{P}^+ + (1+\alpha)r^+/2$, i.e., as a sum of the component $\beta\mathcal{P}^+$ due to the average hadron momentum P (flowing in the s-channel) and the component $(1+\alpha)r^+/2$ due to the t-channel momentum r, see Fig. 2.

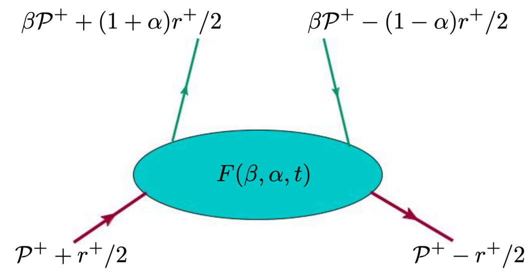

FIG. 2. Flux of the momentum plus-components in terms of DD variables.

Thus, the α-dependence of the DD $F(\beta,\alpha)$ describes the distribution of the momentum transfer r^+ between the initial and final quarks in fractions $(1+\alpha)/2$ and $(1-\alpha)/2$. One may expect that it has a shape similar to those of parton distribution amplitudes, i.e., with maximum at $\alpha = 0$ (equal sharing of r^+) and vanishing at kinematical boundaries. These are located at $\alpha = \pm(1-|\beta|)$, since the support region for DDs is $|\alpha| + |\beta| \le 1$ [26].

1. Pion

In terms of DDs, the matrix element (2.3) is written as [1, 3, 26, 27]

$$\langle \mathcal{P}+r/2|z_\lambda\mathcal{O}^\lambda(z,A)|\mathcal{P}-r/2\rangle_{z^2=0} = \int_\Omega d\alpha d\beta\ e^{-i\beta(\mathcal{P}z)-i\alpha(rz)/2} \times \Big\{2(\mathcal{P}z)F(\beta,\alpha,t) + (rz)\,G(\beta,\alpha,t)\Big\}\Big|_{z^2=0}, \tag{2.5}$$

where Ω is the DD support region, i.e., a rhombus in the $(\alpha\beta)$-plane defined by $|\alpha| + |\beta| \le 1$. The time reversal invariance requires that $F(\beta,\alpha,t)$ is an even function of α, while $G(\beta,\alpha,t)$ is odd in α.

Expanding $e^{-i\beta(\mathcal{P}z)-i\alpha(rz)/2}$ in powers of $(\mathcal{P}z)$ and (rz), one observes that the generic term $(\mathcal{P}z)^{N-k}(rz)^k$ may be obtained both from F- and G-parts [28], with two exceptions. Namely, one cannot obtain the $(\mathcal{P}z)^N$ term from the G-part, and one cannot obtain the $(rz)^N$ term from the F-part. The usual convention is to absorb all the $(\mathcal{P}z)^{N-k}(rz)^k$ terms with $k < N$ into the F-function, leaving all the $(rz)^N$ terms in the G-function [27]. As a result, the G-part would not depend on $(\mathcal{P}z)$, and one can write

$$\begin{aligned}&\langle \mathcal{P}+r/2|z_\lambda \mathcal{O}^\lambda(z,A)|\mathcal{P}-r/2\rangle_{z^2=0}\\&=\Big\{2(\mathcal{P}z)\int_\Omega d\alpha d\beta\, e^{-i\beta(\mathcal{P}z)-i\alpha(rz)/2}\,F(\beta,\alpha,t)\\&+(rz)\int_{-1}^{1} d\alpha e^{-i\alpha(rz)/2}\,D(\,\alpha,t)\Big\}\Big|_{z^2=0}\,, \qquad (2.6)\end{aligned}$$

where $D(\alpha,t)$ is the D-term function introduced in Ref. [27]. It is odd in α.

Comparing the two parameterizations (2.3) and (2.6) we get the relation between the pion GPD and DD [1, 6, 27]

$$\begin{aligned}H\,(x,\xi,t)=&\int_\Omega d\alpha d\beta\,\delta(x-\beta-\alpha\xi)F(\beta,\alpha,t)\\&+\mathrm{sgn}(\xi)D(x/\xi,t;\mu^2)\\&\equiv H_{DD}+D \qquad (2.7)\end{aligned}$$

As noticed in Ref. [28], the $(\alpha\beta)$-integral above, i.e. the "DD part" $H_{DD}(x,\xi,t)$, may be treated as the Radon transform of F.

2. Nucleon

In the nucleon case, we have the following representation [1, 3, 26]

$$\begin{aligned}&\langle \mathcal{P}-r/2,s'|z_\lambda \mathcal{O}^\lambda(z,A))|\mathcal{P}+r/2,s\rangle_{z^2=0}\\&=\int_\Omega d\alpha d\beta\, e^{-i\beta(\mathcal{P}z)-i\alpha(rz)/2}\\&\times\left[(\bar u'\not z u)\,h(\beta,\alpha,t)\;-\frac{1}{2M}(\bar u' i\sigma^{zr}u)e(\beta,\alpha,t)\right]\\&+(rz)\frac{(\bar u' u)}{M}\int_{-1}^{1} d\alpha\, e^{-i\alpha(rz)/2}\,D(\alpha,t)\;. \qquad (2.8)\end{aligned}$$

Here, $h(\beta,\alpha,t)$ and $e(\beta,\alpha,t)$ are even functions of α, while $D(\alpha)$ is odd. Using Gordon decomposition

$$\frac{\mathcal{P}^\lambda}{M}\bar u' u=\frac{1}{2M}\bar u' i\sigma^{\lambda r}u+\bar u'\gamma^\lambda u\;, \qquad (2.9)$$

and comparing (2.8) with the GPD representation (2.4), gives relation between the nucleon GPDs, DDs and D-term [27]

$$\begin{aligned}H\,(x,\xi,t)=&\int_\Omega d\alpha d\beta\,\delta(x-\beta-\alpha\xi)h(\beta,\alpha,t)\\&+\mathrm{sgn}(\xi)D(x/\xi,t)\\&\equiv H_{DD}+D\,, \qquad (2.10)\end{aligned}$$

$$\begin{aligned}E\,(x,\xi,t)=&\int_\Omega d\alpha d\beta\,\delta(x-\beta-\alpha\xi)e(\beta,\alpha,t)\\&-\mathrm{sgn}(\xi)D(x/\xi,t)\\&\equiv E_{DD}-D\,. \qquad (2.11)\end{aligned}$$

Again, we may talk about the "DD parts" $H_{DD}(x,\xi,t)$ and $E_{DD}(x,\xi,t)$ of the corresponding GPDs. Note that the D-term cancels in the sum $H(x,\xi,t)+E(x,\xi,t)\equiv A(x,\xi,t)$. So, $A(x,\xi,t)$ is built purely from the DD $a(\beta,\alpha,t)\equiv h(\beta,\alpha,t)+e(\beta,\alpha,t)$.

C. Fixed parity cases

Usually we are interested in the functions corresponding to operators

$$\mathcal{O}^\lambda_\pm(z,A)=\frac{1}{2}\left[\mathcal{O}^\lambda(z,A)\pm\mathcal{O}^\lambda(-z,A)\right] \qquad (2.12)$$

that are symmetric or antisymmetric with respect to the inversion of z. These combinations appear when we consider "nonsinglet" $q-\bar q$ or "singlet" $q+\bar q$ parton distributions, respectively. Since the D-term contribution (without the overall (rz) factor) is odd in z, it appears in the "singlet" case only. However, the $H+E$ sum does not contain the D-term even in the singlet case.

In fact, it is sufficient to consider matrix element of the original $\mathcal{O}^\lambda(z,A)$ operator. The real part of this matrix element is even in z while its imaginary part is odd in z.

III. SOME PROPERTIES OF GPDS AND DDS

A. DD-parts of GPDs

In this section, we consider the relations between the DDs and the "DD parts" of GPDs which they generate, thus ignoring for a while the D-term contributions to GPDs (we remind that it is absent in the nonsinglet case). The D-term will be discussed later in the section. For definiteness, we will have in mind relations between the DD part of the pion GPD and its DD. All the relations are equally applicable to the DD parts of the nucleon GPDs.

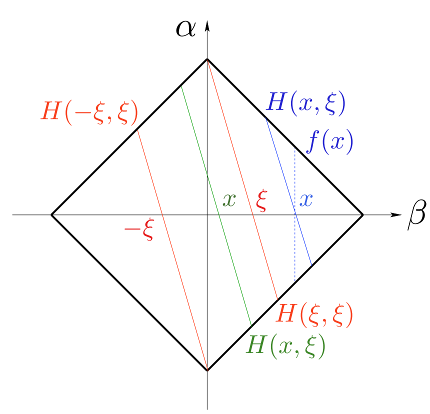

FIG. 3. DD support rhombus and integration lines producing the DD parts of $H(\xi,\xi)$, $H(-\xi,\xi)$, $H(x,\xi=0)=f(x)$ and $H(x,\xi)$ in DGLAP ($|x|>\xi$) and ERBL ($|x|<\xi$) regions.

B. $\xi = 0$ limit

Taking $\xi = 0$, we have

$$H(x, \xi = 0, t) = \int_{-1}^{1} d\beta\, \delta(x - \beta) \int_{-1+|\beta|}^{1-|\beta|} d\alpha F(\beta, \alpha, t)$$
$$= \int_{-1+|x|}^{1-|x|} d\alpha F(x, \alpha, t) \equiv f(x, t)\ . \quad (3.1)$$

This means that integrating $F(\beta, \alpha, t)$ over vertical lines $\beta = x$ gives the $\xi = 0$ ("non-skewed") GPD $\mathcal{H}(x, \xi = 0, t)$, which we will also denote as $f(x, t)$. It is the simplest GPD, that was called "nonforward parton density" in the paper [29], where it has been introduced. It differs from the forward PDF $f(x)$ by the presence of the t-dependence and satisfies $f(x, t = 0) = f(x)$.

C. Polynomiality

The DD representation automatically produces a GPD satisfying the polynomiality property. Indeed,

$$\int_{-1}^{1} dx\, x^n \mathcal{H}(x, \xi, t)$$
$$= \int_{-1}^{1} dx\, x^n \int_{\Omega} d\alpha d\beta\, \delta(x - \beta - \alpha\xi) F(\beta, \alpha, t)$$
$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, \xi^k \int_{\Omega} d\alpha d\beta\, \beta^{n-k} \alpha^k F(\beta, \alpha, t)\,, \quad (3.2)$$

i.e., the $n^{\rm th}$ x-moment of GPD $\mathcal{H}(x, \xi, t)$ is a polynomial in ξ of the order not exceeding n.

Note that, since $F(\beta, \alpha, t)$ is even in α, the summation over k involves even k only, i.e. (3.1) is in fact an expansion in powers of ξ^2.

D. Ioffe-time distributions

By Lorentz invariance, the matrix element (2.3) defining GPD is a function of the scalars $(p_1 z) \equiv -\nu_1$ and $(p_2 z) \equiv -\nu_2$, two Ioffe-time parameters, so we may write

$$\langle p_2 | \bar{\psi}(-z/2) \not{z} \psi(z/2) | p_1 \rangle = 2(\mathcal{P} z) I(\nu_1, \nu_2, t)\,, \quad (3.3)$$

where $I(\nu_1, \nu_2, t)$ is the double Ioffe-time distribution (ITD). Since $z = z_-$ is assumed, only the value of the plus components of momenta are essential in the scalar products $(p_1 z)$ and $(p_2 z)$. The skewness variable ξ in this case is given by

$$\xi = \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} \equiv \frac{\nu_1 - \nu_2}{2\nu}\,. \quad (3.4)$$

We have introduced here the variable $\nu = (\nu_1 + \nu_2)/2$. Treating ν and ξ as independent variables, we define the *generalized Ioffe-time distribution* (GITD) as

$$I(\nu_1, \nu_2, t) = \mathcal{I}(\nu, \xi, t)\ . \quad (3.5)$$

According to (2.1), it is a Fourier transform of the GPD

$$\mathcal{I}(\nu, \xi, t) = \int_{-1}^{1} dx\, e^{ix\nu}\, H(x, \xi, t)\ . \quad (3.6)$$

Using Eq. (2.5), we can write GITD in terms of DD

$$\mathcal{I}(\nu, \xi, t) = \int_{-1}^{1} d\beta\, e^{i\nu\beta} \int_{-1+|\beta|}^{1-|\beta|} d\alpha e^{i\nu\alpha\xi} F(\beta, \alpha, t)\ . \quad (3.7)$$

E. DD profile and ξ-dependence

If $F(\beta, \alpha, t)$ has an infinitely narrow profile in the α-direction, i.e. if $F(\beta, \alpha, t) = f(\beta, t)\delta(\alpha)$, then the ξ-dependence disappears, and we deal with the simplest GPD $f(x, t)$. A nontrivial dependence on the skewness ξ is obtained if DD has a nonzero-width profile in the α-direction.

Using the DD representation (3.7) for the GITD and expanding $e^{i\nu\alpha\xi}$ into Taylor series, we get the following expansion in powers of ξ^2

$$\mathcal{I}(\nu, \xi, t) = \int_{-1}^{1} d\beta\, e^{i\nu\beta} \int_{-1+|\beta|}^{1-|\beta|} d\alpha F(\beta, \alpha, t)$$
$$- \frac{\xi^2 \nu^2}{2} \int_{-1}^{1} d\beta\, e^{i\nu\beta} \int_{-1+|\beta|}^{1-|\beta|} d\alpha\, \alpha^2 F(\beta, \alpha, t) + \ldots \quad (3.8)$$

(odd powers of ξ do not appear because $F(\beta, \alpha, t)$ is even in α). By analogy with (3.1), we will use notation $f_2(\beta, t)$ for the second α-moment of $F(\beta, \alpha, t)$

$$\int_{-1+|\beta|}^{1-|\beta|} d\alpha\, \alpha^2\, F(\beta, \alpha, t) \equiv f_2(\beta, t) \quad (3.9)$$

As a result, we write

$$\mathcal{I}(\nu, \xi, t) = \int_{-1}^{1} d\beta\, e^{i\nu\beta} \left\{ f(\beta, t) - \frac{\xi^2 \nu^2}{2} f_2(\beta, t) \right\} + \mathcal{O}(\xi^4)$$
$$= \mathcal{I}_0(\nu, t) - \frac{\xi^2 \nu^2}{2} \mathcal{I}_2(\nu, t) + \mathcal{O}(\xi^4)\ . \quad (3.10)$$

IV. PSEUDODISTRIBUTIONS

A. Definitions

On the lattice, we choose $z = z_3$, and introduce pseudo-GPDs $\mathcal{H}\left(x, \xi, t; z_3^2\right)$ (and also $\mathcal{E}\left(x, \xi, t; z_3^2\right)$ in the nucleon case),

The two Ioffe-time parameters are given now by $\nu_1 = p_1^{(3)} z_3 \equiv P_1 z_3$ and $\nu_1 = p_2^{(3)} z_3 \equiv P_2 z_3$. In terms of momenta $P_{1,2}$, the skewness ξ is given by

$$\xi = \frac{P_1 - P_2}{P_1 + P_2}\ . \quad (4.11)$$

The pseudo-GITD will be denoted as $\mathcal{M}(\nu,\xi,t;z_3^2)$, e.g., the inverse transformation for $\mathcal{H}$ is written as

$$\mathcal{H}\left(x,\xi,t;z_3^2\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\, e^{-ix\nu}\,\mathcal{M}(\nu,\xi,t;z_3^2)\ . \quad (4.12)$$

Similarly, to denote pseudo-DDs, we will just add z_3^2 to their arguments.

B. Contaminations

On the lattice, we have $z^2\neq 0$, and we need to add extra z-dependent structures to the original parameterization

$$\begin{aligned} z_\lambda M^\lambda &\equiv \langle \mathcal{P}-r/2,s'|z_\lambda\mathcal{O}^\lambda(z,A)|\mathcal{P}+r/2,s\rangle \\ &= \int_\Omega d\alpha d\beta\ e^{-i\beta(\mathcal{P}z)-i\alpha(rz)/2} \\ &\times\left\{(\bar{u}'\not{z}u)\,h(\beta,\alpha,t)-\frac{1}{2M}(\bar{u}'i\sigma^{zr}u)e(\beta,\alpha,t)\right. \\ &\qquad\left.+\frac{\bar{u}'u}{M}(rz)\delta(\beta)D(\alpha,t)\right\} \\ &\equiv (\bar{u}'\not{z}u)H_{DD}-\frac{1}{2M}(\bar{u}'i\sigma^{zr}u)E_{DD}+(rz)\frac{\bar{u}'u}{M}D\ , \end{aligned} \quad (4.13)$$

where $z^2=0$ and the ITDs H_{DD},E_{DD} and D are functions of ν, ξ and t.

Also, for lattice extractions, we need to parametrize the "non-contracted" matrix element M^λ. In this case, the index λ in local operators $\bar{\psi}\gamma^\lambda(zD)^N\psi$ is not symmetrized with the indices $D^{\mu_1}\dots D^{\mu_N}$ in covariant derivatives. A way to perform symmetrization on the level of bilocal operators was indicated in Ref. [30].

Further studies of parameterizations for matrix elements with an open index have been performed in Refs. [31–36]. An important observation made there is that M^λ *should* contain terms that vanish when contracted with z_λ, such as $r^\lambda(\mathcal{P}z)-\mathcal{P}^\lambda(rz)$. One can see that $r^\lambda-\mathcal{P}^\lambda(rz)/(\mathcal{P}z)\equiv\Delta_\perp^\lambda$ is the part of the momentum transfer r that is transverse to z. As shown in these papers, one should add Wandzura-Wilczek-type (WW) terms [38] to the parametrizations of GPDs to secure electromagnetic gauge invariance of the DVCS amplitude [37] with $\mathcal{O}(\Delta_\perp)$ accuracy. While the WW terms are "kinematical twist-3" contributions built from twist-2 GPDs, one cannot exclude non-perturbative (dynamical) twist-3 terms accompanied by the $\Delta_\perp^\lambda$ factor.

A possible parametrization with extra terms is

$$\begin{aligned} M^\lambda &= (\bar{u}'\gamma^\lambda u)H_{DD}-\frac{1}{2M}(\bar{u}'i\sigma^{\lambda r}u)E_{DD}+r^\lambda\frac{\bar{u}'u}{M}D \\ &+i(\bar{u}'u)Mz^\lambda Z_1+(\bar{u}'i\sigma^{\lambda z}u)MZ_2 \\ &-\frac{(\bar{u}'i\sigma^{zr}u)}{M}[\mathcal{P}^\lambda X_1+r^\lambda X_2+z^\lambda M^2X_3] \\ &+\left[r^\lambda(\mathcal{P}z)-\mathcal{P}^\lambda(rz)\right]\frac{\bar{u}'u}{M}Y\ . \end{aligned} \quad (4.14)$$

Using Gordon decomposition

$$\frac{\mathcal{P}^\lambda}{M}\bar{u}'u=\frac{1}{2M}\bar{u}'i\sigma^{\lambda r}u+\bar{u}'\gamma^\lambda u\ , \quad (4.15)$$

we can re-write (4.14) as

$$\begin{aligned} M^\lambda &= (\bar{u}'\gamma^\lambda u)\left[H_{DD}-(rz)Y\right]+r^\lambda\frac{\bar{u}'u}{M}[D+(\mathcal{P}z)Y] \\ &-\frac{1}{2M}(\bar{u}'i\sigma^{\lambda r}u)[E_{DD}+(rz)Y] \\ &+i(\bar{u}'u)Mz^\lambda Z_1+(\bar{u}'i\sigma^{\lambda z}u)MZ_2 \\ &-\frac{(\bar{u}'i\sigma^{zr}u)}{M}[\mathcal{P}^\lambda X_1+r^\lambda X_2+z^\lambda M^2X_3]\ , \end{aligned} \quad (4.16)$$

There are eight spin/tensor structures in total, just as in Ref. [39]. However, in the basis of Ref. [39] $(\bar{u}'\gamma^\lambda u)$ is substituted by two other structures that appear in the Gordon decomposition (4.15). Using this basis, we have

$$\begin{aligned} M^\lambda &= \frac{\mathcal{P}^\lambda}{M}(\bar{u}'u)[H_{DD}-(rz)Y]+r^\lambda\frac{\bar{u}'u}{M}[D+(\mathcal{P}z)Y] \\ &-\frac{1}{2M}(\bar{u}'i\sigma^{\lambda r}u)[H_{DD}+E_{DD}] \\ &+i(\bar{u}'u)Mz^\lambda Z_1+(\bar{u}'i\sigma^{\lambda z}u)MZ_2 \\ &-\frac{(\bar{u}'i\sigma^{zr}u)}{M}[\mathcal{P}^\lambda X_1+r^\lambda X_2+z^\lambda M^2X_3]\ , \end{aligned} \quad (4.17)$$

Comparing Eq. (4.17) with the coefficients A_i in Eq. (35) of Ref. [39], we establish the correspondence $A_1=[H_{DD}-(rz)Y]$, $A_2=iZ_1$, $A_3=D+(\mathcal{P}z)Y$, $A_4=Z_2$, $A_5=(H_{DD}+E_{DD})/2$, $A_6=X_1$, $A_7=X_3$, $A_8=-X_2$. The main difference is that H_{DD} and D in Eq. (4.17) come with the contamination from the Y-function, the 9th ITD.

As one can see, the Y-term in Eq. (4.17) comes with the $\mathcal{P}^\lambda(rz)-r^\lambda(\mathcal{P}z)$ factor that vanishes after contraction with z_λ. Similarly, if we contract matrix element M^λ of Eq. (4.16) with z_λ, we get

$$\begin{aligned} z_\lambda M^\lambda &= (\bar{u}'\not{z}u)H_{DD}+[(rz)D+iz^2M^2Z_1]\frac{\bar{u}'u}{M} \\ &-\frac{(\bar{u}'i\sigma^{zr}u)}{2M}[E_{DD}+(z\mathcal{P})X_1+(rz)X_2+z^2X_3] \\ &-(rz)\left[(\bar{u}'\not{z}u)+\frac{(\bar{u}'i\sigma^{zr}u)}{2M}-(\mathcal{P}z)\frac{\bar{u}'u}{M}\right]Y\ , \end{aligned} \quad (4.18)$$

where the factor accompanying Y vanishes by Gordon decomposition, as expected. Projecting $z_\lambda M^\lambda$ on the light cone, we obtain

$$z_\lambda M^\lambda|_{z^2=0} = (\bar{u}'\not{z}u)H_{DD} + (rz)\frac{\bar{u}'u}{M}D - \frac{(\bar{u}'i\sigma^{zr}u)}{2M}[E_{DD} + \nu(X_1 + 2\xi X_2)] \ . \tag{4.19}$$

Thus, GITD E comes from 3 DDs and the D-term: $E = E_{DD} - D + \nu(X_1 + 2\xi X_2)]$.

V. FITTING PSEUDODISTRIBUTIONS

A. Nonforward parton pseudo-density $f(\beta, t, z_3^2)$

Taking $\xi = 0$ we have

$$\mathcal{M}(\nu, \xi = 0, t; z_3^2) = \int_{-1}^{1} d\beta \, e^{i\nu\beta} f(\beta, t, z_3^2) \ , \tag{5.1}$$

where $\nu = P_1 z_3 = P_2 z_3$. An important point is that $\xi = 0$ may be achieved for different pairs of equal initial and final momenta $P_1 = P_2 \equiv P$. One should check that lattice gives the same curve for different P's, up to evolution-type dependence on z_3^2.

One can use relation (5.1) to fit $f(\beta, t, z_3^2)$. First, taking $t = 0$, we fit the forward pseudodistribution $f(\beta, z_3^2)$, just as a pseudo-PDF. After that, one can vary t, by changing the transverse components $\Delta_\perp^{1,2}$, for several fixed ν. In this way, one can study what kind of dependence on t we have (dipole, monopole, etc.), and how it changes with ν.

B. α^2-moment function $f_2(\beta, t, z_3^2)$

The next step is to check if the ξ-dependence of the lattice data for $\mathcal{M}(\nu, \xi, t; z_3^2)$ agrees with the form

$$\mathcal{M}(\nu, \xi, t; z_3^2) = \mathcal{M}(\nu, \xi = 0, t; z_3^2) - \frac{\xi^2\nu^2}{2}\mathcal{M}_2(\nu, t; z_3^2) + \mathcal{O}(\xi^4) \ , \tag{5.2}$$

and extract $f_2(\beta, t, z_3^2)$ using

$$\mathcal{M}_2(\nu, \xi, t; z_3^2) = \int_{-1}^{1} d\beta \, e^{i\nu\beta} f_2(\beta, t; z_3^2) \ . \tag{5.3}$$

The α-dependence of the DD $F(\beta, \alpha)$ describes the distribution of the momentum transfer $r = P_1 - P_2$ between the initial and final quarks. It is expected that it has a shape similar to those of parton distribution amplitudes.

C. Factorized DD Ansatz

A nonzero-width profile of DD in the α-direction may be modeled by using the Factorized Ansatz [25, 26]

$$F_N(\beta, \alpha, t) = f(\beta, t)\frac{\Gamma\left(N + \frac{3}{2}\right)}{\sqrt{\pi}\Gamma(N+1)}\frac{[(1-|\beta|)^2 - \alpha^2]^N}{(1-|\beta|)^{2N+1}} \tag{5.4}$$

The $[(1-|\beta|)^2 - \alpha^2]$ factor reflects the support properties of the DD, which vanishes if $|\beta| + |\alpha| > 1$. The Ansatz also complies with the requirement that $F(\beta, \alpha)$ should be an even function of α.

For $f(\beta, t)$ one can also take a factorized form $f(\beta, t) = f(\beta)F(t)$, where $f(\beta)$ is the forward PDF, and $F(t)$ some form factor.

Combining (3.7) and (5.4) gives

$$\mathcal{M}(\nu, \xi, t; z_3^2) = \int_{-1}^{1} d\beta \, e^{i\nu\beta} f(\beta, t; z_3^2) \times \int_{-1+|\beta|}^{1-|\beta|} d\alpha \, e^{i\nu\alpha\xi} \frac{\Gamma\left(N + \frac{3}{2}\right)}{\sqrt{\pi}\Gamma(N+1)}\frac{[(1-|\beta|)^2 - \alpha^2]^N}{(1-|\beta|)^{2N+1}} \ . \tag{5.5}$$

Integral over α can be taken

$$A_N(\beta) = \int_{-1+|\beta|}^{1-|\beta|} d\alpha \, e^{i\nu\alpha\xi} \frac{[(1-|\beta|)^2 - \alpha^2]^N}{(1-|\beta|)^{2N+1}} = \int_{-1}^{1} d\eta \, e^{i\nu\xi(1-|\beta|)\eta}(1-\eta^2)^N = {}_0\tilde{F}_1\left(; N + \frac{3}{2}; -\frac{\nu^2\xi^2(1-|\beta|)^2}{4}\right)\sqrt{\pi}\Gamma(N+1) \tag{5.6}$$

where

$${}_0\tilde{F}_1(; b; z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+k)k!} \tag{5.7}$$

is a hypergeometric function.

So, we have a model for pseudo-GITD

$$\mathcal{M}(\nu, \xi, t; z_3^2; N) = \int_{-1}^{1} d\beta \, e^{i\nu\beta} f(\beta, t) \times {}_0\tilde{F}_1\left(; N + \frac{3}{2}; -\frac{\nu^2\xi^2(1-|\beta|)^2}{4}\right)\Gamma\left(N + \frac{3}{2}\right) \ , \tag{5.8}$$

or, expanding in ξ,

$$\mathcal{M}(\nu, \xi, t; z_3^2; N) = \int_{-1}^{1} d\beta \, e^{i\nu\beta} f(\beta, t) \times \sum_{k=0}^{\infty}\left(-\frac{\nu^2\xi^2(1-|\beta|)^2}{4}\right)^k \frac{\Gamma\left(N + \frac{3}{2}\right)}{k!\Gamma(N + 3/2 + k)} \ . \tag{5.9}$$

This expansion may be also obtained by taking Taylor series of $e^{i\nu\xi(1-|\beta|)\eta}$ in Eq. (5.6), and integrating over η.

D. Check of polynomiality

Getting GPDs from a DD representation guarantees that the resulting GPD has the polynomiality property. Still, we can double-check this. Note that the $N^{\rm th}$ moment $\mathcal{M}_N$ of a pseudo-GPD $\mathcal{H}\left(x,\xi,t;z_3^2\right)$ is proportional to the coefficient accompanying ν^N in the Taylor expansion

$$\mathcal{M}(\nu,\xi,t;z_3^2;N) = \sum_{N=0}^{\infty}\frac{i^N\nu^N}{n!}\mathcal{M}_N \ . \tag{5.10}$$

Now, from

$$\mathcal{M}(\nu,\xi,t;z_3^2;N) = \int_{-1}^{1} d\beta \sum_{m=0}^{\infty}\frac{(i\nu\beta)^m}{m!}f(\beta,t;z_3^2) \\ \times\sum_{k=0}^{\infty}\left(-\frac{\nu^2\xi^2(1-|\beta|)^2}{4}\right)^k\frac{\Gamma\left(N+\frac{3}{2}\right)}{k!\Gamma(N+3/2+k)} \tag{5.11}$$

we see that $\mathcal{M}_N$ is a polynomial in ξ of order equal or smaller than N.

E. Fitting α-profile width

After fixing $f(\beta,t;z_3^2)$ that gives the profile of DD in the β-direction, we may quantify what kind of profile it has in the α-direction. The presence of a nontrivial profile is indicated by the presence of ξ-dependence. Using the first terms of the series for ${}_0\tilde{F}_1\left(;b;z\right)$

$$\Gamma(b){}_0\tilde{F}_1\left(;b;z\right) = \sum_{k=0}^{\infty}\frac{z^k\Gamma(b)}{\Gamma(b+k)k!} = 1+\frac{z}{b}+\frac{z^2}{2b(b+1)}+\ldots \tag{5.12}$$

we write (5.9) as

$$\mathcal{M}(\nu,\xi,t;z_3^2;N) = \int_{-1}^{1} d\beta\, e^{i\nu\beta}\, f(\beta,t)\Bigg\{1-\frac{\nu^2\xi^2(1-|\beta|)^2}{4(N+3/2)} \\ +\left(\frac{\nu^2\xi^2(1-|\beta|)^2}{4}\right)^2\frac{1}{2(N+3/2)(N+5/2)}+\ldots\Bigg\} \ . \tag{5.13}$$

In Eq. (5.13), ξ appears through the combination $\xi\nu=(\nu_1-\nu_2)/2$. On the lattice, we have $\nu_1 = P_1z_3$, $\nu_2 = P_2z_3$. Hence, the presence of a nontrivial profile should be reflected by the dependence of the data on the difference P_1-P_2 for a fixed sum P_1+P_2. The first correction in Eq. (5.13) is given by

$$\delta_1\mathcal{M}(\nu,\xi,t;z_3^2;N) = -\int_{-1}^{1} d\beta\, e^{i\nu\beta}\, f(\beta,t;z_3^2)(1-|\beta|)^2 \\ \times\frac{\xi^2\nu^2}{4(N+3/2)} \ . \tag{5.14}$$

Using this expression, one can try to determine the profile parameter N. This task probably will not be easy, since the correction looks rather small due to a small overall factor $\sim\xi^2/4$.

We may also estimate the extra suppression due to the $(1-|\beta|)^2$ factor in the integrand of (5.14). For a simple illustration, take $f(\beta,t)=4(1-|\beta|)^3$. In this case,

$$\int_{-1}^{1} d\beta\, e^{i\nu\beta}\, f(\beta,t) = \frac{48}{\nu^4}\left(\cos(\nu)-1+\frac{\nu^2}{2}\right) \\ = 2-\frac{\nu^2}{15}+\frac{\nu^4}{840}+O\left(\nu^5\right) \tag{5.15}$$

while

$$\int_{-1}^{1} d\beta\, e^{i\nu\beta}\, f(\beta,t)(1-|\beta|)^2 \\ = \frac{960}{\nu^6}\left(-\cos(\nu)+1-\frac{\nu^2}{2}+\frac{\nu^4}{24}\right) \\ = \frac{4}{3}-\frac{\nu^2}{42}+\frac{\nu^4}{3780}+O\left(\nu^5\right) \ . \tag{5.16}$$

Thus, the additional suppression is by about $2/3$ for small ν, i.e., not very strong.

F. D-term

When we take the z-odd part $\mathcal{O}_-^\lambda$ of the operator $\mathcal{O}^\lambda(z,A)$, its parametrization contains a nonzero D-term. In GPD description, it appears in a mixture with H_{DD} (and also E_{DD} in the nucleon case) GPDs. However, using all possible helicity states for nucleons and various values of λ, one can construct sufficient number of linearly independent relations and separate the DDs that appear in the parametrization of Eq. (4.16) by using, e.g., singular value decomposition technique. Unfortunately, as seen from Eq. (4.16), the D-term obtained in this way comes together with the Y-contamination.

Another way is to eliminate H_{DD}, E_{DD}, etc. contributions from the matrix element of $\mathcal{O}_-^\lambda$ by taking kinematics in which $(\mathcal{P}z)=0$. As a result, α-even DD $h(\beta,\alpha)$ will be integrated with the α-odd function $\sin(\alpha(rz)/$, etc., so that we will have

$$\langle\mathcal{P}-r/2,s'|\mathcal{O}_-^\lambda(z,A)|\mathcal{P}+r/2,s\rangle|_{(\mathcal{P}z)=0} \\ = r^\lambda\frac{(\bar{u}'u)}{M}\int_{-1}^{1}d\alpha\ e^{-i\alpha(rz)/2}\,D(\alpha,t) \\ +(\bar{u}'u)Mz^\lambda\int_\Omega d\alpha d\beta\ z_1(\beta,\alpha,t)\,\cos(\alpha(rz)/2)\,. \tag{5.17}$$

On the lattice, choosing $z = z_3$, we can arrange $(\mathcal{P}z)=0$, i.e. $\mathcal{P}_3=0$, by taking p_1 and p_2 with opposite components in z-direction, namely $p_1=(E_1,\mathbf{p}_{1T},P)$ and $p_2=(E_2,\mathbf{p}_{2T},-P)$. Introducing the relevant Ioffe time $\nu_D\equiv-(rz)\Rightarrow 2Pz_3$, we deal with the ITD

$$\mathcal{I}_D(\nu_D,t) = \int_{-1}^{1}d\alpha\, e^{i\alpha\nu_D}\,D\left(\alpha,t\right) \ . \tag{5.18}$$

However, if we choose $\lambda = 0$, we get $r^0 = E_1 - E_2$ as the accompanying factor. It vanishes for purely longitudinal momenta $p_1 = (E_1, \mathbf{0}_T, P)$, $p_2 = (E_2, \mathbf{0}_T, -P)$, and remains rather small when one takes non-equal transverse momenta $\mathbf{p}_{1T}, \mathbf{p}_{2T}$.

Another choice is to take $\lambda = 3$. In this case, we have $\sim z_3$ contamination

$$\frac{1}{i}\langle (E_2, \mathbf{p}_{2T}, -P)|\mathcal{O}_-^3(z,A)|(E_1, \mathbf{p}_{1T}, P)\rangle \\ = 2P\int_{-1}^{1} d\alpha\, \sin(\nu_D \alpha)\, D(\alpha, t) \\ + z^{(3)} M^2 \int_{-1}^{1} d\alpha\, \cos(\nu_D\alpha)\, Z_1(\alpha, t)\ , \tag{5.19}$$

where the "Z-term" function $Z_1(\alpha, t)$ is even in α. Multiplying by $z_\lambda = z_3$, we have

$$i\langle (E_2, \mathbf{p}_{2T}, -P)|z_\lambda \mathcal{O}_-^\lambda(z,A)|(E_1, \mathbf{p}_{1T}, P)\rangle \\ = \nu_D\int_{-1}^{1} d\alpha\, \sin(\nu_D \alpha)\, D(\alpha, t) \\ + \frac{\nu_D^2}{4P^2}\int_{-1}^{1} d\alpha\, \cos(\nu_D\alpha)\, Z(\alpha, t) \\ = \nu_D \mathcal{I}_D(\nu_D, t) + \frac{\nu_D^2}{4P^2}\mathcal{I}_Z(\nu_D, t) \tag{5.20}$$

As we see, for a fixed ν, the contamination term decreases with P. In principle, one may try to extract $\mathcal{I}_D(\nu_D, t)$ by fitting the P-dependence of the matrix element.

VI. SUMMARY

In the present we outlined an approach of lattice extraction of GPDs based on a combined use of the double distributions formalism and pseudo-PDF approach. The use of DDs guarantees that GPDs obtained through them have the required polynomiality property that imposes a non-trivial correlation between x- and ξ-dependences of GPDs. We have introduced Ioffe-time distributions writing them directly in terms of DDs, and generalized them onto correlators off the light cone. An important advantage of using DDs is that the D-term appears then as an independent quantity rather than an non-separable part of GPDs H and E. We discussed relation of the ξ-dependence of GPDS with the width of the α-profiles of the corresponding DDs, and discussed strategies for fitting lattice-extracted pseudo-distributions by DDs. The approach described in the present paper is used already in ongoing lattice extractions of GPDs by HadStruc collaboration.

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