Demonstration of the Equivalence of Soft and Zero-Bin Subtractions

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Abstract

Calculations of collinear correlation functions in perturbative QCD and Soft-Collinear Effective Theory (SCET) require a prescription for subtracting soft or zero-bin contributions in order to avoid double counting the contributions from soft modes. At leading order in λ, where λ is the SCET expansion parameter, the zero-bin subtractions have been argued to be equivalent to convolution with soft Wilson lines. We give a proof of the factorization of naive collinear Wilson lines that is crucial for the derivation of the equivalence. We then check the equivalence by computing the non-Abelian two-loop mixed collinear-soft contribution to the jet function in the quark form factor. These results demonstrate the equivalence, which can be used to give a nonperturbative definition of the zero-bin subtraction at lowest order in λ.

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In perturbative QCD (pQCD) factorization theorems [1], a cross section is expressed as a convolution of several distinct functions, each of which captures physics at a given scale. Suppose a process contains a hard scattering characterized by a scale Q which is much greater than $\Lambda_{\rm QCD}$. The factorization theorem typically contains perturbatively calculable hard coefficients, which capture physics at the scale Q, jet or collinear functions, which describe the propagation of particles in the initial or final state with energies of order Q but whose invariant mass is typically $O(\Lambda_{\rm QCD})$ or $O(\sqrt{\Lambda_{\rm QCD}Q})$, and soft functions which describe low energy quanta emitted in the process.

Soft-Collinear Effective Theory (SCET) [2, 3, 4] is an effective theory that can be used to derive factorization theorems in QCD. In this approach to QCD factorization, QCD is matched onto an effective theory that contains collinear and soft degrees of freedom, whose momentum components in light-cone coordinates scale as

$$\begin{aligned} \text{collinear}: &\quad (\bar{n}\cdot p, n\cdot p, p^{\perp}) \sim Q(1,\lambda^2,\lambda) \\ \text{soft}: &\quad (\bar{n}\cdot p, n\cdot p, p^{\perp}) \sim Q(\lambda^2,\lambda^2,\lambda^2)\,, \end{aligned} \tag{1}$$

where $\lambda \sim \sqrt{\Lambda_{\rm QCD}/Q}$. Here the light-like vectors are $\bar{n}^\mu = (1,0,0,-1)$ and $n^\mu = (1,0,0,1)$. This is the power counting of $\text{SCET}_{\rm I}$ which is relevant for inclusive processes. For exclusive processes, a different power counting is needed, and the appropriate effective theory is $\text{SCET}_{\rm II}$ [5]. Below when we refer to SCET, it is implied that we are discussing $\text{SCET}_{\rm I}$. Factorization theorems in this approach are obtained by matching QCD onto SCET at the hard scale, Q, then decoupling the soft and collinear degrees of freedom in the effective theory by a field redefinition [4]. The matching coefficients in the effective field theory are the hard coefficients of the traditional pQCD factorization theorems. Correlation functions of the collinear and soft fields correspond to the jet functions and soft functions, respectively. SCET is formulated as a systematic expansion in λ and therefore provides a framework for analyzing power corrections to leading twist pQCD factorization theorems.

In the evaluation of loop or phase space integrals that arise in the perturbative calculation of soft or jet functions, one could impose cutoffs to enforce the constraints of Eq. (1) so that no virtual soft modes contribute to the calculation of a jet function or vice versa. In practice this would make integrals horribly complicated and one almost always integrates over all momentum space [6]. This then raises the important question of avoiding double counting of soft contributions in both collinear and soft functions. This problem arises in both pQCD and SCET formulations of the problem. The traditional approach in pQCD has been to argue that jet functions need to be convolved with soft Wilson lines [7, 8, 9, 10, 11], while in SCET this is implemented by "zero-bin" subtractions [12].

An argument for the equivalence (to lowest order in λ) of the two formalisms was first given in Ref. [13]. The essence of the argument of Ref. [13] is that the zero-bin mode of the naive collinear field can be decoupled from the purely collinear field by a field redefinition similar to that used to decouple soft modes from collinear modes in SCET. Performing this field redefinition on the naive collinear matrix element, one finds that the naive jet function factorizes into a purely collinear function convolved with a vacuum matrix element of a zero-bin or soft Wilson line. In a previous paper [14], we studied the equivalence of soft and zero-bin subtractions in the quark form factor and in deeply inelastic scattering (DIS) near $x \to 1$, where x is the Bjorken variable. We emphasized the importance of using an infrared (IR) regulator such as dimensional regularization (DR) which does not take external particles off-shell and therefore spoil the field redefinition that relates the naive and purely collinear functions. We checked the equivalence at one-loop for both the quark form factor and DIS as $x \to 1$. We also verified the equivalence of soft and zero-bin subtractions for the two-loop Abelian diagrams contributing to the jet function appearing in the quark form factor.

In this paper we complete the analysis initiated in Ref. [14]. Our first objective is to complete the argument for the equivalence of soft and zero-bin subtractions presented in Ref. [13]. A crucial step in the argument is that the Wilson line of the naive collinear gluons factorizes into the product of a Wilson line constructed from the collinear zero-bin and a purely collinear Wilson line. This factorization was assumed in the derivation of Ref. [13] but not derived. In this paper, we provide a derivation of this factorization, valid up to corrections of order λ^2. Our second objective is to complete the analysis of zero-bin subtractions in the jet function of the quark form factor [14]. In particular, we verify the equivalence of soft and zero-bin subtractions to two-loops in the non-Abelian theory.

The results of this paper demonstrate the equivalence of the soft Wilson line subtraction of pQCD and the zero-bin subtractions in SCET. It is satisfying to understand the relationship between the two approaches to avoiding double counting. The equivalence of soft and zero-bin subtractions may simplify the calculation of higher order loop diagrams in SCET. In addition, the equivalence provides a nonperturbative operator definition of the SCET zero-bin subtraction, which was defined diagrammatically in Ref. [12].

The paper is organized as follows. In the next section we review the argument of Ref. [13] for the equivalence and provide the proof required for the factorization of the naive collinear Wilson line. In Section III, we complete the analysis of the two-loop zero-bin contribution to the jet function in the factorization theorem for the quark form factor in non-Abelian gauge theory. The equivalence of the zero-bin subtraction and dividing by the soft Wilson lines requires that mixed collinear and soft zero-bin contributions from certain Feynman diagrams proportional to $C_F C_A$ must add up to zero. This cancellation is verified in this section. Section IV contains our conclusions. In the Appendix we discuss some subtleties in evaluating two-loop zero-bin subtractions that arise in the L_2 limit [14].

I. FACTORIZATION OF COLLINEAR WILSON LINES

In this section we briefly review the argument of Ref. [13] and supply a proof of the factorization of the naive collinear Wilson line. Consider the naive collinear matrix element,

$$\langle X_n | \bar{\xi}''_n W''_n | 0 \rangle \, . \tag{2}$$

Here ξ''_n is the naive collinear field and W''_n is the naive collinear Wilson line. By naive we mean that the zero-bin mode has not been removed, either from the definitions of the fields or the SCET Lagrangian. The final state, $\langle X_n|$, contains collinear quanta. For the (unphysical) quark form factor, $\langle X_n|$ contains a single collinear quark, while for a physical quantity there will be a sum over infinitely many particles, weighted by a shape variable. We will not specify any properties of $\langle X_n|$ since they are not required. Lee and Sterman [13] argue that the zero-bin mode couples to the purely collinear modes in the same way as a soft mode, so the zero-bin modes can be decoupled from purely collinear modes by a similar field redefinition. Let

$$U_n(x) = P \exp\left[ig \int_0^\infty ds \, n \cdot A''_{n,0}(ns + x) \right] \, , \tag{3}$$

be the Wilson line constructed from the zero-bin mode of the collinear gauge field. The zero-bin mode is decoupled from other collinear modes by the field redefinition

$$\xi''_n = U_n^\dagger \xi'_n, \qquad A''_n = U_n^\dagger A'_n U_n, \qquad W''_n = U_n^\dagger \tilde{W}'_n U_n \,. \tag{4}$$

Though the $n \cdot A''_{n,0}$ collinear zero-bin has been decoupled, $\tilde{W}'_n$ still contains zero-bin modes of the field $\bar{n} \cdot A'_n$. To make the dependence of these modes explicit in what follows, we define $\bar{n} \cdot A_{n,0}$ (without a prime) to be the zero-bin mode of $\bar{n} \cdot A'_n$ and from now on use the notation $\bar{n} \cdot A'_n$ to refer only to the purely collinear contribution, i.e., $\bar{n} \cdot A'_n = \sum_{q \neq 0} \bar{n} \cdot A_{n,q}$. The explicit dependence on $\bar{n} \cdot A_{n,0}$ can be extracted because of the following property of $\tilde{W}'_n$,

$$\tilde{W}'_n(x) = W'_n(x)\Omega_n(x) \,, \tag{5}$$

which, as we will see below, holds up to corrections of $O(\lambda^2)$. Here, W'_n is the purely collinear Wilson line which does not contain zero-bin modes, and Ω_n is another zero-bin Wilson line defined by

$$\Omega_n(x) = P \exp\left[ig \int_{-\infty}^{0} ds \, \bar{n} \cdot A_{n,0}(\bar{n}s + x) \right] . \tag{6}$$

Using Eq. (5) one finds that the naive collinear matrix of Eq. (2) factorizes as

$$\langle X_n | \bar{\xi}''_n W''_n | 0 \rangle = \langle X'_n | \bar{\xi}'_n W'_n | 0 \rangle \, \langle X^0_n | \Omega_n U_n | 0 \rangle \,. \tag{7}$$

We have factorized the final state, $\langle X_n|$, into the product of a state that contains purely collinear degrees of freedom, $\langle X'_n|$, and a state containing zero-bin modes only, $\langle X^0_n|$. We see that the naive collinear matrix element factors into the product of a purely collinear matrix element and a matrix element of zero-bin Wilson lines.

In Ref. [13] the factorization in Eq. (5) is simply assumed, no argument for its validity is given. Here we fill this gap. The Wilson lines $\tilde{W}'_n$ and Ω_n obey the following differential equations:

$$(i\bar{n} \cdot \partial + g \, \bar{n} \cdot A_{n,0} + g \, \bar{n} \cdot A'_n)\tilde{W}'_n \; = \; 0 \tag{8}$$

$$(i\bar{n} \cdot \partial + g \, \bar{n} \cdot A_{n,0})\Omega_n \; = \; 0 \tag{9}$$

Note that Eq. (8) is not homogeneous in the power counting parameter λ. The field $\bar{n} \cdot A_{n,0}$ is $O(\lambda^2)$, $\bar{n} \cdot A'_n$ is $O(1)$, and $i\bar{n} \cdot \partial$ has no definite scaling with λ since it can act on either the $\bar{n} \cdot A_{n,0}$ or $\bar{n} \cdot A'_n$ present in $\tilde{W}'_n$. The factorization of Eq. (5) separates $\tilde{W}'_n$ into Wilson lines that are solutions to first order equations that are homogeneous in λ. To obtain this factorization, we define the function W^t_n by

$$\tilde{W}'_n(x) = \Omega_n(x) W^t_n(x) \,. \tag{10}$$

Applying the chain rule to $W^t_n(x) = \Omega^\dagger_n(x)\tilde{W}'_n(x)$, it is straightforward to show that

$$(i\bar{n} \cdot \partial + g \, \Omega^\dagger_n \bar{n} \cdot A'_n \Omega_n) W^t_n \; = \; 0 \,, \tag{11}$$

so that

$$\begin{aligned}
W_n^t(x) &= P\exp\left[ig\int_{-\infty}^{0} ds\, \Omega_n^\dagger(\bar{n}s+x)\bar{n}\cdot A_n'(\bar{n}s+x)\Omega_n(\bar{n}s+x)\right] \\
&= P\exp\left[ig\int_{-\infty}^{0} ds\, \Omega_n^\dagger(x)\bar{n}\cdot A_n'(\bar{n}s+x)\Omega_n(x)\right] + O(\lambda^2) \\
&= \Omega_n^\dagger(x) P\exp\left[ig\int_{-\infty}^{0} ds\, \bar{n}\cdot A_n'(\bar{n}s+x)\right]\Omega_n(x) + O(\lambda^2) \\
&= \Omega_n^\dagger(x)\, W_n'(x)\, \Omega_n(x) \, + O(\lambda^2).
\end{aligned} \tag{12}$$

Dropping terms supressed by λ^2 and plugging the result back into Eq. (10), we obtain the desired factorization formula for $\tilde{W}_n'$. The second line of Eq. (12) follows as a consequence of power counting, since

$$\Omega_n(\bar{n}s+x) = \exp(s\bar{n}\cdot\partial)\Omega_n(x) = \Omega_n(x) + O(\lambda^2)\,. \tag{13}$$

If we try to write $\bar{n}\cdot A_n'(\bar{n}s+x)$ as a power series in s, $\bar{n}\cdot A_n'(\bar{n}s+x) = \exp(s\bar{n}\cdot\partial)\bar{n}\cdot A_n'(x)$, all terms in the exponential are $O(1)$ since $\bar{n}\cdot A_n'$ is purely collinear. The approximation of Eq. (13) is equivalent to dropping $O(\lambda^2)$ zero-bin momenta relative to $O(1)$ purely collinear label momenta in the evaluation of the Wilson line in momentum space.

A physical jet function is not of the form of Eq. (2), but rather the square of such a matrix element with a sum over final states. For example, the jet function in a factorization theorem for an event shape cross section takes the form [13],

$$J_n'(e) = \sum_{X_n} |\langle X_n|\bar{\xi}' W_n'|0\rangle|^2 \delta(e - e(X_n))\,, \tag{14}$$

where e is the event shape variable and the delta-function ensures that only final states with e are summed over. Applying the factorization of the collinear matrix element in Eq. (2) to jet functions, one finds that the naively evaluated jet function can be expressed as a convolution of the purely collinear jet function in Eq. (14) and an eikonal jet function, which is defined in terms of matrix elements of the product of zero-bin Wilson lines, $\Omega_n U_n$ [13]. This convolution can be rendered a simple product in moment space, so one obtains

$$\tilde{J}_n'(N) = \frac{\tilde{J}_n''(N)}{\tilde{J}_n^{\rm eik}(N)}\,, \tag{15}$$

where $\tilde{J}_n'(N)$, $\tilde{J}_n''(N)$, and $\tilde{J}_n^{\rm eik}(N)$ are the Nth-moments of the purely collinear, naive collinear, and eikonal (i.e. soft) jet functions. If the jet function of interest is sufficiently inclusive that the event shape only depends on the total momentum of the final state collinear particles (e.g., the factorization theorem for DIS as $x \to 1$), then one can use translation invariance and completeness to write the jet function as the Fourier transform of a T-ordered product of the operators $\bar{\xi}_n' W_n'$ and $W_n'^\dagger \xi_n'$. The naive and purely collinear jet functions in this case are related by an equation analogous to Eq. (15), where all jet functions are now defined by the corresponding T-ordered products. For sufficiently inclusive processes the analog of $\tilde{J}_n^{\rm eik}(N)$ is the well-known soft function. Therefore the work of Ref. [13] combined with the results of this section provides an effective field theory demonstration of how dividing by the soft function eliminates double counting in pQCD factorization theorems for inclusive processes.

II. TWO-LOOP ZERO-BIN SUBTRACTIONS FOR THE JET FUNCTION

Applying the arguments of the last section to the incoming jet function that appears in the SCET factorization theorem for the quark form factor [14], one finds that

$$\langle 0|W_n'^{\dagger}\xi_n'|q(p)\rangle = \frac{\langle 0|W_n''^{\dagger}\xi_n''|q(p)\rangle}{\langle 0|U_n^{\dagger}\Omega_n^{\dagger}|0\rangle}\,. \tag{16}$$

Note that, unlike Section I, we are considering a jet function with a particle in the initial rather than final state. This quantity is clearly unphysical, however, it can be used to test the equivalence of soft and zero-bin subtractions since the argument for the factorization of the naively collinear matrix element into purely collinear and zero-bin matrix elements is independent of the initial and final states. The zero-bin subtractions needed for the purely collinear matrix element, $\langle 0|W_n'^{\dagger}\xi_n'|q(p)\rangle$, are the same zero-bin subtractions needed for virtual contributions to a physical jet function.

The remainder of this paper is devoted to completing the two-loop check of Eq. (16), initiated in Ref. [14]. Specifically, we wish to calculate the two-loop zero-bin subtractions for the left hand side of Eq. (16) and verify that they are reproduced by the right hand side. To $O(\alpha_s^2)$ the matrix elements on the right hand side of Eq. (16) can be parameterized as:

$$\begin{aligned}\frac{\langle 0|W_n''^{\dagger}\xi_n''|q(p)\rangle}{\langle 0|U_n^{\dagger}\Omega_n^{\dagger}|0\rangle} &= \frac{1+\alpha_s\,C_F\,I_n^{(1)}+\alpha_s^2\,C_F^2 I_{n,C_F^2}^{(2)}+\alpha_s^2\,C_F C_A I_{n,C_FC_A}^{(2)}+O(\alpha_s^3)}{1+\alpha_s\,C_F I_s^{(1)}+\frac{1}{2}\alpha_s^2\,C_F^2[I_s^{(1)}]^2+\alpha_s^2\,C_FC_A I_{s,C_FC_A}^{(2)}+O(\alpha_s^3)} \\ &= 1+\alpha_s\,C_F\,(I_n^{(1)}-I_s^{(1)})+\alpha_s^2\,C_F^2\left(I_{n,C_F^2}^{(2)}-I_n^{(1)}\cdot I_s^{(1)}+\frac{1}{2}[I_s^{(1)}]^2\right) \\ &\quad +\alpha_s^2\,C_FC_A\left(I_{n,C_FC_A}^{(2)}-I_{s,C_FC_A}^{(2)}\right)+O(\alpha_s^3)\,. \end{aligned}\tag{17}$$

The $I_n^{(k)}$ and $I_s^{(k)}$ are the $O(\alpha_s^k)$ contributions to the naive collinear and soft matrix elements, respectively. We have made powers of α_s and the color factors accompanying the diagrams explicit. $I_{n,C_F^2}^{(2)}$ and $I_{n,C_FC_A}^{(2)}$ denote the contributions from two-loop collinear diagrams proportional to C_F^2 and C_FC_A, respectively. $I_{s,C_FC_A}^{(2)}$ is the contribution from two-loop soft diagrams that are proportional to C_FC_A. The two-loop soft contribution multiplying the C_F^2 term is given by $[I_s^{(1)}]^2/2$ according to the exponentiation theorem. In Ref. [14], we verified that the zero-bin subtractions for the purely collinear matrix element reproduced the terms in Eq. (17) proportional to C_F and C_F^2. In this section we will check the final term in Eq. (17), which indicates that the two-loop terms proportional to C_FC_A in the zero-bin subtraction and the soft function must be equal.

The two-loop zero-bin contribution to the naive collinear matrix element contains diagrams where all momenta are soft and also mixed collinear-soft zero-bins where one momentum is collinear and the other is soft. Therefore, Eq. (17) implies a nontrivial cancellation among Feynman diagrams with mixed collinear-soft momenta. We will verify this cancellation below. For the remainder of this section we will drop C_FC_A from the subscript on $I_n^{(2)}$ and $I_s^{(2)}$ since this is the only contribution we are concerned with.

The purely collinear contribution to a two-loop integral with integrand $\mathcal{I}(k,l)$ is given by [14]

$$\int_{k,l}\left(\mathcal{I}(k^c,l^c)-\Big[\mathcal{I}(k^c,l^s)-\mathcal{I}_{L_3}(k^s,l^s)\Big]-\Big[\mathcal{I}(k^s,l^c)-\mathcal{I}_{L_2}(k^s,l^s)\Big]-\mathcal{I}_{L_1}(k^s,l^s)\right)\,. \tag{18}$$

Here $\int_{k,l} \equiv \int \frac{d^D k}{(2\pi)^D} \frac{d^D l}{(2\pi)^D}$, and we use the notation of Ref. [14]. The term $\int_{k,l} \mathcal{I}(k^c, l^c)$ is the naive collinear contribution. For this contribution, the integrand is evaluated assuming collinear scaling for both momenta, which is denoted by the superscript c on k and l in Eq. (18). The remaining terms in Eq. (18) are the zero-bin contributions which must be subtracted. There is the zero-bin arising when both k and l are soft, denoted by

$$\int_{k,l} \mathcal{I}_{L_1}(k^s, l^s) \,. \tag{19}$$

The superscript s denotes that the momenta are taken to be soft, and L_1 indicates that we are taking k and l to the soft region simultaneously [14]. There are also mixed collinear-soft zero-bins that arise when either k is collinear and l is soft,

$$\int_{k,l} \left[\mathcal{I}(k^c, l^s) - \mathcal{I}_{L_3}(k^s, l^s) \right] , \tag{20}$$

or when l is collinear and k is soft,

$$\int_{k,l} \left[\mathcal{I}(k^s, l^c) - \mathcal{I}_{L_2}(k^s, l^s) \right] . \tag{21}$$

The L_2 limit is defined by taking k soft while l is collinear, then taking l to be soft. The L_3 limit is the same with k and l interchanged. To see that the limits are in general different, consider what happens to a propagator with momentum $k + l$ in the three limits:

$$\begin{aligned}
\lim_{L_1} \frac{1}{(k+l)^2} &= \frac{1}{(k+l)^2} \,, \\
\lim_{L_2} \frac{1}{(k+l)^2} &= \frac{1}{l^2 + \bar{n} \cdot l \, n \cdot k} \,, \\
\lim_{L_3} \frac{1}{(k+l)^2} &= \frac{1}{k^2 + \bar{n} \cdot k \, n \cdot l} \,.
\end{aligned} \tag{22}$$

When k and l are collinear, k^2, $2k \cdot l$, and l^2 are all $O(\lambda^2)$. Taking k and l to the soft region simultaneously, k^2, $2k \cdot l$, and l^2 are all $O(\lambda^4)$. The propagator in Eq. (22) is $O(\lambda^{-4})$ in the L_1 limit, as opposed to $O(\lambda^{-2})$ when k and l are collinear. However, the relative importance of all three terms in the denominator remains the same, so the form of the propagator is unchanged. In the L_2 limit, we first take k soft while keeping l collinear. Then l^2 and $\bar{n} \cdot l \, n \cdot k$ are $O(\lambda^2)$ but $k^\perp \cdot l^\perp$ is $O(\lambda^3)$ and $n \cdot l \, \bar{n} \cdot k$ and k^2 are $O(\lambda^4)$. The $O(\lambda^3, \lambda^4)$ terms in the denominator are dropped. Next, we take l soft. The propagator denominator is still $l^2 + \bar{n} \cdot l \, n \cdot k$, but now l^2 and $\bar{n} \cdot l \, n \cdot k$ are $O(\lambda^4)$. The propagator scales as $O(\lambda^{-4})$ in the L_1, L_2, and L_3 limits, but the form of the propagator is different in each case.

Strictly speaking, Eq. (18) is naive since it does not account for the possibility that a mixed collinear-soft zero-bin arises when one linear combination of k and l becomes soft and the orthogonal linear combination stays collinear. One must check for a zero-bin for every linear combination of loop momenta that appears in a propagator of the Feynman diagram. However, in practice we find that the zero-bin contribution is subleading in λ unless the gluon propagator connects with the Wilson line. Therefore if we route the momenta so that the momenta of each gluon connected to the Wilson line coincides with one of the loop momenta, then Eq. (18) is sufficient.

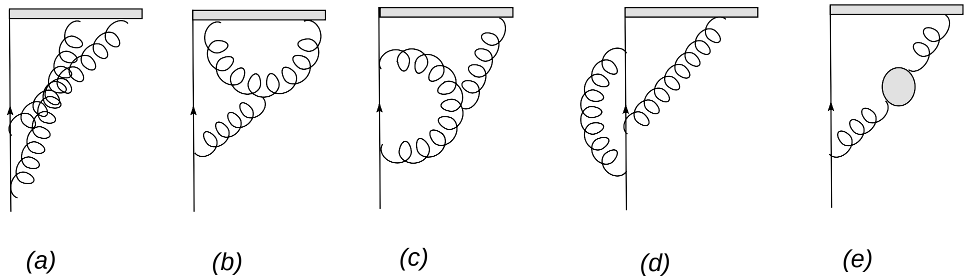

FIG. 1: Two-loop SCET diagrams contributing to $I^{(2)}_{n,C_F C_A}$. The solid line with an arrow is the incoming massless quark. The double line is the Wilson line. All gluons are collinear.

Next we turn to the two-loop collinear SCET diagrams that give a contribution proportional to $C_F C_A$. Two-loop QCD-like graphs are depicted in Fig. 1, while two-loop graphs with the SCET seagull vertex are shown in Fig. 2. We will analyze diagrams in Fig. 1 first, and start by considering the zero-bin from the L_1 limit. It is easy to see that this contribution is identical to the contribution of the two-loop soft matrix element proportional to $C_F C_A$. All calculations are performed in Feynman gauge. The external quark is on-shell and massless. Because the equivalence relies on field redefinitions, we cannot regulate IR divergences by taking external particles off-shell. We will use DR to regulate both UV (ultraviolet) and IR divergences. We denote the momentum of the outermost gluon connected to the Wilson line as k. In Figs. 1(a) and (b), the inner gluon connected to the Wilson line has momentum l. In Figs. 1(c) and (d), l is the momentum of one of the gluons in the quark-gluon vertex correction subgraph. In Fig. 1(e), the blob represents all possible contributions to the gluon self-energy that give rise to a color factor $C_F C_A$. The loop momentum l is the momentum of one of the gluons or ghosts in the self-energy subgraph. Then the propagator denominators of all gluons (or ghosts) in Fig. 1 are l^2, k^2, or $(l+k)^2$. From Eq. (22) we see that these propagators are unchanged in the L_1 limit. The coupling of the gluons to the Wilson line is also unaffected by taking the L_1 limit. Finally, we need to consider the modification to the coupling of the collinear quark to the soft gluon. In the L_1 limit, expanding to lowest order in λ, it is easy to show that the collinear quark coupling and propagator give the same Feynman rules as the eikonal Wilson line, $U_n^\dagger$. No analog of Fig. 1(d) exists in the soft matrix element. In the L_1 limit of the collinear matrix element, Fig. 1(d) gives a contribution that is subleading in λ. The L_1 limit of the other diagrams in Fig. 1 are exactly the same as the corresponding two-loop diagrams contributing to the $C_F C_A$ term in the calculation of the soft matrix element, shown in Fig. 3. It is then obvious that

$$\int_{k,l} \mathcal{I}^{(2,a-e)}_{L_1}(k^s, l^s) = I_s^{(2)} \,, \tag{23}$$

where $\mathcal{I}^{(2,a-e)}(k,l)$ denotes the sum of the integrands coming from the two-loop diagrams in Fig. 1(a)-(e). Thus the zero-bin subtraction and soft Wilson line subtraction give equivalent results if the remaining zero-bin contributions all vanish. The remaining possible zero-bin contributions are the mixed collinear-soft zero-bins of the diagrams in Fig. 1 and zero-bin contributions from diagrams with the SCET seagull vertex depicted in Fig. 2. Note that the mixed collinear-soft zero-bin subtraction does not vanish for the two-loop Abelian diagrams and is necessary to reproduce the $-\alpha_s^2 C_F^2 I_n^{(1)} \cdot I_s^{(1)}$ term in Eq. (17).

We next analyze the mixed collinear-soft zero-bin contributions from Fig. 1. With the routing described above it is straightforward to show that the zero-bin subtraction is sub-

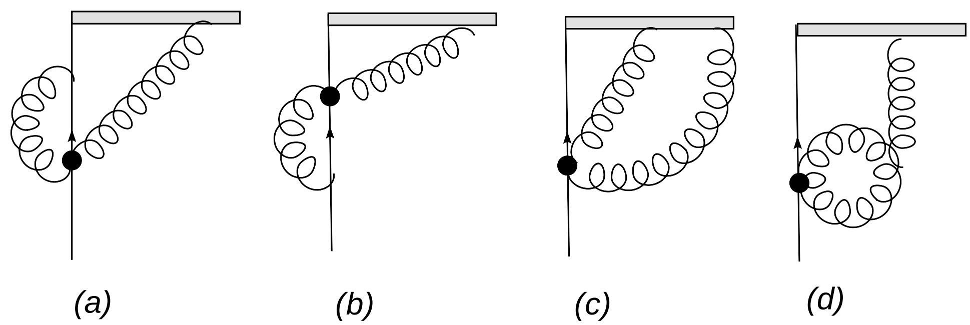

FIG. 2: Two-loop SCET diagrams involving the seagull vertex.

leading in λ unless it comes from the region where k is soft and l is collinear. There is no mixed collinear-soft zero-bin contribution from Fig. 1(e). The reason is that when l is collinear and k soft, Fig. 1(e) gives a contribution proportional to

$$\bar{n}^\mu n^\nu \int_{k,l} \frac{k^2 g^{\mu\nu} - k^\mu k^\nu}{k^2\, \bar{n}\cdot k\, n\cdot k\, l^2\, (l^2 + \bar{n}\cdot l\, n\cdot k)} \tag{24}$$

From the naive collinear integrand in Eq. (24) one must subtract the integrand in the L_2 limit. However, the integrand in Eq. (24) does not change in the L_2 limit, so the integrand when l is collinear and k is soft vanishes,

$$\mathcal{I}^{(2,e)}(k^s, l^c) - \mathcal{I}^{(2,e)}_{L_2}(k^s, l^s) = 0\,. \tag{25}$$

The remaining diagrams, Figs. 1(a)-(d), have nonvanishing mixed collinear-soft zero-bin contributions. There is a nontrivial cancellation among the four diagrams that ensures that the two-loop zero-bin subtraction proportional to $C_F C_A$ reproduces the right hand side of Eq.(17). We will describe the evaluation of Fig. 1(b) in some detail and quote our results for the remaining diagrams. Evaluation of Fig. 1(b) gives

$$g_s^4 C_A C_F \int_{k,l} \frac{\bar{n}\cdot(l-k)\,\bar{n}\cdot(p+l+k)}{l^2\, k^2\, (l+k)^2\, \bar{n}\cdot(l+k)\, \bar{n}\cdot k\, (p+l+k)^2}\,. \tag{26}$$

When k and l are taken to be soft and collinear, respectively, we get the naive collinear contributions to $I^{(2)}$,

$$\begin{aligned} I^{(2,b)}_{n,\mathrm{nc}} &\equiv \int_{k,l} \mathcal{I}^{(2,b)}(k^s, l^c) \\ &= (4\pi)^2 \int_{k,l} \frac{1}{k^2\, \bar{n}\cdot k}\, \frac{\bar{n}\cdot(p+l)}{l^2\,(l^2+\bar{n}\cdot l\, n\cdot k)\,((p+l)^2 + \bar{n}\cdot(p+l)\, n\cdot k)}\,. \end{aligned} \tag{27}$$

The factor of $(4\pi)^2$ arises because $I^{(2,b)}_{n,\mathrm{nc}}$ is defined to be the amplitude of the Feynman diagram divided by $\alpha_s^2 C_F C_A$. The l-integral is easily evaluated using Feynman parameters and the result is

$$I^{(2,b)}_{\mathrm{nc}} = I^{(1)}_s \times \frac{1}{\varepsilon_{\mathrm{IR}}^2} \frac{\Gamma[1-\varepsilon_{\mathrm{IR}}]\Gamma[2-\varepsilon_{\mathrm{IR}}]\Gamma[1+\varepsilon_{\mathrm{IR}}]}{\Gamma[2-2\varepsilon_{\mathrm{IR}}]}\,, \tag{28}$$

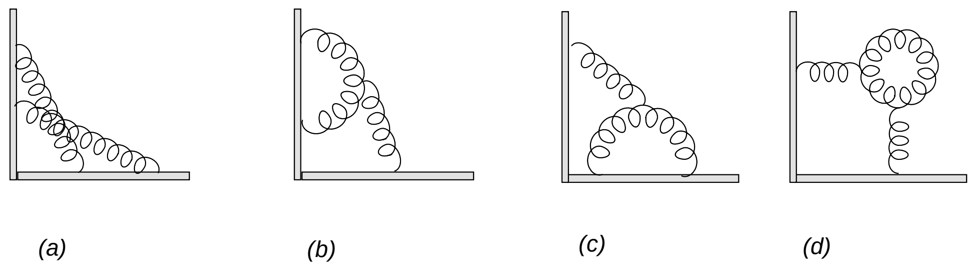

FIG. 3: Two-loop diagrams contributing to $I_s^{(2)}$.

where

$$I_s^{(1)} \equiv i \int_k \frac{1}{k^2\,\bar{n}\cdot k\, n\cdot k}\left(\frac{\bar{n}\cdot p\, n\cdot k}{4\pi\mu^2}\right)^{-\varepsilon} . \tag{29}$$

The divergences are of IR origin and this is denoted by the subscript IR on the DR parameter $\varepsilon = (4-D)/2$.

Note that the subgraph containing the virtual l momentum gives rise to a double IR pole, $1/\varepsilon_{\rm IR}^2$. In a one-loop diagram, double $1/\varepsilon_{\rm IR}^2$ poles come from regions where the virtual momenta is both soft and collinear. If l is purely collinear we do not expect to see a double IR divergence in this subgraph. This arises in the naive collinear graph because we have not excluded the region where l is soft. Once we perform the zero-bin subtraction (by removing the L_2 region) the final answer must be free from $1/\varepsilon_{\rm IR}^2$ poles. We will see that this is the case below.

Taking the L_2 limit of the integrand in Eq. (27) yields

$$\begin{aligned} I_{n,L_2}^{(2,b)} \equiv \int_{k,l} \mathcal{I}_{L_2}^{(2,b)}(k^s,l^s) &= (4\pi)^2 \int_{k,l} \frac{1}{k^2\,\bar{n}\cdot k}\,\frac{1}{l^2\,(l^2+\bar{n}\cdot l\, n\cdot k)\, n\cdot(l+k)} \\ &= -(4\pi)^2 \int_{k,l} \frac{1}{k^2\,\bar{n}\cdot k}\,\frac{1}{l^2\,(l^2+\bar{n}\cdot l\, n\cdot k)\, n\cdot l} , \end{aligned} \tag{30}$$

where the last line follows from the change of variables $\bar{n}\cdot l \to -\bar{n}\cdot l$ and $n\cdot l \to -n\cdot l - n\cdot k$. The same integral arises in the evaluation of the L_2 limit of Fig. 1(c). The evaluation of this integral is subtle and is discussed in the Appendix. The result is

$$I_{L_2}^{(2,b)} = -I_s^{(1)} \times \left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\frac{1}{\varepsilon_{\rm IR}} , \tag{31}$$

so the mixed collinear-soft zero-bin contribution from Fig. 1(b) is

$$\begin{aligned} I_{\rm pc}^{(2,b)} &= I_{\rm nc}^{(2,b)} - I_{L_2}^{(2,b)} \\ &= I_s^{(1)} \times \left[\frac{1}{\varepsilon_{\rm IR}^2}\frac{\Gamma[1-\varepsilon_{\rm IR}]\Gamma[2-\varepsilon_{\rm IR}]\Gamma[1+\varepsilon_{\rm IR}]}{\Gamma[2-2\varepsilon_{\rm IR}]} + \left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\frac{1}{\varepsilon_{\rm IR}}\right] . \end{aligned} \tag{32}$$

As expected the $1/\varepsilon_{\rm IR}^2$ poles cancel in the purely collinear loop integral.

We obtain a similar result for Fig. 1(a),

$$I_{\rm pc}^{(2,a)} = I_s^{(1)} \times \left[-\frac{2}{\varepsilon_{\rm IR}^2}\frac{\Gamma[1-\varepsilon_{\rm IR}]\Gamma[2-\varepsilon_{\rm IR}]\Gamma[1+\varepsilon_{\rm IR}]}{\Gamma[2-2\varepsilon_{\rm IR}]} - 2\left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\frac{1}{\varepsilon_{\rm IR}}\right] . \tag{33}$$

When evaluating Fig. 1(c) in the limit that k is soft and l is collinear, we find that the l-integral can be split into a UV divergent term, an IR divergent term, and a term that is identical to the l-integral of Eq. (24). The last term gives a vanishing contribution once the L_2 limit of the integrand is subtracted. The remaining terms give

$$I_{\rm pc}^{(2,c)} = I_s^{(1)} \times \left[-\frac{\Gamma[\varepsilon_{\rm UV}]\Gamma[2-\varepsilon_{\rm UV}]\Gamma[1-\varepsilon_{\rm UV}]}{\Gamma[2-2\varepsilon_{\rm UV}]} + \frac{1}{\varepsilon_{\rm IR}^2}\frac{\Gamma[1-\varepsilon_{\rm IR}]\Gamma[2-\varepsilon_{\rm IR}]\Gamma[1+\varepsilon_{\rm IR}]}{\Gamma[2-2\varepsilon_{\rm IR}]} \right. \\ \left. + \left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\frac{1}{\varepsilon_{\rm IR}} \right] . \tag{34}$$

Finally, Fig. 1(d) gives

$$I_{\rm pc}^{(2,d)} \; = \; I_s^{(1)} \times \frac{\Gamma[\varepsilon_{\rm UV}]\Gamma[2-\varepsilon_{\rm UV}]\Gamma[1-\varepsilon_{\rm UV}]}{\Gamma[2-2\varepsilon_{\rm UV}]} . \tag{35}$$

We see that

$$I_{\rm pc}^{(2,a)} + I_{\rm pc}^{(2,b)} + I_{\rm pc}^{(2,c)} + I_{\rm pc}^{(2,d)} = 0 , \tag{36}$$

so the mixed collinear-soft zero-bin contributions from Figs. 1(a)-(e) add up to zero.

In addition to the mixed collinear-soft zero-bin discussed above there are two-loop SCET diagrams that have no QCD analog. These graphs, given in Fig. 2, involve the seagull interactions with two collinear quarks and two collinear gluons (see Fig. 1 of Ref. [3]). The Feynman rule for the SCET seagull interaction is such that if a collinear gluon from the seagull vertex is contracted with the gluon coming from the Wilson line, $W_n^\dagger$, the graph vanishes, so the diagrams of Fig. 2(a)-(c) are zero. For Fig. 2(d) we obtain

$$-\frac{1}{2} g_s^4 C_F C_A \int_{k,l} \frac{\bar n\cdot(p-k)\,\bar n\cdot(2\,l-k)}{\bar n\cdot k\,k^2\,l^2\,(l-k)^2\,(p-k)^2} \left[\frac{1}{\bar n\cdot(p-l)} - \frac{1}{\bar n\cdot(p+l-k)}\right] . \tag{37}$$

When k is soft and l collinear one obtains the naive collinear contribution to $I_n^{(2)}$:

$$I_{\rm nc}^{(2,3d)} = (4\pi)^2 \int_{k,l} \frac{1}{\bar n\cdot k\, n\cdot k\, k^2} \frac{\bar n\cdot l}{l^2\,(l^2-\bar n\cdot l n\cdot k)} \left[\frac{1}{\bar n\cdot(p-l)} - \frac{1}{\bar n\cdot(p+l)}\right] , \tag{38}$$

which by SCET power counting is $\mathcal{O}(1)$. The integration over l can be performed by combining denominators using the standard Feynman parameterization and after completing the square the resulting integral vanishes by symmetry. Therefore this zero-bin contribution vanishes. It is also easy to check that the zero-bin contributions from both the L_1 and L_2 regions are subleading in the λ expansion and can be ignored. This result is essential as Fig. 2(d) has no analog in the soft function diagrams shown in Fig. 3.

III. CONCLUSION

In this paper, we completed the two-loop analysis of zero-bin subtractions to the jet function appearing in the quark form factor initiated in Ref. [14]. We verified that the zero-bin subtraction is equivalent to dividing by a matrix element of soft Wilson lines to two-loop order. We also supplied a proof of the factorization of the naive collinear Wilson line into a purely collinear Wilson line and a soft Wilson line, valid to $O(\lambda^2)$. This property is essential for the argument for the equivalence of soft and zero-bin subtractions first presented in Ref. [13]. These arguments imply that the equivalence of soft and zero-bin subtractions should hold to all orders in perturbation theory.

One important consequence of this equivalence (to lowest order in λ) is that it provides an operator definition for the zero-bin subtraction procedure that goes beyond perturbation theory. The equivalence may also help to simplify higher order calculations in SCET. For example, in the calculation of the jet function in this paper, we saw that the equivalence led us to anticipate a cancellation between mixed collinear-soft zero-bin subtractions coming from several different SCET diagrams. We expect that the equivalence will simplify SCET calculations of collinear correlation functions, like parton distribution functions, jet functions, and fragmentation functions, since it allows one to work with the naive collinear correlation functions and soft matrix elements, rather than having to remove zero-bins in collinear correlation functions diagram by diagram.

Though the analysis of this paper, as well as Refs. [13] and [14], focuses on avoiding double counting in factorization theorems for inclusive processes, the issue of double counting arises in exclusive processes as well [12]. It would be interesting to see if one could find an operator definition of the zero-bin subtraction in $\mathrm{SCET_{II}}$.

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IV. APPENDIX

In this Appendix, we discuss our evaluation of the l integral in Eq. (30). First, we use the identity

$$\frac{1}{l^2\,(l^2+\bar{n}\cdot l\, n\cdot k) n\cdot l} = 2\int_0^\infty d\lambda \int_0^1 dx \frac{x}{(l^2 + x\,\lambda\, n\cdot l + (1-x)\,\bar{n}\cdot l\, n\cdot k)^3}\,. \tag{39}$$

The parameter λ is dimensionful and can be made dimensionless by rescaling $\lambda \to \bar{n}\cdot p\,\lambda$. The l integral is then straightforward to evaluate by completing the square. The result is

$$\begin{aligned} I_{L_2}^{(2,b)} &= I_s^{(1)} \times \int_0^\infty d\lambda\, \lambda^{-1-\varepsilon} \int_0^1 dx\, x^{-\varepsilon}(1-x)^{-1-\varepsilon}\,\Gamma[1+\varepsilon] \\ &= -I_s^{(1)} \times \left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\frac{1}{\varepsilon_{\rm IR}}\,. \end{aligned} \tag{40}$$

The λ integral is proportional to $1/\varepsilon_{\rm UV} - 1/\varepsilon_{\rm IR}$ while the $1/\varepsilon$ from the x integral is clearly IR in origin. Note that we have set a factor of

$$\frac{\Gamma[1-\varepsilon]^2\Gamma[1+\varepsilon]}{\Gamma[1-2\varepsilon]}\,, \tag{41}$$

equal to 1. Anything besides the double $1/\varepsilon$ poles in the evaluation of the scaleless integral is ambiguous. One source of ambiguity is the freedom to rescale λ. We chose to rescale λ so as to obtain a prefactor proportional to $(\bar{n}\cdot p\, n\cdot k/(4\pi\mu^2))^{-\varepsilon}$, but other rescalings are possible. The other source of ambiguity is due to the ambiguity in expanding any function of ε when multiplying a factor of $1/\varepsilon_{\rm UV} - 1/\varepsilon_{\rm IR}$.

However, there is even more ambiguity in the result because the coefficient of the double ε poles actually depends on how one choses to combine denominators using Feynman parameters. To see this we now evaluate Eq. (30) using the identity

$$\frac{1}{l^2\,(l^2+\bar{n}\cdot l\, n\cdot k)n\cdot l} = 2\int_0^\infty d\lambda \int_0^1 dx \frac{x}{(l^2 + x\,\lambda\, n\cdot l + x\,\bar{n}\cdot l\, n\cdot k)^3}\,. \tag{42}$$

The result is then

$$\begin{aligned} I_{L_2}^{(2,b)} &= I_s^{(1)} \times \int_0^\infty d\lambda\, \lambda^{-1-\varepsilon} \int_0^1 dx\, x^{-1-2\varepsilon}\, \Gamma[1+\varepsilon] \\ &= -\frac{1}{2} I_s^{(1)} \times \left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\frac{1}{\varepsilon_{\rm IR}}\,, \end{aligned} \tag{43}$$

which differs from Eq. (40) by a factor of 1/2!

Mathematically, there is no inconsistency here, since $\varepsilon_{\rm UV} = \varepsilon_{\rm IR} = 2 - D/2$ so the right hand sides of Eq. (40) and Eq. (43) are both equal to zero. Both results are consistent with the well-known result that scaleless integrals vanish in DR. However, if a physical regulator is used, a scaleless integral would be the sum of UV and IR divergent terms. Since the cancellation of UV and IR divergences in physical observables is handled differently in quantum field theory, one often wishes to separate scaleless integrals into their UV and IR divergent parts, even if DR is being used to regulate both. Evidently, for the double $1/\varepsilon$ poles appearing in the integral of Eq. (30), such a separation is ambiguous.

Faced with this situation, one is forced to invoke a prescription for handling the zero-bin integrals in the L_2 limit. One physically motivated prescription was mentioned earlier, that is to fix the overall coefficient so that the $1/\varepsilon_{\rm IR}^2$ poles cancel in the purely collinear integral. This leads to the result of Eq. (40). Another way to resolve the ambiguity is to modify the integral by including a λ-suppressed correction. For instance, we can replace $l^2 + \bar{n}\cdot l\, n\cdot k$ with $(l+k)^2$ with $k^2 \neq 0$. Now the result of the integral is completely unambiguous and equal to the result of Eq. (40) with the following modification

$$\left(\frac{1}{\varepsilon_{\rm UV}} - \frac{1}{\varepsilon_{\rm IR}}\right)\frac{1}{\varepsilon_{\rm IR}} \to \left(\frac{1}{\varepsilon_{\rm UV}} - \log\left(\frac{-k^2}{4\pi\mu^2}\right) + \frac{\pi^2}{12}\right)\frac{1}{\varepsilon_{\rm IR}}\,. \tag{44}$$

This prescription unambiguously fixes the coefficient of the mixed $1/(\varepsilon_{\rm UV}\varepsilon_{\rm IR})$ pole and is in agreement with the result of Eq. (40). We will use the result of Eq. (40) for the evaluation of the l-integral for the L_2 zero-bin integral in Eq. (30).

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